# Renormalization-group analysis on fractals: Ising spin-glass and the Schrödinger equation

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A renormalization-group analysis of random frustrated Ising models on a $d=2$ Sierpinski gasket is carried out. A numerical study of the recursion relations shows that in the spin-glass phase, the width of the probability distribution of the renormalized exchange couplings and the characteristic energy sensitivity to the boundary conditions decay algebraically with the size of the system. In the paramagnetic phase the two quantities decay exponentially. An exact renormalization-group analysis of the tight-binding Hamiltonian on a $d$-dimensional Sierpinski gasket is carried out. In any $d$, for almost all energies, the hopping matrix element renormalizes to zero faster than exponentially, showing that the corresponding eigenstates, if any, are localized.

## I. INTRODUCTION

In this paper we study two unrelated problems on a fractal lattice. The lattice is a finitely ramified Sierpinksi gasket and thus enables us to construct exact renormalization-group recursion relations. Section II treats the problem of frustration on the gasket and studies the equilibrium characteristics of different kinds of ordering possible on the gasket. Section III describes the derivation of the exact recursion for the problem of an electron, described by the tight-binding Hamiltonian, on a $d$-dimensional gasket.

## II. FRUSTRATION ON FRACTALS

Understanding the nature of ordering in spin-glasses has remained an intriguing and almost insoluble problem. The novel properties of spin-glasses arise from the presence of randomness and frustration. A model incorporating both features has been proposed by Edwards and Anderson.[1] However, this model is not amenable to an exact analysis and its physics is not yet fully understood. Attempts have been made to characterize spin-glass behavior by considering models which have frustration but no randomness[2] or have randomness but no frustration.[3]

In this section we carry out a renormalization-group analysis of the Edwards-Anderson model with Ising spins on the Sierpinski gasket. The analysis is made feasible because the gasket is finitely ramified. However, the gasket is not quasi-one-dimensional and allows for frustration. Owing to its finite order of ramification, only zero-temperature phase transitions can occur in spin systems on the gasket.[4] A real spin-glass is also expected to undergo a zero-temperature equilibrium phase transition in three dimensions, since the lower critical dimensionality (LCD) for the spin-glass order is bigger than three.[5]

Recently fractal lattices have been the subject of many studies[4,6,7] and it has been suggested that they may represent the geometrical features of the backbone of a percolating cluster.[6] To construct the two-dimensional gasket, we begin with a triangle. The three midpoints of its edges are connected, creating four smaller triangles. The central triangle is then removed and the same procedure is continued for each of the new triangles down to the microscopic lattice constant. The fractal dimensionality of such a gasket is equal to ln3/ln2.

Consider an Ising model on the $d=2$ Sierpinski gasket. The Hamiltonian is given by

$$\mathcal{H}=-\sum_{\langle ij\rangle} J_{ij}S_iS_j\ . \tag{1}$$

The spins $S_i$ take on the values $+1$ or $-1$ and the exchange constants $J_{ij}$ couple nearest-neighbor spins. Figure 1 shows a fragment of the system. The geometry of the gasket is such that $S_2$, $S_4$, and $S_6$ are connected by the exchange interactions only to each other and to $S_1$, $S_3$, and $S_5$. This simplifying feature is due to the finite order of ramification of the gasket.

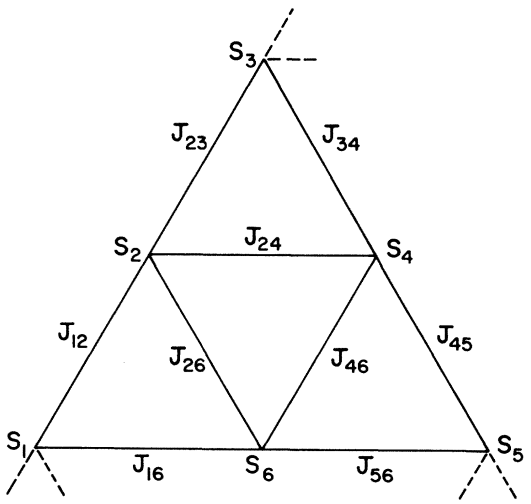


FIG. 1. Fragment of the Sierpinski gasket in $d=2$.

It is straightforward to carry out an exact decimation procedure in which the midpoint spins $S_2$, $S_4$, and $S_6$ are eliminated and effective interactions $J'_{13}$, $J'_{35}$, and $J'_{15}$ between the corner spins are found. In each such decimation step, nine exchange interactions are replaced by three effective exchange interactions and a constant term which is of interest only when calculating the free energy. The self-similarity of the gasket enables one to iterate this process recursively and to obtain renormalization values of the exchange couplings at each length scale. The calculation may be performed either at finite temperatures or at zero temperature. In the former case a partial trace over the eight states of the midpoint spins is carried out, whereas in the latter only the lowest-energy state contributes.

Let the exchange couplings $J_{ij}$ in Eq. (1) be independent random variables governed by the probability distribution $P_0(J)$. Upon carrying out the decimation transformation the renormalized interactions are not all independent random variables. As mentioned previously, each unit of three of the renormalized couplings is obtained from a given set of nine exchange interactions. This results in exchange interactions within a unit being correlated but there is no correlation between exchange couplings in different units. Therefore, when one considers many units, it is meaningful to ascribe a new probability distribution for the renormalized exchange interactions at a given length scale. This probability distribution can be characterized[8] by a mean value $J_{\rm av}$ and its dispersion $\sigma$. The fixed probability distributions (FPD) $P^*$ would determine the critical behavior of random models.

Another physical quantity of interest is the sensitivity of the free energy to changes in the boundary conditions, $\Delta E_s$. Recently, it has been suggested that the size dependence of $\Delta E_s$ allows one to identify the equilibrium LCD of an ordered phase.[5] In the paramagnetic phase $\Delta E_s$ decays *exponentially* with the size of the system. On the other hand, in an ordered phase such as an equilibrium spin-glass phase, if indeed such a phase exists, $\Delta E_s$ would be expected to depend *algebraically* on the size of the system. At zero temperature let the algebraic form be given by

$$\Delta E_s \sim N_s^x , \tag{2}$$

where $N_s$ denotes the number of spins. A positive exponent $x$ would lead to $\Delta E_s \to \infty$ as $N_s \to \infty$, which is characteristic of a system above its LCD, i.e., long-range order would persist even at finite temperatures. On the other hand, a negative $x$ would result in $\Delta E_s \to 0$ meaning that the system was below its LCD. It is important to note, however, that the negative-$x$ system is not a paramagnet at zero temperature (otherwise $\Delta E_s$ would decay exponentially) and becomes paramagnetic at infinitesimal temperatures. Finally, a zero value of $x$ would correspond to a system at its LCD and it may or may not undergo a finite-temperature phase transition. A numerical analysis[5] of the $d=3$ Heisenberg spin-glass with Gaussian couplings shows that for sufficiently short time scales $x=0$, whereas in equilibrium $x$ is negative.

In order to define a characteristic $\Delta E_s$ for Ising spins on the gasket, we note, that for any particular length scale the Hamiltonian for the gasket is equivalent to that for a triangle with renormalized exchange couplings. Since there are only eight energy states for the Ising spins on the triangle, $\Delta E_s$ may be defined as the energy difference[9] between the highest and the lowest of these eight states. While this definition of $\Delta E_s$ is not unique, adopting other reasonable measures of $\Delta E_s$ leads to virtually the same exponent $x$.

We have carried out a numerical analysis of the recursion relations for systems of $N=3^m$ triangles of side equal to the lattice parameter, with $m=0,1,2,\ldots,9$. Note $N_s=1.5(N+1)$. The procedure was repeated several times to ensure sufficient statistics. The largest system studied had 29 526 spins. We consider the following cases.

(1) $P_0(J)=\delta(J)$. This is an FPD describing a decoupled paramagnetic system.

(2) $P_0(J)=\delta(J-J_0)$, $J_0>0$. At zero temperature this is also an FPD corresponding to a uniform ferromagnet. On successive iterations the probability distribution is unchanged, $\Delta E_s$ is scale invariant, and $x=0$. At any finite temperature this FPD is unstable and the paramagnetic FPD is approached exponentially fast for large systems. The system undergoes a zero-temperature phase transition.[4]

(3) $P_0(J)=\delta(J-J_0)$, $J_0<0$. The system is fully frustrated and, even at $T=0$, upon one decimation all of the renormalized exchange constants are zero. The system, therefore, is paramagnetic at all temperature.

(4) Random ferromagnet, $T=0$. The exchange couplings were chosen to be absolute values of Gaussian numbers, characterized by zero mean and unit variance. A numerical analysis of the recursion relations shows that the decays of $\sigma$, $J_{\rm av}$, and $\Delta E_s$ are all algebraic and have virtually the same exponent $x \approx -0.04$. The system is at or slightly below its LCD.

(5) Random antiferromagnet, $T=0$ (Fig. 2). The distribution of the couplings was as described in case (4) except that they were all chosen to be negative. The probability

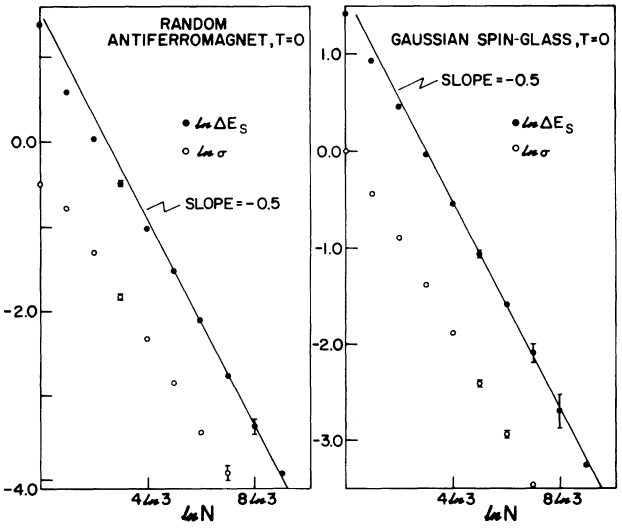


FIG. 2. Plots of $\ln\Delta E_s$ and $\ln\sigma$ vs $\ln N$ for the random antiferromagnetic and the Gaussian spin-glass at $T=0$.

distribution seems to approach the paramagnetic FPD but only in an algebraic fashion. The system is frustrated and $J_{av}$ is found to renormalize to zero after one iteration. The exponent $x \approx -0.5$. The system is below its LCD and it undergoes a $T=0$ phase transition.

(6) Gaussian spin-glass, $T=0$ (Fig. 2). This is also a frustrated system and its behavior is very similar to that of the random antiferromagnet. Note that the algebraic decays of $\sigma$ and $\Delta E_s$ are characterized by the same exponent $x \approx -0.5$.

(7) Gaussian "spin-glass," $T=1$, in units in which the probability distribution has unit variance (Fig. 3). As expected, the system is paramagnetic. The quantities $\sigma$ and $\Delta E_s$ decay according to the $\exp(-N/\xi)/N^{1/2}$ law. This is consistent with $\xi \to \infty$ as $T \to 0$.

(8) $P_0(J)=\frac{1}{2}[\delta(J-J_0)+\delta(J+J_0)]$, $T=0$ (Fig. 3). On an average half of the triangles are fully frustrated. One would therefore expect that even at zero temperature, following one iteration, a finite fraction of the renormalized bonds would become zero, decoupling the system into subsystems of finite size. In fact, $\Delta E_s$ and $\sigma$ decay exponentially and reach the paramagnetic FPD after only five iterations.

While all of the calculations presented above have been carried out for the artificial fractal lattice they nevertheless reproduce many of the properties of frustrated systems on regular lattices. For example, in $d=2$, the Gaussian Ising system is believed to undergo a zero-temperature phase transition whereas the $\pm J$ Ising system is paramagnetic even at $T=0$.[10,5] If real spin-glasses are below their LCD in $d=3$ it is tempting to speculate that their equilibrium behavior is similar to that described in cases (6) and (7).

## III. ELECTRONIC STATES ON FRACTALS

The energy spectrum of an electron in a periodic potential consists of bands of extended eigenstates. On the other hand, the electronic states in a random potential may, in general, be either extended or localized.[11] A scaling analysis[12] of the localization problem has shown that all eigenstates in $d=1$ and 2 are localized, whereas the existence of a mobility edge, separating extended and localized states, is predicted in higher dimensionalities. Physically, one may understand the absence of extended states as arising from destructive quantum interference caused by the random potential. It is an intriguing possibility that a similar destructive interference may appear in systems without any randomness. We show, in this section, that this indeed is the case for an electron described by the tight-binding Hamiltonian on a $d$-dimensional Sierpinski gasket. For example, in $d=2$ the gasket can be visualized as being made up of a triangular lattice with a fraction of the bonds removed in a *nonrandom self-similar* manner. It is important to note that the removal is done in such a way that the gasket is finitely ramified but is not quasi-one-dimensional.

We carry out an exact decimation procedure for the tight-binding Hamiltonian on the $d$-dimensional gasket. For any $d$ the recursion relations may be cast in the form of a logistic map[13]

$$t'=\lambda t(1-t) \tag{3}$$

with $\lambda=d+3$. An analysis of the recursion relations shows that, in any $d$, there are no brands of extended eigenstates.

To construct a $d$-dimensional gasket[4,6] we begin with a $d$-dimensional hypertetrahedron. The $d(d+1)/2$ midpoints of its edges are connected, creating $(d+1)$ smaller hypertetrahedra and a hypervolume in the center. The central hypervolume is removed and the same procedure is continued for each of the new hypertetrahedra down to the microscopic lattice constant. We note that the fractal dimensionality of the gasket is equal to $\ln(d+1)/\ln 2$.

For concreteness, let us consider the $d=2$ gasket, a fragment of which is shown in Fig. 4. The Schrödinger equation for the wave function at site 1 is

$$xa_1=(a_2+a_6)+(a_{2'}+a_{6'}) . \tag{4}$$

Here $x$ is the energy of the electron measured in units of the hopping matrix element. Similar equations may be written down for the wave functions at other sites. The geometry of the gasket is such that $a_2$, $a_4$, and $a_6$ are con-

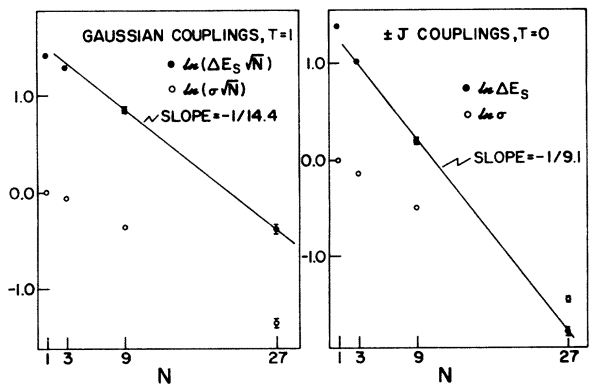


FIG. 3. Plots of $\ln(\Delta E_s\sqrt{N})$ and $\ln(\sigma\sqrt{N})$ vs $N$ for the Gaussian "spin-glass" at $T=1$ and plots of $\ln\Delta E_s$ and $\ln\sigma$ vs $N$ for the system with the $\pm J$ couplings at $T=0$.

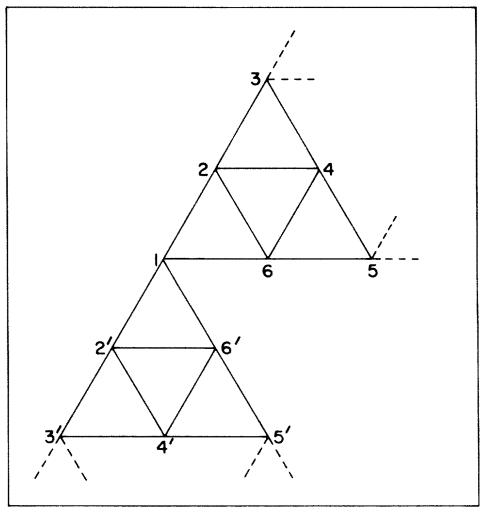


FIG. 4. Fragment of the Sierpinski gasket in $d=2$.

nected in the three corresponding Schrödinger equations only to each other and to $a_1$, $a_3$, and $a_5$. This simplifying feature is due to the finite order of ramification of the gasket. The three linear equations may be solved yielding equations of the form $a_i = a_i(a_1, a_3, a_5; x)$, $i = 2,4,6$. A similar analysis may be carried out in the lower triangle. Substituting the expressions for $a_2$, $a_6$, $a_{2'}$, and $a_{6'}$ into Eq. (4) leads to the equation

$$x' a_1 = a_3 + a_5 + a_{3'} + a_{5'} , \tag{5}$$

where

$$x' = x(x-3) . \tag{6}$$

The self-similarity of the gasket enables one to iterate this process recursively and to obtain renormalized values of $x$ at each length scale. An analogous decimation procedure has been carried out by José[14] in a $d=1$ tight-binding model for both periodic and random cases. For the periodic system the recursion relation is

$$x' = x^2 - 2 . \tag{7}$$

It is straightforward to perform the decimation procedure for the $d=3$ gasket, yielding

$$x' = x^2 - 6x + 6 . \tag{8}$$

We now generalize our results to arbitrary dimensionality $d$. In any decimation step, we eliminate $d(d+1)/2$ midpoints from each hypertetrahedron. The Schrödinger equations for these sites may be written in matrix form as $\underline{A}\vec{P} = \vec{Q}$, where

$$\vec{P} = \begin{pmatrix} a_2 \\ a_6 \\ \vdots \\ a_4 \\ \vdots \end{pmatrix}, \quad \vec{Q} = \begin{pmatrix} a_1 + a_3 \\ a_1 + a_5 \\ \vdots \\ a_3 + a_5 \\ \vdots \end{pmatrix} . \tag{9}$$

Here $\vec{P}$ is a vector constructed of the wave functions at the midpoints which are to be decimated. The vector $\vec{Q}$ is made up of pairs of wave functions at the corners of the hypertetrahedron added together. For convenience the elements of $\vec{P}$ are arranged in such a way that all the $d$ nearest-neighbor midpoints of corner 1 are written first. The remaining $(d-1)$ nearest-neighbor midpoints of corner 3 are written next, and so on.

The matrix elements of $\underline{A}$ take on only three possible values: $x$, $-1$, and 0. Similarly, on very general symmetry grounds, the matrix $A^{-1}$ is obtainable from $\underline{A}$ by making the replacements $x \to \alpha$, $-1 \to \beta$, and $0 \to \gamma$. This is related to the fact that all of the midpoints are either nearest neighbors or next-nearest neighbors to any given corner site. By the choice of the vectors $\vec{P}$ and $\vec{Q}$ the first row of the matrix $A$ is $x, -1, -1, -1, -1, \ldots, -1, 0, 0, \ldots, 0, 0$, where there are $2(d-1)$ elements equal to $-1$ and $(d-1)(d-2)/2$ elements equal to zero. The general structure of the second row of the matrix $\underline{A}$ is as follows: The first element is $-1$, the second (diagonal) element is $x$, and the next $(d-2)$ elements are $-1$; the next $(d-1)$ elements are made up of $(d-2)$ zeros and one element equal to $-1$; we need not concern ourselves with the remaining elements of this row. The last row of the matrix $\underline{A}$ has 0 as its first element; the next $(d-1)$ elements are all zeros except for two which are equal to $-1$. The same holds for the next $(d-1)$ elements, and as before we are not interested in the remaining elements.

Since $\underline{A}^{-1}\underline{A} = \mathbb{1}$ we may solve for $\alpha$, $\beta$, and $\gamma$ as functions of $x$ and $d$. We have therefore succeeded in inverting the matrix $\underline{A}$. It should be noted that this inversion is not possible for a finite, discrete set of values of $x$ (henceforth referred to as $\tilde{x}$). For these values, the determinant of the matrix vanishes and the recursion relation derived below [Eq. (11)] does not hold.

Following the procedure in $d=2$ we express the sum of the midpoint wave functions in the Schrödinger equation for corner 1 as a linear combination of the corner wave functions. By taking into account that the corner 1 belongs to two hypertetrahedra we get a new Schrödinger equation on a different length scale. The recursion relation is

$$x' = \frac{x - 2d[\alpha + (d-1)\beta]}{\alpha + 3(d-1)\beta + (d-1)(d-2)\gamma} , \tag{10}$$

which upon substituting the expressions for $\alpha$, $\beta$, and $\gamma$ becomes

$$x' = x^2 + 3(1-d)x + 2d(d-2) , \tag{11}$$

where we have assumed that the denominator of Eq. (10) is nonzero. For special values of $x$ where the denominator is zero, $x' = \infty$, and the hopping matrix element becomes zero upon one iteration. Equation (11) reduces to Eqs. (6), (7), and (8) for $d=2$, 1, and 3, respectively.

Upon making the transformation,

$$x = -(d+3)t + 2d , \tag{12}$$

Eq. (11) can be cast in the quadratic form [Eq. (3)] with $\lambda = d+3$. Since the exact recursion relation is known, it is straightforward to iterate the decimation procedure to obtain the effective Hamiltonian for large length scales. The iterative properties of the one-dimensional map given by Eq. (1) have been studied extensively.[13,15] For $d > 1$, $\lambda$ is larger than 4 and the logistic map does not have any stable fixed points or cycles. The two fixed points of the transformation [$t=0$ and $(\lambda-1)/\lambda$] are unstable and any initial value of $t$ leads to a faster than exponentially growing sequence of $t$'s. This is equivalent to a decreasing value for the effective hopping matrix element on successively larger length scales and shows that the electronic eigenstates, if any, are localized stronger than exponentially. We note, however, that there is a set of initial values of $t$ for which the transients go to the unstable fixed points but this set is of measure zero.[15]

Two points deserve further clarification. First, we have not treated the set of points $\tilde{x}$ for which Eq. (11) does not hold. It is indeed possible that the set of $\tilde{x}$'s and of those values of $x$ which renormalize to $\tilde{x}$ may have degenerate eigenstates. However, we stress again that this set is discrete. Second, for values of $x$, for which the recursion relation [Eq. (11)] does hold, $x'$ eventually renormalizes to $+\infty$. This does not necessarily distinguish between localized eigenstates and the absence of any states (band gaps).

It is a possibility that the gap states are in fact of measure 1 in this problem.

This behavior is in sharp contrast to that of the $d=1$ periodic tight-binding model [Eq. (7)]. In this case $\lambda=4$ and there is a basin of attraction between $t=0$ and 1. One may identify this basin with a band of extended states and the frequency with which a given energy occurs on successively iterating the recursion relation, starting from any point within the basin, is proportional to the density of states within the band.[14]

The method and results of this note are applicable to a system of point masses $m$ placed on the sites of the gasket and connected by springs of spring constants $k$, provided we make the simplifying (albeit artificial) assumption that the masses are allowed to move only in a direction orthogonal to the $d$-dimensional space of the gasket. The parameter $t$ in this case is given by $(-m\omega^2/k\lambda)$. The $\omega\rightarrow 0$ limit corresponds to the $t\rightarrow 0-$ limit. We may then make the linear approximation for the recursion relation yielding $t'=\lambda t$. Upon making the decimation, $\omega$ renormalizes to $\lambda^{1/2}\omega$ while a fraction $(d+1)^{-1}$ of the sites are left intact. It follows from a straightforward scaling argument that in the limit of $\omega\rightarrow 0$ the number of modes with frequency less than $\omega$ is proportional to $\omega^y$ with $y=\ln(d+1)/\ln[(d+3)^{1/2}]$. Thus the low-temperature specific heat of the system obeys the $T^y$ law with $y\rightarrow 2$ as $d\rightarrow\infty$. Following Dhar,[16] who studied the truncated tetrahedron lattice, we note that the fractal dimensionality is not equal to the exponent $y$.

*Note added.* After the work presented in this section was completed we learned of similar work by Domany *et al.*[17] They study the solution of the Schrödinger equation on a variety of fractal lattices and in particular on the Sierpinski gasket in $d=2$ and 3. Their analysis of the recursion relations goes much beyond our own. For the $d=2$ gasket, they show the existence of localized eigenstates and explicitly construct them, they show that the gaps are of measure one, and they explicitly construct the density of states function. We were informed by Kadanoff that Eq. (4) has apparently existed in unpublished form and was first derived by Alexander.[18] (We are grateful to Stephen for bringing to our attention the work of Domany *et al.*[17] and to Kadanoff for useful correspondence.)

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