Large-size critical behavior of infinitely coordinated systems

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Details are presented of an extension of the size-scaling hypothesis to systems in which each element interacts equally with all others (systems for which the mean-field approximation is valid in the thermodynamic limit). A simple argument, which relates the large-size critical behavior of physical quantities with the upper critical dimensionality of the corresponding short-range system, already presented in a Letter, is here made precise and checked either analytically or numerically on several examples. In particular, the scaling form for the magnetization is explicitly derived in the case of the infinitely coordinated Ising model, and a numerical study is presented of the infinitely coordinated *XY*-Ising quantum model in a transverse field, with its extension in the presence of an imaginary longitudinal field (a model exhibiting a Yang-Lee edge singularity).

I. INTRODUCTION

This paper presents details and further applications of a work on the extension of the finite-size scaling hypothesis of Fisher and Barber to infinitely coordinated systems.[1] We define as an "infinitely coordinated system" a thermodynamical system of N elements, each of which is coupled to all others with a strength independent of the position and the nature of the elements. One of the most simple examples of such a system is the infinitely coordinated ferromagnetic Ising model defined by the Hamiltonian

$$H=-JN^{-1}\sum_{i\neq j}S_iS_j\ , \tag{1}$$

where $S_i=\pm\frac{1}{2}$ are spin variables, the summation covers the whole range of i and j values from 1 to N without any restriction (except $i=j$), and J is a positive constant. The N^{-1} prefactor appearing in (1) is essential to ensure the convergence of the free energy per spin in the limit where N tends to infinity (thermodynamic limit). For all other examples this prefactor must not be forgotten unless some of our general conclusions become wrong. This simple example (as well as the Heisenberg case) has been first treated analytically for any N by Kittel and Shore.[2] We will reproduce these calculations in a closer form as a check of our general predictions in Sec. III.

Both the simplicity of these models and the fact that they can often be studied analytically[2,3] explain why they have been widely studied in the past.[2–5] For finite N, they have been often used as models for the nucleus, as in the Lipkin model.[4] For infinite N, they have been used to describe dense gases, and the study of the so-called "spin van der Waals models" has been the subject of a great interest.[5] When such systems present a second-order phase transition in the thermodynamic limit ($N\rightarrow\infty$), it is well known that the mean-field approximation is justified due to the long-range nature of the interactions.[2–5] It is for this reason that such systems have often been introduced as a simple "mean-field" model of real short-range d-dimensional systems. This is, for example, the case in the model of Sherrington and Kirkpatrick for spin-glasses.[6]

This paper is devoted to the behavior of infinitely coordinated systems when N is large but finite. When the system develops a second-order phase transition for $N=\infty$, the nature of the transition, and the value of the exponents are generally well known through the mean-field approximation. However, when N is large but finite, generally no true transition exists. For very large N there is a critical scaling of the thermodynamical quantities with N, which depends on the nature of the system and which has not been systematically studied up to now. In the simple example of the Ising model described by Eq. (1), Kittel and Shore[2] found analytically that the magnetization goes to zero as $N^{-1/4}$ at the critical temperature, but no general argument was given to explain this characteristic exponent. By extending the scaling hypothesis of Fisher and Barber[7] to these systems we are able to give some general conclusions which explain very simply this result as well as other numerical results that we present in the paper. In Sec. II we present the extension of the scaling hypothesis; in Sec. III we give some analytical examples, and in Sec. IV we present some numerical calculations for infinitely coordinated quantum-spin systems. On all these examples the general conclusions of Sec. II are checked.

II. SIZE SCALING OF INFINITELY COORDINATED SYSTEMS

A. Size scaling in the short-range case

Let us recall briefly the scaling hypothesis as introduced by Fisher and Barber[7] and Nightingale[8] for systems with short-range interactions (such as nearest-neighbor interactions). Let us consider a homogeneous and isotropic d-dimensional system, contained in a hypercube of length L, which develops in the thermodynamic limit $L=\infty$ a regular second-order phase transition at $T=T_c$. As usual we define a characteristic coherence length ξ for the infinite system ($L=\infty$) which diverges at T_c as

$$\xi \underset{L=\infty}{\sim} |T-T_c|^{-\nu} . \tag{2}$$

Let us consider a given thermodynamical quantity A which is singular at the transition in the infinite system. Its critical exponent a is defined by

$$A \underset{L=\infty}{\sim} |T-T_c|^{a} . \tag{3}$$

The scaling hypothesis postulates the existence of a regular function $F_a(x)$ such that, for large L, A can be written as

$$A \sim |T-T_c|^{a} F_a(L/\xi) . \tag{4}$$

$F_a(x)$ is such that $F_a(x) \to \text{const}$ when $x \to \infty$ in order that (3) is recovered, and $F_a(x) \sim x^{\omega_a}$ when $x \to 0$ with $\omega_a = -a/\nu$ in order that A must be regular at $T=T_c$ for L finite. As a consequence, the critical scaling of A at $T=T_c$ is given by

$$A \underset{T=T_c}{\sim} L^{\omega_a} \sim L^{-a/\nu} . \tag{5}$$

The scaling hypothesis has been checked in many cases and is currently used in numerical methods such as the "phenomenological renormalization group,"[8,9] which has been applied to various model Hamiltonians. More precisely, this hypothesis is valid for the magnetization, the susceptibility, the free energy per site, etc., for dimensionalities smaller than the upper critical dimensionality d_c at which the short-range system starts to have a mean-field behavior. At $d=d_c$ some extra logarithmic terms appear in the scaling (L/ξ is replaced by some $(L/\xi)[\ln(L/L_0)]^{\alpha}$), as shown in the case of the spherical model.[10] These logarithmic terms can be attributed to fluctuations which, even if they are not able to change the exponents from their mean-field values, can however affect the critical behavior.

B. Extension to infinitely coordinated systems

The usual scaling cannot be straightforwardly extended to infinitely coordinated systems since, in these systems, the usual concepts of "length" as well as "dimensionality" have completely lost their meanings. Each element interacting equally with all others, it does not matter if they are disposed on a line, on a plane, etc. The coherence length ξ, which plays an essential role in the usual scaling is then conveniently replaced by a more general quantity N_c, a "coherence number," independent of the dimensionality. We suppose that this coherence number diverges at the transition in the infinite system ($N=\infty$) with a characteristic exponent ν^*:

$$N_c \underset{N=\infty}{\sim} |T-T_c|^{-\nu^*} . \tag{6}$$

The physical meaning of N_c is less clear than in the short-range case, where it can be defined as the number of elements contained in the volume ξ^d. Here, all the elements are completely delocalized and we cannot refer to a well-defined volume in the space. The existence of N_c is here a simple conjecture which will be tested by the consequences.

As above, we consider a given thermodynamical quantity which behaves near T_c in the infinite system as

$$A \underset{N=\infty}{\sim} |T-T_c|^{a_{\rm MF}} , \tag{7}$$

where $a_{\rm MF}$ denotes the mean-field value of the exponent a. The scaling hypothesis is extended by postulating the existence of a regular function $F_a^*(x)$ such that, near T_c and for large N, A behaves as

$$A \sim |T-T_c|^{a_{\rm MF}} F_a^*(N/N_c) . \tag{8}$$

As above, it is necessary that $F_a^*(x) \to \text{const}$ when $x \to \infty$ and that $F_a^*(x) \sim x^{\omega_a}$ with $\omega_a = -a_{\rm MF}/\nu^*$ when $x \to 0$. Then at $T=T_c$, one has

$$A \underset{T=T_c}{\sim} N^{\omega_a} \sim N^{-a_{\rm MF}/\nu^*} . \tag{9}$$

Thus if one knows ν^* as well as all the mean-field exponents, the critical scaling of all thermodynamical quantities is known.

C. General argument giving the exponent ν^*

Let us now show how ν^* can be simply linked with the upper critical dimensionality d_c of the corresponding finite-ranged model by the following reasoning. The main idea is to compare the infinitely coordinated system of N elements with the corresponding short-range system at $d=d_c$ and with finite length $L=N^{1/d_c}$ in each space direction. By definition of d_c, this short-range system is supposed to behave with mean-field behavior for L infinite. A first assumption is that the scaling of Fisher and Barber applies, *even at* $d=d_c$, to this short-range system; i.e., there exists a function F_a such that Eq. (4) can be written where a takes its mean-field value $a_{\rm MF}$. A second assumption is that both systems have the same scaling exponents. Then the correspondence between Eqs. (4) and (8) is made by assuming $N_c \sim \xi^{d_c}$. As a direct consequence it follows, by comparing Eqs. (2) and (6) that

$$\nu^* = \nu_{\rm MF} d_c \ . \qquad (10)$$

We would like to emphasize that this derivation of ν^* is based on many assumptions. In particular it is known that the first assumption is simply not true since some logarithmic corrections must appear in the scaling of the finite ranged system at $d=d_c$.[10] These logarithmic terms are due to fluctuations. It is reasonable to suppose that the fluctuations disappear completely in the uniform infinite ranged limit where the system becomes infinitely coordinated, so that the logarithmic terms would not appear in Eq. (8). In fact, this assumption will be checked in all examples of infinitely coordinated models that we will consider below.

As a direct consequence of formula (10) one gets for any infinitely coordinated system

$$A \underset{T=T_c}{\sim} N^{\omega_a} \ \text{ with } \omega_a = -\frac{a_{\rm MF}}{\nu_{\rm MF} d_c} \ , \qquad (11)$$

so that the critical scaling for large N of any quantity is known if one knows the mean-field exponents and the upper critical dimensionality of the corresponding short-ranged system.

Applied to the case of the magnetization in the infinitely coordinated Ising model (as well as the Heisenberg model) for which $\beta_{\rm MF}=\nu_{\rm MF}=\frac{1}{2}$ and $d_c=4$,[11] this general argument explains very simply the result of Kittel and Shore,[2] $m \sim N^{-1/4}$. We would like to give more details on this simple example by showing that the scaling form (8) can be analytically derived.

III. ANALYTICAL EXAMPLES

A. Infinitely coordinated Ising model

On the very simple example of the infinitely coordinated Ising model, described by the Hamiltonian (1), we would like to show, by following Kittel and Shore[2] and Niemeijer,[5] how a scaling form of the type (8) can be derived explicitly for the magnetization.

Note that the Hamiltonian (1) can be also written

$$H = -\frac{J}{N}\left[\sum_i S_i\right]^2 + \frac{J}{4} \ . \qquad (12)$$

The energies are

$$E_p = -J(N-1)/4 + Jp(N-p)/N \ , \qquad (13)$$

$p=0,1,2,\ldots,N$ with degeneracy

$$G(E_p) = \frac{N!}{p!(N-p)!} \ . \qquad (14)$$

The partition function can be calculated by

$$Z_N = \sum_{p=0}^{N} G(E_p)\exp(-\beta E_p) \ . \qquad (15)$$

with the use of the procedure of Niemeijer,[5] the summation can be *exactly* transformed into an integral, for any value of N,

$$Z_N = K_N \int_{-\infty}^{+\infty} \exp[-Nf(\eta)]d\eta \qquad (16)$$

with

$$K_N = 2^N (N\beta J/\pi)^{1/2} \exp(-\beta J/4) \qquad (17a)$$

and

$$f(\eta) = \beta J \eta^2 - \ln\cosh(\beta J \eta) \ . \qquad (17b)$$

Let us define here the thermal average of the magnetization by

$$m = \left\langle \left[\frac{1}{N}\sum_i S_i\right]^2 \right\rangle^{1/2} = \left[\frac{1}{N}\frac{\partial \ln Z_N}{\partial(\beta J)}\right]^{1/2} \ . \qquad (18)$$

Then, an integration by parts gives

$$m^2 = \langle \eta^2 \rangle - \frac{1}{2\beta J N} \ , \qquad (19)$$

and, if we ignore $1/N$ corrections, we can work with

$$m^2 = \frac{\int \eta^2 \exp[-Nf(\eta)]d\eta}{\int \exp[-Nf(\eta)]d\eta} \ . \qquad (20)$$

With this formula, m is different from the spontaneous magnetization defined from the average of $|\eta|$ as in the original papers.[2,4] Our m^2 is more exactly proportional to the zero-field susceptibility multiplied by the temperature. However, for large N our m^2 does approach the square of the spontaneous magnetization with corrections of order N^{-1} which do not affect the slower critical behavior that we will find in Eq. (24). Exact relations (20) and (17b) allow one to determine implicitly $m(T)$ *for any N.*

In the thermodynamic limit $N\rightarrow\infty$, we recover the result that the spontaneous magnetization corresponds to the minimum of the function $f(\eta)$ and is thus given by the implicit equation

$$m_{\infty}=\tfrac{1}{2}\tanh(\beta J m_{\infty}) , \tag{21}$$

which corresponds to the mean-field approximation.

The Curie temperature is thus given by $\beta J=2$, i.e., $k_B T_c=J/2$, $m_{\infty}=0$ for $T>T_c$, $m_{\infty}\neq 0$ for $T<T_c$.

Let us now consider the case where N is large but not infinite and where T is in the vicinity of T_c. The spontaneous magnetization is very small, and $f(\eta)$ can be conveniently expanded as

$$f(\eta)\sim 2\frac{T-T_c}{T_c}\eta^2+\tfrac{4}{3}\eta^4+O(\eta^6) . \tag{22}$$

At this stage it is interesting to notice that $f(\eta)$ can be recognized as a simple Ginzburg-Landau expression for the free-energy[11] without any gradient term, i.e., if there was only one uniform order parameter η. The absence of d-dependent gradient terms is reasonable since the dimensionality does not play any role here.

Thus for large N, m is given by

$$m^2\sim\frac{\int \eta^2\exp\left[2N\frac{T_c-T}{T_c}\eta^2-\tfrac{4}{3}N\eta^4+NO(\eta^6)\right]d\eta}{\int \exp\left[2N\frac{T_c-T}{T_c}\eta^2-\tfrac{4}{3}N\eta^4+NO(\eta^6)\right]d\eta} . \tag{23}$$

The large-N behavior of m can be then formed by standard methods and we find

$$\begin{aligned} m &\sim m_{\infty}(T)+O\left[\frac{1}{N}\right] \quad \text{for } T<T_c , \\ m &\sim N^{-1/4} \quad \text{for } T=T_c , \\ m &\sim N^{-1/2} \quad \text{for } T>T_c . \end{aligned} \tag{24}$$

This is the result already found by Kittel and Shore.[2] However, we would like to determine precisely the behavior of m in the large-N critical region by neglecting the η^6 terms and defining

$$N_c=\left[\frac{T_c-T}{T_c}\right]^2 . \tag{25}$$

Then performing the change of variable

$$u=\eta N^{1/4} , \tag{26}$$

Eq. (23) becomes

$$m^2\sim\frac{|T_c-T|}{T_c}\left[\frac{N}{N_c}\right]^{-1/2}\times\frac{\int u^2\exp\left[\pm 2\left[\frac{N}{N_c}\right]^{1/2}u^2-\tfrac{4}{3}u^4\right]du}{\int \exp\left[\pm 2\left[\frac{N}{N_c}\right]^{1/2}u^2-\tfrac{4}{3}u^4\right]du} , \tag{27}$$

where $\pm$ is the sign of T_c-T.

This is precisely a form like (8) with

$$[F_a(x)]^2=x^{-1/2}\frac{\int u^2\exp(\pm 2x^{1/2}u^2-\tfrac{4}{3}u^4)du}{\int \exp(\pm 2x^{1/2}u^2-\tfrac{4}{3}u^4)du} \tag{28}$$

and

$$a_{\rm MF}=\tfrac{1}{2} . \tag{29}$$

As expected we recognize the mean-field exponent for the magnetization of the Ising model.[11]

Thus the scaling hypothesis as postulated above is here completely justified. Equation (25) gives $\nu^*=2$, a value consistent with our argument giving $\nu^*=\nu_{\rm MF}d_c$ since it is well known that $\nu_{\rm MF}=\frac{1}{2}$ and $d_c=4$ for the short-range Ising model.[11]

It can be shown that the corrective η^6 term of $f(\eta)$ does not affect the asymptotic scaling. In particular it does not introduce any logarithmic term in the scaling form.

B. Infinitely coordinated Heisenberg model

Similar calculations can be performed in the case of the infinite range Heisenberg model:

$$H=-JN^{-1}\sum_{\substack{i\neq j\\ i,j}}(S_i^xS_j^x+S_i^yS_j^y+S_i^zS_j^z) , \tag{30}$$

where the S_i^{α}'s are now Pauli spin-$\frac{1}{2}$ operators. It has been shown[2,4] that the partition functions of the Ising and Heisenberg models differ only by leading terms of order N^{-1} which do not affect the large-N behavior of the magnetization given by Eq. (24). The same type of calculation, not reproduced here, as in the Ising case can be performed yielding to the same scaling form for N with the same exponent ν^*. It is well known[11] that in the short-range case the transition in temperature of the Heisenberg model is affected by quantum fluctuations and is only driven by thermal fluctuations as in the Ising case. Thus the upper critical dimensionality is $d_c=4$ in both cases which is consistent with the exponent $\nu^*=2$ found in both cases.

C. Infinitely correlated Berlin and Kac model

The scaling form can be also explicitly derived in the case of the infinitely coordinated version of the model introduced by Berlin and Kac,[12] which can be considered as a formulation of the spherical model.[13] In this case, the calculation is even more simple than the Ising case.

The model is described by the Hamiltonian

$$H=-JN^{-1}\left[\sum_i \sigma_i\right]^2 \tag{31}$$

where σ_i are now N continuous variables subject to the condition

$$\sum_i \sigma_i^2=N/4\ . \tag{32}$$

The order parameter is defined between $-\frac{1}{2}$ and $+\frac{1}{2}$ by

$$\eta=\left[\sum_i \sigma_i\right]\Big/ N \tag{33}$$

given η, the number of configurations is proportional to the volume of the intersection of the plane $\eta=$const by the hypersphere in N dimension of radius $N^{1/2}/2$ defined by (32). This is the volume of an hypersphere of radius $N^{1/2}(\frac{1}{4}-\eta^2)^{1/2}$ in $N-2$ dimensions. The partition function is thus given by

$$Z_N=K_N\int_{-1/2}^{+1/2}(1-4\eta^2)^{(N-2)/2} \times\exp(\beta JN\eta^2)d\eta\ , \tag{34}$$

where K_N is only N dependent. This expression can be written as

$$Z_N=K_N\int_{-1/2}^{+1/2}\frac{1}{1-4\eta^2}\exp[-Nf(\eta)]d\eta\ , \tag{35}$$

with

$$f(\eta)=-\tfrac{1}{2}\ln(1-4\eta^2)-\beta J\eta^2\ . \tag{36}$$

For N infinite the mean-field value of η is given by the minimization of $f(\eta)$ and this is given by the self-consistent equation

$$\frac{4\eta_\infty}{1-4\eta_\infty^2}=2\beta J\eta_\infty\ . \tag{37}$$

The critical temperature is given by $\beta J=2$, i.e., $k_BT_C=J/2$, as in the Ising case.

When N is large but not infinite, and when T is in the vicinity of T_c, $f(\eta)$ can be expanded in a Ginzburg-Landau form similar to (22):

$$f(\eta)\sim 2\frac{T-T_c}{T_c}\eta^2+4\eta^4+O(\eta^6)\ , \tag{38}$$

and also the η^2 term in $1/(1-4\eta^2)$ can be neglected in Z_N. Thus a scaling form for $m^2=\langle\eta^2\rangle$ similar to (27) can be explicitly derived with only different numerical coefficients. It can be also shown in that case that there are no logarithmic terms. The large-size critical behavior of the infinitely coordinated Berlin and Kac model is consequently similar to that of the Ising and Heisenberg model. In particular, the exponent $\nu^*=2$ is the same in the three cases consistent with the upper critical dimensionality $d_c=4$.[10]

IV. NUMERICAL EXAMPLES

A. Anisotropic XY model in a transverse field

1. Generalities

In the case of short-range interactions, it is known that quantum systems at zero temperature can behave as classical systems of larger dimensionality at finite temperature.[14] The upper critical dimensionality for a ground-state transition of a quantum system is consequently smaller than for the thermal transition of the classical equivalent. Thus quantum systems provide examples of systems with different d_c values useful to check the generality of the predictions made in Sec. II. A typical example is the XY Ising model in a transverse field. For arbitrary S spins and in the infinitely coordinated limit, the Hamiltonian is given by

$$H=-(NS)^{-1}\sum_{\substack{i<j\\ i,j}}(S_i^xS_j^x+\gamma S_i^yS_j^y)-\Gamma\sum_i S_i^z\ , \tag{39}$$

where S_i are spin-S Pauli operators, γ is an anisotropy parameter ($\gamma=1$ corresponds to the isotropic XY case), and Γ is the transverse field (the coupling constant has been normalized in order that $\Gamma_c=1$). We have limited our study to $0\le\gamma\le 1$.

The short-range version of this Hamiltonian has been widely studied in the past by exact calculations in one dimension[15] and various approximate methods in larger dimensionalities such as series expansions,[16] real-space renormalization group,[17] etc. In the anisotropic case, this d-dimensional quantum model undergoes a second-order phase transition in transverse field at $T=0$ which has the same critical behavior as the transition in temperature of the classical Ising model in $d+1$ dimension.[18] In the equivalence, the gap G between the ground state and the first excited state plays the role of the inverse of the coherence length in the extradimensionality for the classical equivalent.[14] The result is that the gap scales as the inverse of the length of the system at the critical field. In other words, the dynamical exponent z defined, at the critical field, by

$$G \underset{\Gamma=\Gamma_c}{\sim} L^{-z} \tag{40}$$

is exactly equal to 1 in that case. As a direct consequence of this equivalence, the upper critical dimensionality must be $d_c=3$ for the ground-state transition of the quantum system in field, since it is $d_c=4$ for the classical equivalent.

The symmetric XY case ($\gamma=1$) is very peculiar since this equivalence does not hold.[18] The transition is here very different.[19] There is no spontaneous magnetization in the low-field phase, and the exponent z at the critical field is $z=2$ instead of $z=1$. Unless there is no exact transformation, it has been suggested that the quantum XY model in a transverse field would be equivalent to a classical model in $d+2$ dimensions.[20] If we trust this assumption, the upper critical dimensionality would be $d_c=2$ in the isotropic case.

We have performed numerical calculations on the Hamiltonian (39) for $\gamma\neq 1$ as well as analytical calculations for $\gamma=1$. We have calculated the gap G between the ground state and the first excited state as well as the x component of the magnetization m which can be defined as

$$m^2=(NS)^{-2}\left\langle 0\left|\left[\sum_i S_i^x\right]^2\right|0\right\rangle , \tag{41}$$

where the expectation value is taken in the ground state. Note that there are different ways to define m for finite N values. The above one has the advantage of corresponding to the definition of the magnetization in the short-range case, the square of which being the limit of the xx spin-correlation function for infinite distances. Here the distance does not play any role so that a uniform average over all the elements can be made. Generally other relevant definitions differ from (41) by leading terms in N^{-1} in the large-N limit, which do not affect the scaling behaviors found below.

In the isotropic case $\gamma=1$, the direction of the magnetization in the XY plane is not defined and it is better to define m more generally by

$$m^2=(NS)^{-2}\left\langle 0\left|\left[\sum_i S_i^x\right]^2+\left[\sum_i S_i^y\right]^2\right|0\right\rangle . \tag{42}$$

Before presenting the numerical results for N finite, let us give the mean-field results for N infinite which can be obtained analytically.

2. *Mean-field results*

There are different ways in getting the infinite N results for m. A simple one consists in rewriting the Hamiltonian under the form

$$H=-(2NS)^{-1}(J^{x^2}+\gamma J^{y^2}-K^x-\gamma K^y)-\Gamma J^z , \tag{43}$$

with

$$J^\alpha=\sum_i S_i^\alpha,\quad K^\alpha=\sum_i (S_i^\alpha)^2 . \tag{44}$$

The K^α terms become negligible in the thermodynamic limit (they behave as N, while the J^{α^2} behave as N^2). The lowest states of the Hamiltonian belong to the maximum eigenvalue $J=NS$ of the total spin so that in the limit $N=\infty$, the Hamiltonian can be replaced by

$$H=-(2J)^{-1}(J^{x^2}+\gamma J^{y^2})-\Gamma J^z . \tag{45}$$

In the thermodynamic limit the large spin $\vec{J}$ of size NS can be treated as a classical spin by writing

$$J^x=J\sin\theta\cos\phi,\quad J^y=J\sin\theta\sin\phi ,$$
$$J^z=J\cos\theta . \tag{46}$$

Then, we must minimize

$$E(\theta,\phi)=-\frac{J}{2}(\sin^2\theta\cos^2\phi+\gamma\sin^2\theta\sin^2\phi)-\Gamma J\cos\theta \tag{47}$$

and calculate

$$m=J^x/J=\sin\theta\cos\phi . \tag{48}$$

In the anisotropic case $\gamma<1$, one gets $\phi=0$ or π and $\theta=0$ for $\Gamma>1$, $\theta=\cos^{-1}\Gamma$ for $\Gamma<1$ so that, retaining only the positive value of the magnetization,

$$m_\infty=(1-\Gamma^2)^{1/2} \text{ for } \Gamma<1 ,$$
$$m_\infty=0 \text{ for } \Gamma>1 . \tag{49}$$

Thus the system develops a second-order phase transition when $N=\infty$ at $\Gamma_c=1$ with a mean-field exponent $\beta_{\mathrm{MF}}=\frac{1}{2}$ for the magnetization

In the isotropic case $\gamma=1$, we find that ϕ can take any value, which reflects the fact that the direction of the self-consistent field can be chosen arbitrarily in the XY plane. The definition (42), independent of the choice of the polarization, gives exactly the same result as (49). Thus the infinitely correlated Ising and XY models in the transverse field have exactly the same behavior for $N=\infty$. This result is completely different than in the short-range case.

To obtain G in the limit $N=\infty$ one can identify the gap (in units $\hbar=1$) with the frequency ω for the motion of the large spin $\vec{J}$. With the use of the Hamiltonian (45), the equations of motion for J^x and J^y are given by

$$\begin{aligned} \mathring{J}^x &= i\omega J^x = i[H,J^x] \\ &= \frac{\gamma}{2J}(J^yJ^z+J^zJ^y)+\Gamma J^y \ , \\ \mathring{J}^y &= i\omega J^y = i[H,J^y] \\ &= \frac{-1}{2J}(J^xJ^z+J^zJ^x)-\Gamma J^x \ . \end{aligned} \tag{50}$$

In the classical limit where $N\to\infty$ the random-phase approximation, which consists of replacing J^z by its mean-field value, is justified. Replacing $J^z=J\cos\theta$ by Γ for $\Gamma<1$ and by J for $\Gamma>1$, one finds for $G_\infty=\omega$

$$\begin{aligned} G_\infty &= 0 \quad \text{for } \Gamma\le 1 \ , \\ G_\infty &= [(\Gamma-1)(\Gamma-\gamma)]^{1/2} \quad \text{for } \Gamma\ge 1 \ . \end{aligned} \tag{51}$$

Thus at the second-order phase transition the gap opens as

$$G_\infty \sim (\Gamma-\Gamma_c)^{s_{\text{MF}}} \ , \tag{52}$$

with a mean-field exponent s_{MF}

$$\begin{aligned} s_{\text{MF}} &= \tfrac{1}{2} \quad \text{for } \gamma\neq 1 \ , \\ s_{\text{MF}} &= 1 \quad \text{for } \gamma=1 \ . \end{aligned} \tag{53}$$

3. *Finite-size numerical results in the spin-$\frac{1}{2}$ transverse Ising case*

In the spin-$\frac{1}{2}$ case the K^α terms defined in (44) are simple constants so that the Hamiltonian (45) can be used *even for finite N*. This Hamiltonian corresponds exactly to that introduced by Lipkin *et al.*[4] as a model for the nucleus. Some numerical calculations[4] as well as analytical calculations[4,21] are available. Here, we have focused our numerical investigations in the large-N critical region. Since the Hamiltonian involves a unique spin, very large sizes can be reached easily. In the basis $|J,M\rangle$, where $M=-J,-J+1,\ldots,+J$ is the eigenvalue of J^z and where $J=NS=N/2$, it can be easily seen that H connects only M with $M\pm 2$. It appears that the ground state and the first excited state belong to two different subspaces in which the matrices to be diagonalized are of order $N/2+1$ and $N/2$, for N even. We went up to $N\approx 150$. We have calculated G and m for various values of γ and we found that the large-N behavior is the same in the whole anisotropic region $\gamma\neq 1$. We report here the results of the calculations in the simple Ising case $\gamma=0$. The numerical results for G and m are given in Fig. 1 for $N=10,20,60,100$. The asymptotic mean-field results of Sec. IV A 2 are represented by dashed curves. The differences $G-G_\infty$ and $m-m_\infty$ have been studied as a function of N and the following results have been found:

$$\begin{aligned} &G\sim\exp(-aN), \ \ m-m_\infty\sim N^{-1} \ \text{ for } \Gamma<1 \ , \\ &G\sim N^{-1/3}, \ \ m\sim N^{-1/3} \ \text{ for } \Gamma=1 \ , \\ &G-G_\infty\sim N^{-1}, \ \ m\sim N^{-1/2} \ \text{ for } \Gamma>1 \ . \end{aligned} \tag{54}$$

The exponential behavior of G below the transition has been previously derived analytically,[21] however there do not exist, to our knowledge, any analytical derivations of the $\frac{1}{3}$ exponent for both G and m at Γ_c. This critical exponent has been determined here with a great accuracy (within 10^{-3} error). In Fig. 2 we have plotted $\ln G$ and $\ln m$ as a function of $\ln N$ for $\Gamma=\Gamma_c=1$, and one can see, at least in the case $S=\frac{1}{2}$ studied here, that the $N^{-1/3}$ behavior is veri-

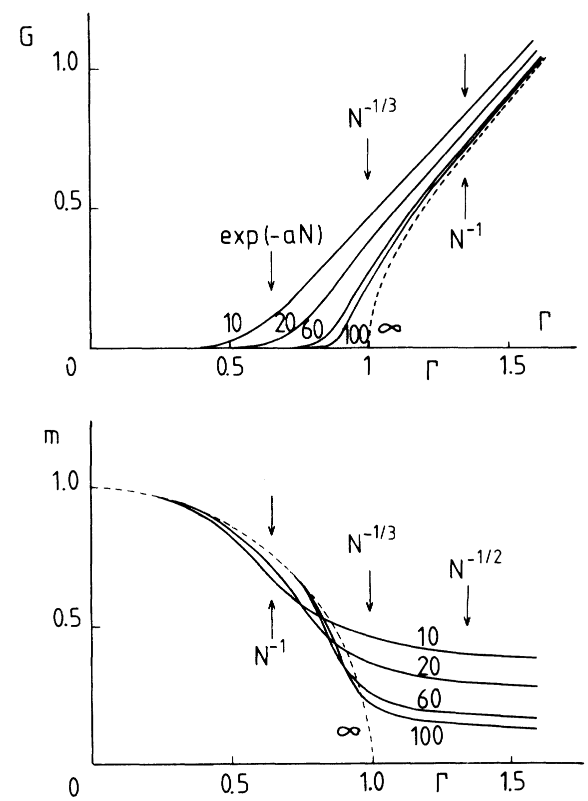

FIG. 1. Finite-size results ($N=10,20,60,100$) for the gap and the magnetization of the infinitely coordinated Ising model in a transverse field. Infinite N mean-field results are represented by dashed curves.

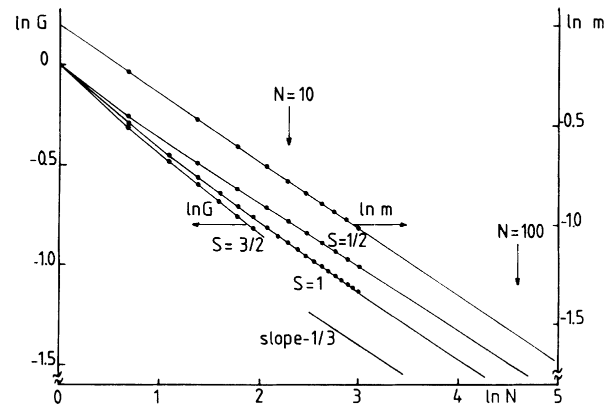

FIG. 2. Double-logarithmic plot of the gap (left scale) and the magnetization (right scale) as the function of size at the critical field for the infinitely coordinated Ising model in a transverse field (the dots are not shown for $N>20$).

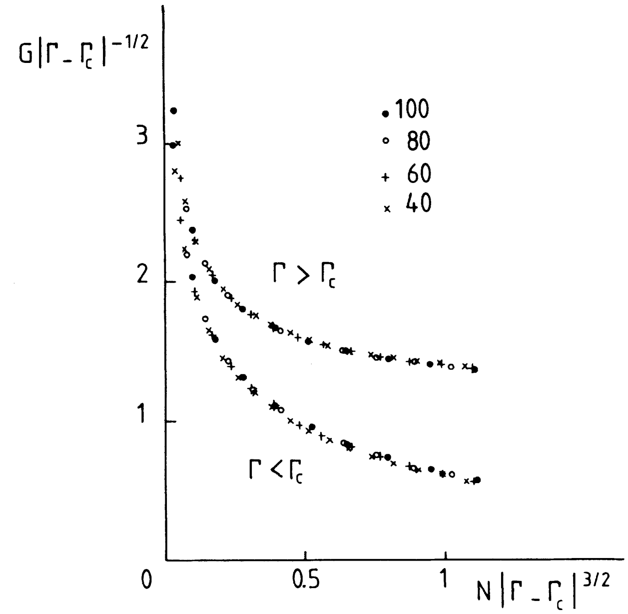

FIG. 3. Results for the gap of the infinitely coordinated Ising model in a transverse field in the critical region ($|\Gamma-\Gamma_c|\lesssim 0.05$) as a plot of $G|\Gamma-\Gamma_c|^{-1/2}$ vs $N|\Gamma-\Gamma_c|^{3/2}$. The points corresponding to $N=40,60,80,100$ are all disposed on the same curve which represents the scaling function F_G^* (see text).

fied in a large range of N values. One can certainly exclude logarithmic terms. Behaviors of the type $N^{-1/3}\ln(N/N_0)$, as in short-range systems at $d=d_c$, are not supported by the numerical results.

The critical exponent $\omega_G=\omega_m=-\frac{1}{3}$ found numerically for both G and m is consistent with the general argument of Sec. II which leads to

$$\omega_m=-\beta_{\rm MF}/(\nu_{\rm MF}d_c) \ ,$$
$$\omega_G=-s_{\rm MF}/(\nu_{\rm MF}d_c) \ . \tag{55}$$

Here for $\gamma=0$ one has $\beta_{\rm MF}=s_{\rm MF}=\nu_{\rm MF}=\frac{1}{2}$, thus $\omega_m=\omega_G=-1/d_c$. Our numerical result at $\Gamma=\Gamma_c$ provides a direct estimation of the critical dimensionality which is $d_c=3$ for this quantum system. Thus here $\nu^*=\frac{3}{2}$, and the scaling forms for G and m must be

$$G\sim|\Gamma-\Gamma_c|^{1/2}F_G^*(N|\Gamma-\Gamma_c|^{3/2}) \ ,$$
$$m\sim|\Gamma-\Gamma_c|^{1/2}F_m^*(N|\Gamma-\Gamma_c|^{3/2}) \ . \tag{56}$$

The scaling hypothesis has been numerically confirmed by computing the functions F_G^* and F_m^*. In Fig. 3 we have plotted $|\Gamma-\Gamma_c|^{-1/2}G$ as a function of $|\Gamma-\Gamma_c|^{3/2}N$. The points lie on the same curves in a large range of N values (60—150) near Γ_c which represent the function F_G^* above as well as below the transition.

4. *Extension to larger spins*

In order to know if the scaling behavior found in the anisotropic case for $S=\frac{1}{2}$ is general and independent of the size of the spin, we have performed other numerical calculations for $S=1$ and $\frac{3}{2}$. The Hamiltonian (45) can no longer be used for $S>\frac{1}{2}$, and we must consider the original form (43). The diagonalization is then more difficult, and we could not reach very large sizes. However, for $S=1$ it appears to be powerful to represent the Hamiltonian in the basis $|J,L,M\rangle$ where, for the highest eigenvalue $J=N$, L, and M are eigenvalues of $\sum_i(S_i^z)^2=K^z$ and $\sum_i S_i^z=J^z$, respectively; with J given, L varies from 0 to N by integer values; with J and L given, M varies from $-L$ to $+L$ by integer values. The matrix to be diagonalized is of order $(N+1)(N+2)/2$. Using this representation and using the Lanczös algorithm[22] to diagonalize the large matrix we were able to reach $N\sim 70$. For $S=\frac{3}{2}$ such a manipulation could not be used, and we were limited to $N\sim 8$. In Fig. 2, we give the result for the critical scaling of the gap as a plot of $\ln G$ vs $\ln N$ for $\Gamma=\Gamma_c$. After a crossover at small sizes the curve for $S=1$ becomes asymptotically parallel to the curve for $S=\frac{1}{2}$. The conclusion is less clear for $S=\frac{3}{2}$ since we could not reach large sizes and we are certainly remaining in the crossover region. It seems, however, reasonable to conclude that the result $\omega_G=-\frac{1}{3}$ is independent of the spin size as expected from the classical-quantum equivalence.

5. *Analytical results in the $S=\frac{1}{2}$ isotropic case ($\gamma=1$)*

For $\gamma=1$, straightforward analytical calculation can be done in the $S=\frac{1}{2}$ case where (45) can be used. The Hamiltonian

$$H=-(2J)^{-1}[(J^x)^2+(J^y)^2]-\Gamma J^z \ , \quad (57)$$

is entirely diagonal in the representation J, M, and the eigenvalues are

$$E(J,M)=-[J(J+1)-M^2]/(2J)-\Gamma M$$
$$=-(J+1)/2+M^2/(2J)-\Gamma M \ . \quad (58)$$

The magnetization as defined in (42) is given by

$$m^2=[J(J+1)-M^2]/J^2 \ . \quad (59)$$

For $\Gamma>1$, the ground state and the first excited state correspond to $M=J$ and $M=J-1$, respectively, so that the gap $G=E(J,J-1)-E(J,J)$ and the magnetization are given by

$$G=\Gamma-1+1/N, \quad m=(2/N)^{1/2} \quad \text{for } \Gamma>1 \ . \quad (60)$$

For $\Gamma<1$, the minimization of $E(J,M)$ implies that M varies discontinuously in the ground state. However, for $J=N/2$ large, M is a quasicontinuous variable given by $M\sim\Gamma J$ in the ground state. Thus the gap $E(J,M-1)-E(J,M)$ and the magnetization are asymptotically given by

$$G=N^{-1}, \quad m=(1-\Gamma^2+J^{-1})^{1/2}=m_\infty+O(N^{-1}) \quad \text{for } \Gamma<1 \ . \quad (61)$$

The mean-field results are recovered when $N\to\infty$. The large-N critical behaviors at $\Gamma=1$ of G and m are of the form $G\sim N^{\omega_G}$, $m\sim N^{\omega_m}$ with

$$\omega_G=-1, \quad \omega_m=-\tfrac{1}{2} \ . \quad (62)$$

The scaling forms here are particularly trivial since G below the transition as well as m above the transition do not depend on Γ. However, for G above the transition and m below the transition, one has

$$G\sim(\Gamma-1)(1+N_c/N), \quad \Gamma>1$$
$$m\sim2(1-\Gamma)^{1/2}(1+N_c/N)^{1/2}, \quad \Gamma<1 \quad (63)$$

with

$$N_c=|\Gamma-1|^{-1} \ .$$

The exponents ω_G, ω_m, and $\nu^*=1$ are consistent with our argument (10) if one takes the mean-field exponents $\beta_{\rm MF}=\frac{1}{2}$, $s_{\rm MF}=1$ found in Sec. IV A 2, if we suppose that $\nu_{\rm MF}=\frac{1}{2}$ as in the Ising case, and if we trust the upper critical dimensionality $d_c=2$ proposed for this model.[20]

B. A quantum model exhibiting a Yang-Lee edge singularity

1. *Generalities*

Yang and Lee[23] have shown that for classical Ising model the zeros of the partition function are distributed on the circle of radius unity in the complex activity plane $z=\exp(-2h/k_BT)$, where h is the applied magnetic field. In the thermodynamic limit, a density of zeros $G(\theta)$ can be defined on the circle $z=\exp(i\theta)$ which determines all the physical quantities. For $T>T_c$, $g(\theta)$ has a gap of width $2\theta_g(T)$ and the edges of this gap, for $h=\pm ihg(T)$, are branch points for the magnetization $m(h,T)$. Near these edges and on the circle the real part of the magnetization m' and the density of zeros $g(\theta)$ are proportional and behave like

$$g(\theta)\sim m'\sim(\theta-\theta_g)^\sigma\sim(h-h_g)^\sigma \ , \quad (64)$$

where the edge exponent σ is different than the equal exponent $1/\delta$ in a real applied field. Fisher[24] has interpreted this singularity as a new critical behavior, corresponding to a transition when a purely imaginary field is applied to the system. This transition shares the properties of ϕ^3 field theory and the upper critical dimensionality is $d_c=6$.[24]

The Yang-Lee edge singularity has been investigated by various approaches.[24,25] In particular an interesting indirect way to study the Yang-Lee edge singularity of a d-dimensional classical Ising model is to study the equivalent quantum model in $d-1$ dimensions: the Ising model in a transverse field with an extra purely imaginary longitudinal field. This has been recently done for $d=2$ by using real-space renormalization-group methods on the one-dimensional quantum model.[25] This quantum model provides a new example for which the upper critical dimensionality must be $d_c=5$.

We would like to report here on numerical calculation performed on the infinite range version of the transverse Ising model in an imaginary longitudinal field in the case of $S=\frac{1}{2}$ spins, where the Hamiltonian is written as

$$H=-(J^x)^2/(2J)-\Gamma J^z-ihJ^x \quad (65)$$

with the notations of Sec. III. The Hamiltonian is no longer Hermitian and some numerical manipulations must be used for $h\neq0$ as explained in Sec. IV B 3. We have computed the real part of the gap between the eigenvalues of lowest real part and the complex magnetization, the square of which being always defined as in (45).

2. *Mean-field results*

The mean-field theory of the Yang-Lee edge singularity has been developed by Suzuki.[26] Unless the critical behavior is exactly the same, the quantitative results are quite different here since we are working on a quantum-equivalent model. The mean-field calculation on (65) can be done as in Sec. III B 2 by introducing an angle θ which is now generally complex $\theta=\theta'+i\theta''$ with

$$J^x=J\sin\theta,\quad J^z=J\cos\theta,\quad m=\sin\theta \tag{66}$$

the form to be minimized is

$$E(\theta)=-J(\tfrac{1}{2}\sin^2\theta+\Gamma\cos\theta+ih\sin\theta)\ . \tag{67}$$

In the spirit of the analytical continuation of the real field results, we simply minimize as if the coefficients would be real. We get

$$-\sin\theta\cos\theta+\Gamma\sin\theta-ih\cos\theta=0\ , \tag{68}$$

i.e.,

$$\sin\theta'[-\cos\theta'\cosh(2\theta'')+\Gamma\cosh\theta''-h\sinh\theta'']=0\ ,$$

$$-\tfrac{1}{2}\cos(2\theta')\sinh\theta''+\Gamma\cos\theta'\sinh\theta''-h\cos\theta'\cosh\theta''=0\ . \tag{69}$$

The magnetization is given by

$$m'=\sin\theta'\cosh\theta'',\quad m''=\cos\theta'\sinh\theta''\ . \tag{70}$$

m' plays the role of an order parameter which is zero in a whole region of the plane h,Γ. The critical line which separates this region from the "ordered" region where $m'\neq 0$ is given by

$$\Gamma_c^{2/3}-h_c^{2/3}=1\ . \tag{71}$$

This line represented in Fig. 4 starts from the critical point $\Gamma_c=1$ for $h_c=0$ and ends in the asymptotic direction $h=\Gamma$ of the (h,Γ) plane. Near $\Gamma_c\sim 1$ one has

$$h_c\sim(\Gamma_c-1)^{\Delta} \tag{72}$$

with

$$\Delta=\tfrac{3}{2}\ .$$

The exponent Δ is related with the usual exponents β and δ of the magnetization without complex field by the relation $\Delta=\beta\delta$.[27] This relation is verified here where β and δ take the mean-field values $\beta_{\rm MF}=\frac{1}{2}$, $\delta_{\rm MF}=3$ of the Ising model.[11]

In the disordered region $\Gamma>\Gamma_c$ the magnetization is given by

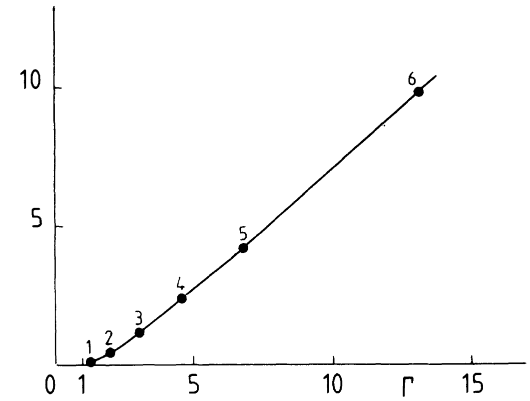

FIG. 4. Mean-field critical curve in the plane h,Γ for the infinitely coordinated Ising model in a transverse field with a purely imaginary longitudinal field ih. The points 1,2, . . . , 6 correspond to h and Γ values where the critical scaling has been numerically investigated (see Fig. 6).

$$m'=0,\quad m''=\sinh\theta''\ , \tag{73}$$

where θ'' is implicitly given as a function of h,Γ by

$$\Gamma\sinh\theta''-h\cosh\theta''=\sinh\theta''\cosh\theta''\ . \tag{74}$$

In the ordered region $\Gamma<\Gamma_c$ the magnetization is given by Eqs. (70) where θ' and θ'' are calculated implicitly with (68) after eliminating the trivial solution $\theta'=0$.

The critical behaviors of G and h near the critical curve can be easily found analytically. For example, when h is taken constant one has

$$m'\sim(\Gamma_c-\Gamma)^{1/2},\quad |m''-m_c''|\sim\Gamma_c-\Gamma \quad \text{for } \Gamma\lesssim\Gamma_c\ ,$$

$$G\sim(\Gamma-\Gamma_c)^{1/2},\quad |m''-m_c''|\sim(\Gamma-\Gamma_c)^{1/2} \quad \text{for } \Gamma\gtrsim\Gamma_c\ . \tag{75}$$

The same behavior is obtained in $h-h_c$ by taking Γ constant. We recover the mean-field value of the exponent σ: $\sigma_{\rm MF}=\frac{1}{2}$ for the real part of the magnetization.[26] We observe that the exponent is the same for the imaginary part of the magnetization above the transition (but not below). Also we find that the mean-field exponent for the gap here is $s_{\rm MF}=\frac{1}{4}$, different from the case $h=0$.

An example of the behavior of G_∞ and m when varying Γ, at constant h value, is given in Fig. 5 where the mean-field result of m'_∞ and G_∞ (a) and m''_∞ (b) are given by the dashed curves.

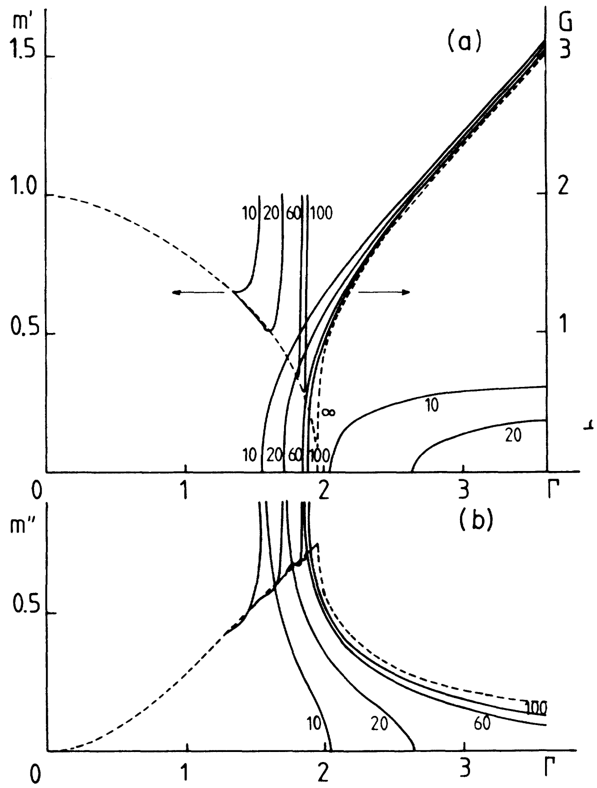

FIG. 5. Finite-size results ($N=10,20,60,100$) for the gap G [(a), right scale], the real part of the magnetization m' [(a), left scale] and the imaginary part of the magnetization m'' (b) of the infintely coordinated Ising model in a transverse field with a purely imaginary longitudinal field ih. The quantities are plotted vs Γ for a constant h value ($h=0.421\,85$). Infinite N mean-field results are represented by dashed curves.

3. Numerical results for finite sizes

The Hamiltonian (65) is represented by a non-Hermitian, but still symmetric, matrix in the representation $|J,M\rangle$. We still keep the definition of the scalar product as the sum of simple products of the coordinates. Note that the square of the norm of a vector can become negative or zero. With this definition the eigenvectors of H can be still orthogonalized. The ground state is then defined as the eigenvalue of lowest real part. The gap is defined as the difference between the eigenvalues of lower real part. We define G as the real part of the gap. The square of the magnetization is calculated as in (45) with the new definition of the scalar product. Numerical results for a constant h value are reported in Fig. 5 as a function of Γ for $N=10,20,60,100$. One observes that there is a singularity for N finite at a given critical value $\Gamma_c(N)$ depending on N, which tends to the mean-field value Γ_c when $N\to\infty$. As in the mean-field case this singularity corresponds to an opening of G for $\Gamma>\Gamma_c(N)$ and separates a region where $m'\neq 0$ [for $\Gamma<\Gamma_c(N)$] from a region where $m'=0$ [$\Gamma>\Gamma_c(N)$]. [Note that m' becomes again different from zero at a given $\Gamma^*(N)>\Gamma_c(N)$, which tends quickly to infinity when $N\to\infty$; this $\Gamma^*(N)$ is not a true critical value. It corresponds to the annulation of m^2 which becomes positive for $\Gamma>\Gamma^*(N)$.] The singularity at $\Gamma_c(N)$ has trivial exponents for $N\neq\infty$:

$$m''\sim m'\sim[\Gamma_c(N)-\Gamma]^{-1/2} \quad \text{for } \Gamma\lesssim\Gamma_c(N) \ ,$$
$$m''\sim[\Gamma-\Gamma_c(N)]^{-1/2}, \quad G\sim[\Gamma-\Gamma_c(N)]^{1/2} \quad \text{for } \Gamma\gtrsim\Gamma_c(N) \ . \tag{76}$$

This singularity corresponds mathematically to the point when the norm of the ground state vanishes. This explains the trivial exponents $\frac{1}{2}$. One can also understand this singularity physically as follows. From the classical-quantum equivalence, the finite system with N elements (disposed in any space dimensionality) corresponds to an infinite one-dimensional classical Ising system for which a Yang Lee edge singularity exists with $\sigma=-\frac{1}{2}$.[23]

When $N\to\infty$, $\Gamma_c(N)$ tends to the mean-field critical value Γ_c calculated above and the exponents change into the mean-field exponents. There is thus a large crossover for the magnetization when increasing N. The exponent goes from a negative value $-\frac{1}{2}$ for finite N to a positive value $+\frac{1}{2}$ for infinite N. The critical scalings of m and G have been studied at $\Gamma=\Gamma_c$ and the results are given in Fig. 6 where we have plotted $\ln G$ and $\ln(m''_\infty-m'')$ as a function of $\ln N$ for various values of Γ_c, h_c. If we forget the kink in the curve for m'' which corresponds to N such that $\Gamma^*(N)=\Gamma_c$ we find that for large N:

$$G\sim N^{\omega_G}, \quad m''_\infty-m''\sim N^{\omega_m} \tag{77}$$

with

$$\omega_G=0.2, \quad \omega_m=0.4 \ .$$

These values are consistent with the general predictions of Sec. II. The mean-field value of the exponent ν is known to be $\nu_{MF}=\frac{1}{4}$ for the Yang-Lee problem,[24] and here the critical dimensionality is $d_c=5$ for the quantum equivalent. Equation (11) explains the results if we consider the mean-field exponents $s_{MF}=\frac{1}{4}$ and $\sigma_{MF}=\frac{1}{2}$ derived above.

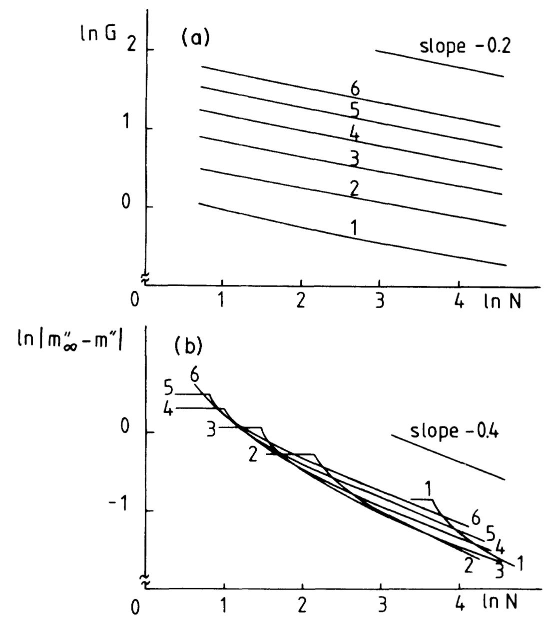

FIG. 6. Double-logarithmic plot for the gap (a) and the imaginary part of the magnetization (b) as a function of size at the different critical points shown on Fig. 4 for the infinitely coordinated Ising model in a transverse field with a purely imaginary longitudinal field.

V. CONCLUSION

In this paper we have extended the finite-size scaling hypothesis of Fisher and Barber to the case of infinitely correlated systems. We have given a very general argument which relates the large size scaling exponents with the upper critical dimensionality of the corresponding short-range system. In some simple examples where $d_c = 4$, such as the Ising model and the spherical model of Berlin and Kac, we have been able to derive the scaling form of the magnetization explicitly. Thus our argument has been proved on the infinitely coordinated version of the $O(n)$ model in the special case $n = 1$ and $n = \infty$. It would be useful to extend the calculation in the general n case. Moreover, we have performed numerical calculations on various infinitely correlated quantum models where $d_c = 2$ (XY model in a transverse field), $d_c = 3$ (Ising model in a transverse field), $d_c = 5$ (Ising model in a transverse field with an imaginary longitudinal field). In each case we have checked our general argument. All these studies give some confidence on the generality of the reasoning of Sec. II C. Our argument can be now applied to more complicated systems. In particular it could give a simple way to estimate d_c in short-range systems where it is not known by studying the finite-size scaling of corresponding infinitely coordinated model. In particular in a disordered infinitely coordinated model, the spin-glass model of Sherrington and Kirkpatrick,[6] we learned[28] that a scaling form can be derived for the order parameter which is consistent with an upper critical dimensionality $d_c = 6$. Other further studies would be to explore systems in which the range of interaction can vary with a given parameter, for example models with interactions varying as r^{-p}. In the one-dimensional Ising case it has been analytically shown that the system behaves as it would be infinitely coordinated for p smaller than a critical value $p_c = 2$.[29] On more complicated systems where no analytical results are available, it could be useful to perform finite-size calculations. Then, the present study could give some help to interpret the numerical results in the parameter region where infinitely coordinated physics is expected. To conclude, we would like to insist on the fact that infinitely coordinated systems are not unrealistic models. Not only is their study useful for obvious fundamental reasons, but also these systems are currently used as a model of real systems in various domains of physics such as field theory, condensed matter, nuclear physics (the Lipkin model for nucleus, etc.), and even atomic physics (superradiance).

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