Computer study of the percolation threshold in a two-dimensional anisotropic system of conducting sticks

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We report here a Monte Carlo study of the percolation threshold in two-dimensional systems of conducting sticks. This is an extension of the work of Pike and Seager, who have considered only the isotropic sample of randomly-oriented—, equal-length—sticks system. Our study is concerned with the dependence of the percolation threshold on the macroscopic anisotropy of systems in which there is a preferred orientation of the sticks ensemble, as well as on the distribution of the sticks' lengths. In particular, we studied systems in which the orientation is determined by random alignments within a given interval or in which the alignments are normally distributed around a given direction. Similarly, for the sticks' lengths we have studied systems of equal lengths, of normally distributed lengths, and of log-normally distributed lengths. The results have shown that the percolation threshold always increases with the macroscopic anisotropy. Extrapolation of the results, from those of the finite sticks ensembles used to the infinite ensemble case, has indicated that in the infinite ensemble the percolation threshold is isotropic. It is found that the broader the stick-length distribution, the lower the mean of the distribution needed for the onset of percolation. Application of the present results for the evaluation of the conductivity indicates that the anisotropy dependence of the conductivity in systems of conducting fibers is determined by both the anisotropy dependence of the percolation threshold and the anisotropy dependence of the critical exponent. If (as found experimentally) a practically infinite two-dimensional system has a conductivity anisotropy, it must be attributed to the anisotropy in the latter parameter.

I. INTRODUCTION

While composites in which conducting fibers are embedded in an insulating matrix are widely used by the electrical industry,[1] the understanding of their electrical properties has only recently emerged.[1–3] This understanding is based on the realization that the electrical conduction in such systems is a percolative process.[4] The experimental evidence for this consists of the existence of a sharp onset of high conductivity[1,2] at a certain concentration ω_c of the conducting member of the composite, ω, as well as of the power-law dependence[3] of the conductivity on $\omega-\omega_c$. The percolation approach has been applied thus far to the experimental studies of aluminum fibers,[1] carbon fibers,[2] and carbon black aggregates,[3] all embedded in insulating polymers. The experimental results have shown that in comparison with systems of spherelike conducting particles, the onset of percolation in the fiber or "sticklike" systems takes place at a lower ω of the conducting filler. In the systems of conducting fibers it was found that the ω for the onset of percolation decreases with increasing aspect ratio (L/D) of the fibers.

In all the experimental works quoted above the composites used were deliberately made to be *macroscopically isotropic,* i.e., the fibers' microscopic orientations were completely random. However, since all the composites mentioned above are prepared by mixing the conducting particles and the polymer in a molten state, it is clear that if the melt is cooled down during its flow there will be an increasing degree of fiber orientation with increasing flow distance.[5,6] It is expected that similar to the dependence of the mechanical properties[6] on the fiber-orientation distribution (FOD) there will be a strong dependence of the electrical properties on the FOD. The simplest argument to appreciate this is to compare a system of randomly oriented sticks with a system of all parallel sticks. It is apparent that the parallel-sticks system will be anisotropic; the ω_c required for the onset of percolation and the resistivity of the system for a given ω will be larger than those corresponding to the randomly oriented (isotropic) sticks system. The above expectations were verified lately[7] by showing that the resistivity as well as the anisotropy of the resistivity of samples having a given conducting-particle concentration increase with the flow distance of the composite melt during the sample preparation.

The simplest simulation of the above composites is a two-dimensional sample of conducting sticks (or straight fibers) embedded randomly in an insulating matrix. The first, and as far as we know the only, analysis of such a system is the study of Pike and Seager[8] who carried out a Monte Carlo study of the two-dimensional stick percolation problem. However, their work was concerned only with the percolation threshold of the macroscopically isotropic case of randomly distributed sticks with a random FOD. The sticks were assumed to be all of the same fixed length L and a zero width D (i.e., $L/D\rightarrow\infty$).

In view of the interest in the above composites and the recent general interest in the percolation in other two-dimensional anisotropic systems,[9–13] we are presenting here an extension of the study of Pike and Seager[8] to *anisotropic* stick systems. This extension consists of the computation of the percolation threshold of stick systems in which there is a preferred orientation of the stick ensemble and in which the sticks' lengths have various distributions. The latter consideration was prompted by the fact that in some cases[3,7] the conducting fibers given are not of the same length. In a following paper we plan to present a Monte Carlo calculation of the electrical conductivity of the systems considered here. Such a calculation has not been carried out before for systems of conducting sticks. We are not concerned here with the effect of the aspect ratio on the percolation threshold, although in three-dimensional composites the threshold is strongly dependent on this parameter.[1] In two-dimensional systems the nature of the problem does not change with the change in aspect ratio, and from the work of Pike and Seager[8] it is apparent that the dependence on L/D is very weak. (They found that the critical size of the square in the "square percolation problem," i.e., $L/D=1$, is only about half the critical stick length in the stick percolation problem where $L/D\rightarrow\infty$.) We shall see here that the effects of the FOD and of the fiber-length distribution (FLD) on the percolation threshold are by far stronger and therefore more interesting.

In the present work we adopt the procedure used by Pike and Seager[8] as a starting point. The method is modified, however, to enable the inclusion of various FOD and FLD for the study of their effect on the percolation threshold. In a study such as ours computer time costs are an important factor. Correspondingly, we have used the smallest stick ensembles necessary to deduce the behavior of an effectively infinite system, as well as to estimate the deviation of the computed results from those of the infinite system.

Section II of this paper gives a simple analytic prediction for the expected dependence of the percolation threshold on the macroscopic anisotropy of the stick ensemble. In Sec. III we present the method used for the graphics and the computations. The results of the computation which are concerned with the effect of various FOD's and FLD's on the percolation threshold of the stick system are given in Sec. IV. In Sec. V we discuss the results and their implication for the prediction of the dependence of the critical conductivity of the system on the FOD and FLD.

II. THE ANISOTROPIC SYSTEM AND ITS EXPECTED THRESHOLD

In this section we describe the systems to be considered in this work, we define the macroscopic anisotropy, and we use this definition to consider the anisotropic case as a simple transformation of the isotropic case. The conclusions of the analysis are found to be quite general and, as will be shown in Sec. IV, in agreement with the computational results.

Let us consider a sample which is a unit-size square. In this square we "plant" N randomly distributed sites. A sample of such sites which is generated by (introducing an initial number called seed to) the computer is shown in Fig. 1. The next stage is to attach a stick of length L to each site as shown in Fig. 2. Since there is no width to the sticks, there can be no intersection of two sticks. Hence, if the stick length is less than the sample size (unity), there can be no continuous path from one boundary of the sample to the opposite boundary. In other words there is no percolation. If, however, we allow the sticks to have a nonzero angle with each other, we may obtain percolation as is shown in Fig. 3. In this figure we present an anisotropic sample in which the orientations of the sticks are either θ (10° in this case) or $-\theta$ (−10°) with respect to a chosen longitudinal (y) direction. As is immediately seen, for the stick length used there is percolation along the longitudinal direction but (in this finite system; see below) there is no percolation in the transverse (x) direction.

It is already apparent from the above description that L must be of the order of the intersite distance. Hence, the choice of L depends on the number of sticks, N, in the sample. We follow Pike and Seager[8] in choosing an effective intersite distance $2r_s$ by attaching to each site a circle with an area πr_s^2 where

$$r_s = 1/\sqrt{\pi N} \ . \qquad (1)$$

Correspondingly, the stick lengths in this study will be expressed in terms of r_s and we may define a stick density by N.

Examining Fig. 3, one notes that the anisotropy of the system is manifested by the fact that a stick has a component $L\cos\theta$ in the longitudinal direction and a component $L\sin\theta$ in the transverse direction. Considering the macroscopic anisotropy of the system, we find that the

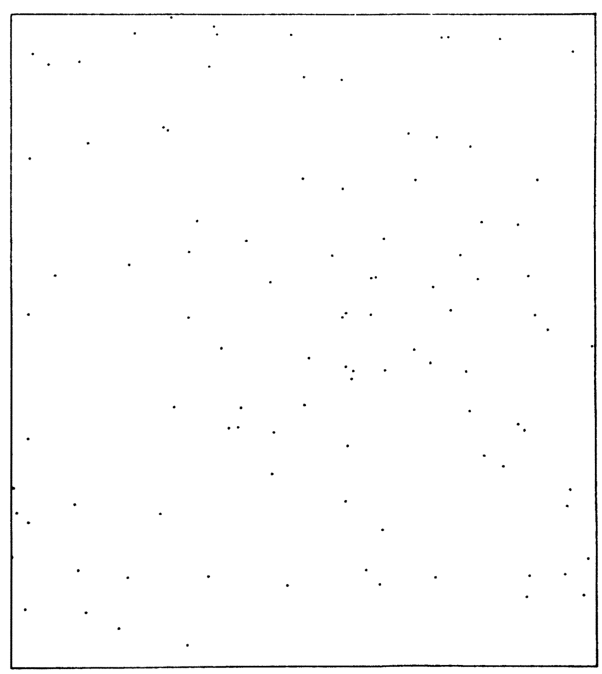

FIG. 1. Sample used for the two-dimensional sticks system. There are 100 sites (stick centers) randomly distributed. Ensemble of stick centers shown here was generated by using seed 7.

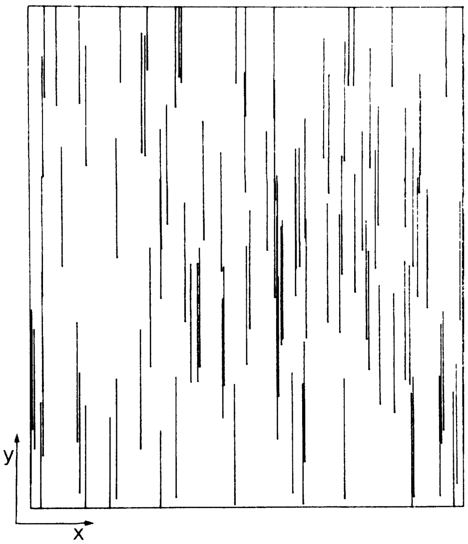

FIG. 2. Sticks of equal length are attached to the sites shown in Fig. 1. Direction of the stick alignments here will be called the longitudinal direction.

sum of the sticks' longitudinal components $P_{\|}$ is $NL\cos\theta$ and that the sum of the sticks' transverse components $P_{\perp}$ is $NL\sin\theta$. These quantities are closely related to parameters used in other anisotropic percolation problems.[10,11] We may define then the macroscopic anisotropy of the sample as $P_{\|}/P_{\perp}$ which in the above simple case amounts to $\cot\theta$. Our definition of macroscopic anisotropy will be shown to be convenient both for the analysis of the problem and for the computation of the macroscopic anisotropy of the various systems considered. This definition is easily generalized. Suppose that in the general case stick i makes an angle θ_i with the longitudinal direction. The macroscopic longitudinal component will then be $\sum_{i=1}^{N} L\,|\cos\theta_i|$ and the transverse component will be $\sum_{i=1}^{N} L\,|\sin\theta_i|$. This yields the macroscopic anisotropy

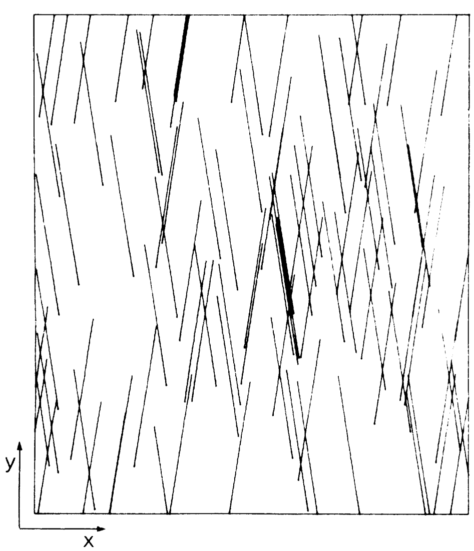

FIG. 3. Sticks of equal length ($L=5.1r_s$) making an angle of 10° or −10° with the longitudinal direction. Note the existence of longitudinal percolation and the nonexistence of transverse percolation.

$$P_{\|}/P_{\perp}=\frac{\sum_{i=1}^{N}|\cos\theta_i|}{\sum_{i=1}^{N}|\sin\theta_i|} . \tag{2}$$

In the completely random isotropic case discussed by Pike and Seager,[8] the angles θ_i lie in the interval $-90°\le\theta_i\le 90°$. For a sufficiently large ensemble the system is expected to be isotropic and thus the critical stick length, necessary to bring about percolation in the longitudinal direction, $L_{c\|}$ is expected to be the same as the critical stick length necessary to bring about percolation in the transverse direction, $L_{c\perp}$. We further expect that for the completely random case, $L_{c\|}=L_{c\perp}=L_c$, where L_c is the value obtained by Pike and Seager.[8] Expressing L_c in terms of r_s as suggested in their work yielded then and in the present work a constant $f_0=4.2$ such that

$$L_c=f_0 r_s . \tag{3}$$

The simplest anisotropic random-orientation distribution is obtained by a generalization of the completely random case, i.e., by allowing θ_i to be chosen randomly in the interval

$$-\theta_\mu\le\theta_i\le\theta_\mu , \tag{4}$$

where $\theta_\mu\le 90°$. It is apparent that the smaller the value of θ_μ, the more anisotropic will be the sample. The orientation distributions obtained for six values of θ_μ, for a sample of 100 sticks, are shown in Fig. 4. One can clearly see that longitudinal percolation sets in for $\theta_\mu\ge 30°$, while transverse percolation exists here only for the $\theta_\mu\ge 70°$ cases.

The aim of this work is to find the dependence of $L_{c\|}$ and $L_{c\perp}$ on $P_{\|}/P_{\perp}$. For the finite ensembles such as those shown in Fig. 4 one can compute these critical values as well as the values of $P_{\|}/P_{\perp}$ for each θ_μ (see Sec. III). The results of such computations can be presented in a form similar to that of Eq. (3) except that now f_0 will be replaced by the functions $f_{\|}(P_{\|}/P_{\perp})$ and $f_{\perp}(P_{\|}/P_{\perp})$ which are defined by

$$L_{c\|}=f_{\|}(P_{\|}/P_{\perp})r_s \tag{5}$$

and

$$L_{c\perp}=f_{\perp}(P_{\|}/P_{\perp})r_s . \tag{6}$$

Turning to the infinite system ($N\to\infty$) and recalling that each angle has the same probability in the random system defined by Eq. (4), we find, using definition (2), that the macroscopic anisotropy will be given by

$$P_{\|}/P_{\perp}=\frac{\int_0^{\theta_\mu}\cos\theta\,d\theta}{\int_0^{\theta_\mu}\sin\theta\,d\theta}=\frac{\sin\theta_\mu}{1-\cos\theta_\mu} . \tag{7}$$

On the other hand, realizing that the angles θ are equally distributed between 0 and θ_μ, we may think of the angle $\theta_\mu/2$ as a "mean angle" or a "mean direction" (in respect to which the stick orientations are distributed symmetrically). Hence, as far as the anisotropy of the system is concerned, we may consider a stick which lies in the $\theta_\mu/2$

direction to be representative of the system. For this representation to be valid the anisotropy associated with the components of this stick,

$$\frac{L\cos(\theta_\mu/2)}{L\sin(\theta_\mu/2)}\ , \tag{8}$$

must be consistent with that of the whole system [Eq. (7)]. Indeed, the expressions given in Eqs. (7) and (8) are the same and thus we may follow the changes in the anisotropy of the system by considering the corresponding changes in the representative stick. In particular the representative stick of the isotropic system ($\theta_\mu=90°$) makes an angle of 45° with the longitudinal and transverse directions. Its longitudinal component is $L/\sqrt{2}$ and its transverse component is also $L/\sqrt{2}$.

For the evaluation of the effect of anisotropy on the percolation threshold let us now introduce the macroscopic anisotropy $P_{\|}/P_{\perp}$ by carrying out a contraction transformation of the system. The transformation is carried out by "squeezing" the transverse direction by a factor of $P_{\|}/P_{\perp}$. All the sticks in the system will now have their transverse components $L\sin\theta_i$ transformed to $(L\sin\theta_i)/(P_{\|}/P_{\perp})$, while their longitudinal component $L\cos\theta_i$ has been kept the same. The density of the sticks has correspondingly increased from N to $N(P_{\|}/P_{\perp})$. The transformed representative stick has a longitudinal component $L/\sqrt{2}$ and a transverse component $L/[\sqrt{2}(P_{\|}/P_{\perp})]$. Correspondingly its transformed length is

$$L'=(L/\sqrt{2})[1+1/(P_{\|}/P_{\perp})^2]^{1/2}\ . \tag{9}$$

From the percolation point of view no changes occurred upon the contraction transformation since the connectedness remained that of the isotropic sample. Hence if L in the isotropic sample was sufficiently long to yield percolation, in both directions, then L' is also sufficiently long to yield percolation in both directions. In particular, if L_c was the critical stick length for the isotropic system, then

$$L_c'=(L_c/\sqrt{2})[1+1/(P_{\|}/P_{\perp})^2]^{1/2} \tag{10}$$

will be the critical stick length in the anisotropic contracted sample. As for the isotropic sample [Eq. (3)] we can present L_c' in terms of the stick density, i.e., as

$$L_c'=f'/[\pi N(P_{\|}/P_{\perp})]^{1/2}\ , \tag{11}$$

where f' depends on $P_{\|}/P_{\perp}$. Since this is the dependence we are looking for, we combine Eqs. (3), (10), and (11) to find that

$$f'=f_0\{[(P_{\|}/P_{\perp})+(P_{\|}/P_{\perp})^{-1}]/2\}^{1/2}\ . \tag{12}$$

Equation (12) predicts that for the infinite ensemble of sticks, in which the orientations are distributed according to Eq. (4), the percolation threshold will increase with increasing anisotropy from its isotropic value f_0. We may further conclude that the difference between the longitudinal threshold and the transverse threshold shown in Fig. 4 is due to the finite number of sticks in the corresponding ensemble. In Sec. IV we show that the computational results confirm these conclusions.

The above simple argument can be readily generalized

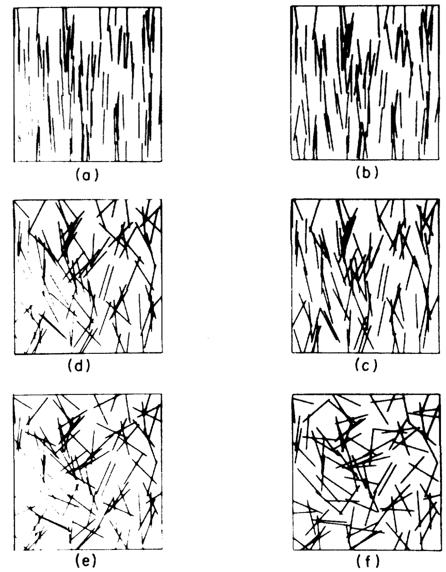

FIG. 4. Two-dimensional sample of 100 sticks of equal length ($L=4.2r_s$) with a sample orientation determined by the cutoff θ_μ of the random angles θ_i. (a) $\theta_\mu=5°$, (b) $\theta_\mu=10°$, (c) $\theta_\mu=30°$, (d) $\theta_\mu=50°$, (e) $\theta_\mu=70°$, and (f) $\theta_\mu=90°$.

to all cases where $P_{\|}/P_{\perp}$ can be determined by computation (for finite systems, as in Sec. III) or by analysis (for an infinite system, as above). This is because for a sufficiently large ($N\to\infty$) system every angle distribution can be mapped into the random case. The corresponding θ_μ can be found from the common $P_{\|}/P_{\perp}$. In Sec. IV we demonstrate that this is true for the normal distribution of angles. Another case of interest is that in which the sticks are of different lengths and their lengths follow a given distribution. In this case the anisotropy of the system is defined by

$$P_{\|}/P_{\perp}=\frac{\sum_{i=1}^{N}|L_i\cos\theta_i|}{\sum_{i=1}^{N}|L_i\sin\theta_i|}\ . \tag{13}$$

If the lengths L_i and orientations θ_i are not correlated and if the ensemble is sufficiently large, the dependence on anisotropy will be the same as in Eq. (12) except that f_0 will be determined by the distribution of L_i. In Sec. IV we define f_0, for all cases where the stick lengths are not equal, as the mean of the stick-length distribution. It will be shown there that the expectation for the above [Eq. (12)] dependence of f' on $P_{\|}/P_{\perp}$ is confirmed by the computation for both normal and log-normal stick-length distributions.

III. COMPUTATIONAL PROCEDURES

The samples used for the study were made of a unit-size square. In this square sites were "planted" by generating random coordinates (x_i,y_i), $i=1,2,\ldots,N$. This was done by using a pseudorandom number generator (UNIFRM) available for the IBM 3670 computer used throughout this work. UNIFRM produces a sequence of psuedorandom numbers uniformly distributed on the interval [0,1]. The starting point, or seed, for UNIFRM is chosen arbitrarily, but once selected we kept it to be the same for all our illustrations and computations. To check the importance of the seed's choice we have varied the seeds when such a check was called for. For the site selection we have obtained all N of the x_i coordinates and then all N of the y_i coordinates. An example of a resulting random array of sites, for $N=100$, was shown in Fig. 1.

We attach a stick to the site (x_i,y_i) by choosing a length l_i and orientation θ_i (with respect to the y axis). For these values of l_i and θ_i a stick is plotted by drawing a line segment between $(x_i+(l_i/2)\sin\theta_i, y_i+(l_i/2)\cos\theta_i)$ and $(x_i-(l_i/2)\sin\theta_i, y_i-(l_i/2)\cos\theta_i)$. The values of l_i and θ_i are chosen according to the desired distribution as will be explained below. The simplest case of $l_i=L$ and $\theta_i=0°$ for $i=1,2,\ldots,100$ was shown in Fig. 2.

In the computation one would like to find whether percolation by the conducting sticks occurs in a given sample; i.e., whether there is a continuous conducting path along the y axis, and whether there is such a path along the x axis. We have called the corresponding paths the longitudinal (y) and the transverse (x) paths. Similarly, we call the conductivity along the y axis the longitudinal conductivity and the conductivity along the x axis the transverse conductivity. The principle of the computation is as follows. Every stick is being checked against another to find whether they intersect. If their point of intersection is within our unit-square sample they are assigned a common cluster number. All sticks within the same cluster have the same cluster number. If in the cluster there is a stick which intersects one boundary and another stick which intersects the opposite boundary we call the cluster a percolating cluster and we say that percolation along the corresponding axis prevails. A simple case was illustrated in Fig. 3 where we have chosen $\theta_i=10°$ for $i=1,3,\ldots,99$ and $\theta_i=-10°$ for $i=2,4,\ldots,100$, and $L=5.1r_s$. There is clearly a percolating cluster along the y axis and no percolating cluster along the x axis. Hence in this finite sample we have longitudinal percolation but no transverse percolation. Although the present procedure is similar to that used by Pike and Seager,[8] we describe here the execution of the above principle in some detail. This is done since in our samples l_i and θ_i are of a given distribution and this requires some modification of their procedure.

The intersection of the sticks is done by first checking whether the distance between sites i and j,

$$d_{ij}=[(x_i-x_j)^2+(y_i-y_j)^2]^{1/2},$$

is larger than $l_i/2+l_j/2$. If this is not the case we have to consider the geometrical construction shown in Fig. 5. Similar to Ref. 8 we define the following distances:

$$A_i=d_{ij}\,|\cos(\theta_j+\gamma)/\sin(\theta_j-\theta_i)|$$

and

$$A_j=d_{ij}\,|\cos(\theta_i+\gamma)/\sin(\theta_j-\theta_i)|,$$

where $\gamma=\arctan[(y_i-y_j)/(x_i-x_j)]$. As is clearly apparent from Fig. 5, the two sticks intersect if both conditions $A_i\le l_i/2$ and $A_j\le l_j/2$ are fulfilled. If, however, the point of intersection is outside the (unit-square) sample, the two sticks are not considered intersecting. A pair of intersecting sticks receives the same cluster number. The cluster number is being updated as each pair of sticks is being checked so that if a new stick is found to intersect two sticks, which previously belonged to different clusters, the two clusters will be assigned the same (one of the previous) cluster number. To check for percolation we find all the sticks which intersect the "longitudinal" boundaries, i.e., $y=0$ and 1, and all the sticks which intersect the "transverse" boundaries, i.e., $x=0$ and 1, and record their cluster numbers. If any cluster number appears for two opposite boundaries we say that the corresponding cluster is a percolating cluster. The program also records the number of percolating clusters and the number of sticks intersecting with each boundary which belongs to a given percolating cluster.

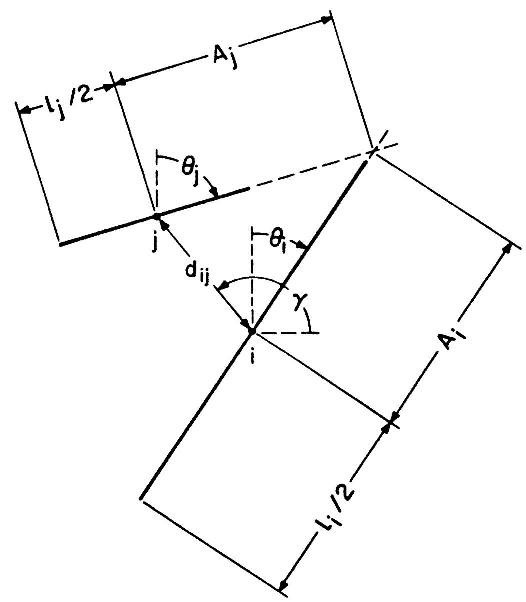

FIG. 5. Diagram of the two-stick intersection system. Geometrical quantities defined in this figure are used in checking whether two sticks intersect.

Now that the criterion for percolation was established we can proceed with the dependence of the percolation threshold on the system's parameters. As in Ref. 8 we predetermine the concentration of sticks in the sample and check the minimum stick length required for percolation. It is obvious from Eq. (1) that the argument can be reversed, and by predetermining the stick length we can find the critical concentration of sticks.

We start from the simple case where all the sticks have the same stick length L. The first trivial case is that of $\theta_i=\theta$ for all the sticks. This means that all the sticks are parallel and of course percolation is achieved only when $L\cos\theta=1$ for longitudinal percolation and $L\sin\theta=1$ for transverse percolation. This trivial example is mentioned in order to illustrate the uninteresting problem (from the percolation point of view) of very long sticks in a sample of a finite size. When one uses finite samples in the calculation one has to ensure that this case is not approached (see below).

The second trivial example, however, carries the nature of the percolation problem; i.e., the conducting particle size is much smaller than the macroscopic size of the sample. In this case the sticks are assigned the orientations $\theta_i=+\theta$ or $\theta_i=-\theta$, where the signs of θ_i are chosen randomly. This situation was shown in Fig. 3 and as will be discussed in Sec. III it demonstrates already the essential properties of the percolation problem in a system of oriented sticks. The completely random orientation case is the case discussed by Pike and Seager.[8] For this case we have used again the UNIFRM generator to generate N random values for the N random alignments of the sticks. The values θ_i, with respect to the y direction, are chosen randomly within the interval $-90°\le\theta_i\le 90°$. This is an isotropic case and thus the critical stick length in the y direction is expected to coincide with the critical stick length in the x direction. Both values are expected to be equal to L_c, the value obtained by Pike and Seager for the completely random-orientation case. As will be shown below this expectation is fulfilled. This fulfillment is important from the computational point of view since it indicates that the results obtained in the present work are not sensitive to the differences between the procedure used here and the procedure used in Ref. 8 and that an ensemble of 1000 sticks is statistically sufficient for the isotropic case. In Fig. 6 we show a computer-generated sample of $N=1000$ sticks of equal size which are randomly oriented. Examining this sample makes one appreciate the computational effort (i.e., cost) needed for the search of percolation in such a sample. Hence we did not carry out computations for larger samples.

The simplest FOD was described already in Sec. II as a generalization of the completely random case. Another possible choice of FOD is that of normally distributed alignments. We have chosen for this case an angle distribution with a mean of $\theta=0°$ and a standard deviation of $2\sigma=\theta_\sigma$. As for the case characterized by Eq. (4), we have studied the dependence of $L_{c\parallel}$ and $L_{c\perp}$ on θ_σ. Again, the smaller the θ_σ the more anisotropic is the FOD. To generate the corresponding θ_i we have used a subroutine (NORMAL) which is also available in our computer library. NORMAL computes a pseudorandom sequence of numbers normally distributed with zero mean and unit variance.

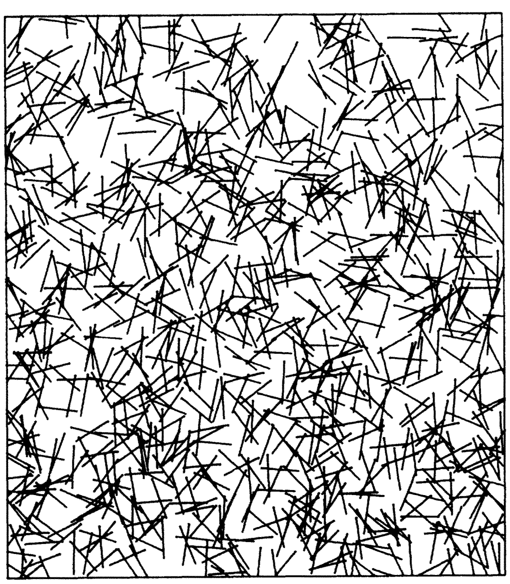

FIG. 6. Two-dimensional sample of 1000 sticks of equal length. Stick alignments are random and the sample is isotropic.

Each of the above (θ_μ or θ_σ) FOD's is associated with a different degree of sample anisotropy. We have thus defined a common criterion to correlate the values of $L_{c\parallel}$ and $L_{c\perp}$ found and the degree of anisotropy (or orientation) of the sample. The common criterion by which we have chosen to characterize the degree of anisotropy is the ratio given in Eq. (2).

It is worth noting that Eq. (2) can also be a good criterion for the statistical reliability of the sample. If for a given FOD two different seeds yield samples which differ substantially from one another in the numbers derived with the use of Eq. (2), the number of sticks is clearly not statistically sufficient. For the normal FOD's generated here with 1000-stick systems of fixed L, the deviations of the $P_\parallel/P_\perp$ values from one seed to another were of the same order of those obtained for the random case, i.e., less than 5%. In the latter case we have also compared the computed $P_\parallel/P_\perp$ values with the prediction given in Eqs. (7) and (8). As can be seen in Fig. 7, the deviations of the computed values (points) from the predicted values (curve)

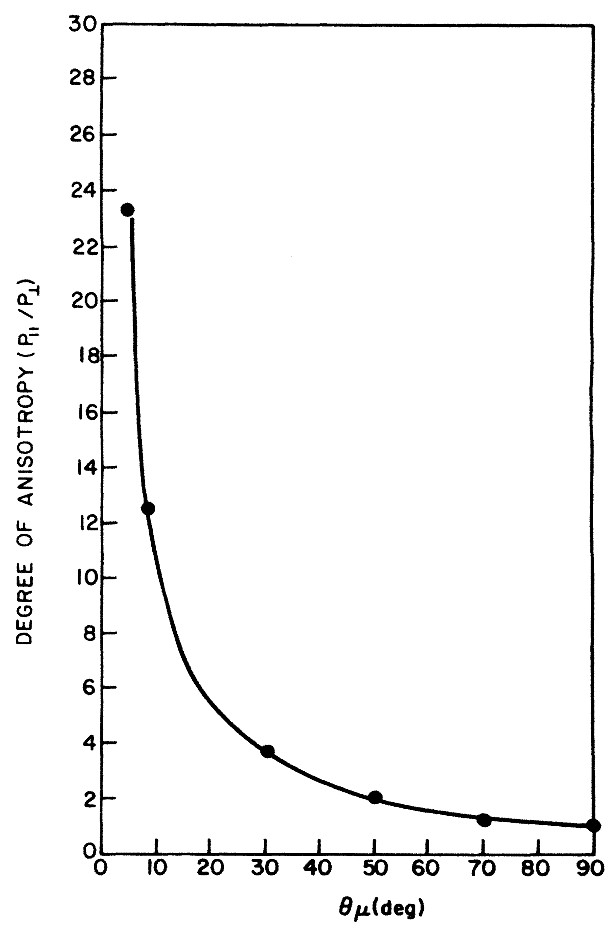

FIG. 7. Degree of macroscopic anisotropy as a function of the cutoff angle θ_μ of the random angle distribution. Curve is derived from Eq. (7) and the points are computed for the, 100, θ_i values of seed 7 with the use of Eq. (2).

derived from these equations were also of the same order as the deviations between various seeds.

Completing the study of the orientation effects on samples of sticks with the same length, we turned to the case where the stick lengths l_i are normally distributed around a mean L_M. For this purpose we have used the normal-distribution procedure which has been described above for the distribution of the angles. However, in this case $L_M \neq 0$ and we have chosen the standard deviation $2\sigma = 2r_s$. The $L_{c\parallel}$ and $L_{c\perp}$ determined in this case correspond to the minimum values of L_M which yield percolation in the respective direction. Another FLD which is of interest in view of its existence in some composites[3,7] is the log-normal distribution of sticks with a mean L_M and a standard deviation $2\sigma = 1$ (i.e., 95% of the sticks are between $0.1L_M$ and $10L_M$). The corresponding lengths were obtained again using the subroutine NORMAL, but the lengths l_i were chosen so that $\log l_i$ is normally distributed. The effect of these two FLD's were studied for the various FOD's which were described above for the case of the same fixed L.

The corresponding macroscopic anisotropy of this system is determined with the use of definition (13). If the choices of l_i and θ_i are not independent, Eq. (13) should yield the overall "macroscopic anisotropy" of the system which will be different from the "directional anisotropy" given by Eq. (2). In the cases studied here l_i and θ_i are chosen independently and the anisotropies derived from the two definitions should be equal. A comparison between the results obtained by the use of Eq. (2) and Eq. (13) indicates here the statistical sufficiency of the samples used. In the cases where the stick lengths were distributed we found that the deviations between the results based on Eq. (2) and those based on Eq. (13) were again less than about 5% for $N = 1000$. An example of a sample having a random-isotropic orientation distribution of the sticks, and a log-normal distribution of their length is shown in Fig. 8. It is quite apparent that the long sticks in the system will determine the onset of percolation as well as the conductance of the sample.

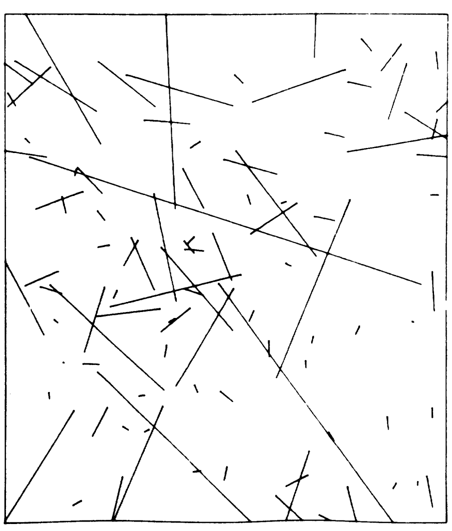

FIG. 8. Two-dimensional sample of randomly aligned sticks having a log-normal distribution of lengths. Mean stick length is $1.2r_s$ and the standard deviation is $\frac{1}{2}$.

Finally, in the case of a sample with a given fixed stick length L we can use the relations (5) and (6) to deduce the critical stick concentrations required for the onset of percolation. This is done by combining these relations and Eq. (1). The critical stick concentration required for longitudinal percolation $N_{c\parallel}$ will then be given by

$$N_{c\parallel} = f_{\parallel}^2/(\pi L^2) \ . \tag{14}$$

Similarly, the critical stick concentration for transverse percolation $N_{c\perp}$ will be given by

$$N_{c\perp} = f_{\perp}^2/(\pi L^2) \ . \tag{15}$$

One should note that while we present our results below in terms of Eqs. (5) and (6), in studied composites[1–3] it is usually the value of L or its FLD which is predetermined while the experimental variable is usually N.

IV. RESULTS OF COMPUTATIONS

As was pointed out above, the simplest random anisotropic system of fixed-length sticks is that of sticks of random sites but with an alignment θ (with respect to a given axis—y in our case), where only the sign of θ is chosen randomly. Taking samples of this kind (Fig. 3), we have considered the cases $\theta = \pm 5°, \pm 10°, \ldots, \pm 85°$ and found the corresponding values for $L_{c\parallel}$ and $L_{c\perp}$. These values were determined for this sample as well as for all subsequent samples studied here to within $\pm 0.1r_s$. With the use of our definition of the macroscopic anisotropy [Eq. (2)] we were able to deduce the dependences given by Eqs. (5) and (6). The results are shown in Fig. 9. Two features are conspicuous: (a) The critical stick length increases with increasing anisotropy, and (b) the ratio $L_{c\perp}/L_{c\parallel}$ increases with increasing anisotropy. As will be seen below, these features are reproducible for all the FOD's and FLD's considered in this study, indicating that these features are general and that the qualitative character of the dependences is already apparent in this simple quasirandom case. Note that the case $\theta = \pm 45°$ ($P_{\parallel}/P_{\perp} = 1$) is an isotropic case and that $P_{\parallel}/P_{\perp} < 1$ simply means the change of the longitudinal and transverse directions, hence the increase of $L_{c\parallel}$ and $L_{c\perp}$ when $P_{\parallel}/P_{\perp} \to 0$. We may further note that the values obtained in the present isotropic case ($L_{c\parallel} \approx L_{c\perp} \approx 5.5r_s$) are not too far off from the completely random, and therefore isotropic, FOD ($L_c = 4.2r_s$) obtained by Pike and Seager[8] and in the present work (see below). While, as will be shown below, such a small sample ($N = 100$) yields already the gross features of the functions $f_{\parallel}$ and $f_{\perp}$, the details disclose the limitations imposed on the computations by the use of small-size samples (small N). For example, we see in Fig. 9 that the minimum of $f_{\parallel}$ occurs around $P_{\parallel}/P_{\perp} = 2$, in contrast with the expectations that it will coincide with the minimum of $f_{\perp}$ and that it will be found for $P_{\parallel}/P_{\perp} = 1$. Let us turn now to the more physical cases of oriented stick ensembles in which the stick alignments are distributed randomly around a given direction.

The most thoroughly FOD investigated here is that of an ensemble of equal-length sticks with fixed length L

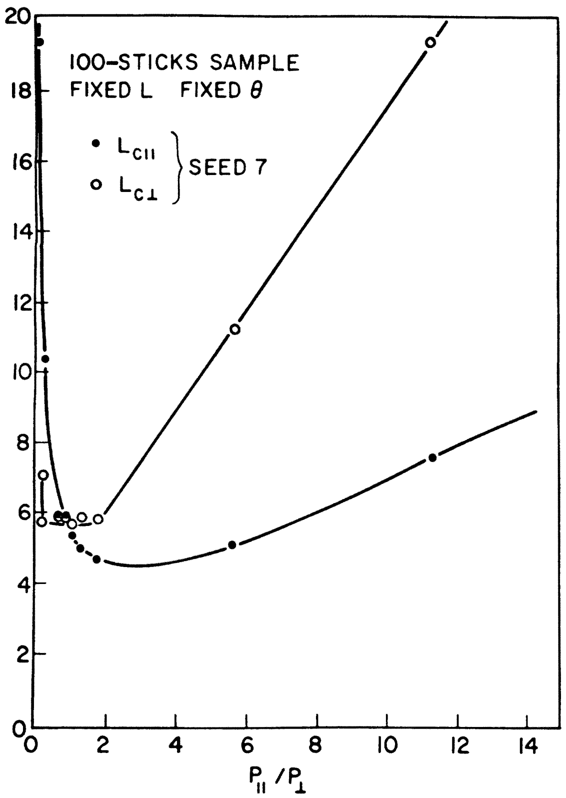

FIG. 9. Results of the percolation threshold computations for a sample of 100 sticks with fixed alignments and random sites. Threshold is determined by the minimum stick length $L_{c\|}$ which yields percolation along the y axis and the minimum stick length $L_{c\perp}$ which yields percolation along the x axis. (Typical sample used for this case was shown in Fig. 3 for $P_\|/P_\perp = 5.67$.)

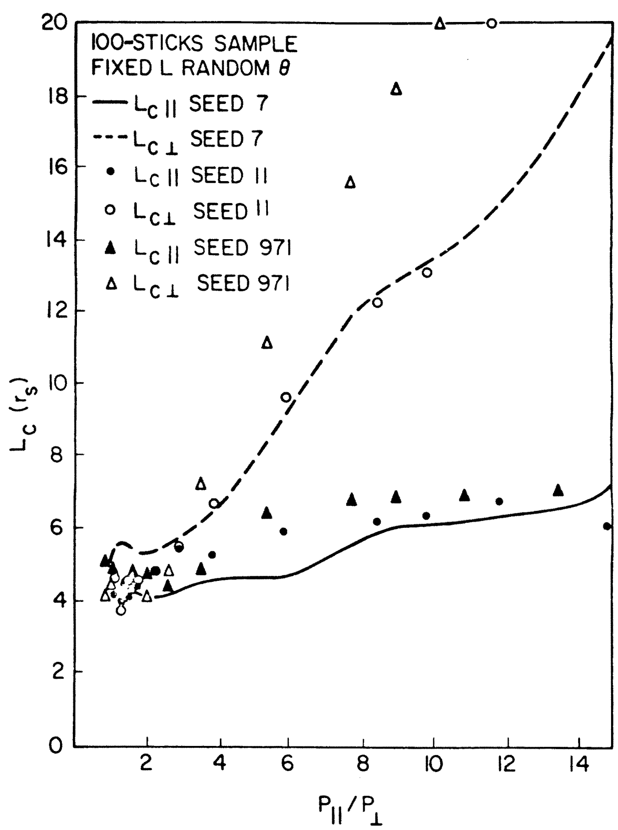

FIG. 10. Results of the percolation threshold computations for a sample of 100 randomly oriented sticks. Alignments are limited by the cutoff angle θ_μ. The computation was done for three different samples (seeds) in order to appreciate the effect of stick site-selection on the critical values $L_{c\|}$ and $L_{c\perp}$.

having random alignments θ_i within an interval $-\theta_\mu \le \theta_i \le \theta_\mu$. The value of θ_μ determines the degree of orientation of the macroscopic sample. Hence, the first stage is the computation of the corresponding anisotropy, $P_\|/P_\perp$, as a function of the cutoff angle θ_μ. This is done by using Eq. (2). The relatively high accuracy of the determination of $P_\|/P_\perp$ (see above) indicates that the error intervals and the faults in the results, which are associated with the ensembles being too small, will be manifested mainly in the computed values of $L_{c\|}$ and $L_{c\perp}$. Having the calibration of Fig. 7 we turned to compute $L_{c\|}$ and $L_{c\perp}$ for various samples of 100 sticks. First, we have carried out detailed computations for the seed shown in Fig. 1 (seed 7) and have summarized them by the corresponding curves of Fig. 10. The effect of varying the seeds is shown clearly by the results obtained using two other seeds. As expected from the increase of $L_{c\|}$ and $L_{c\perp}$ with increasing anisotropy, this effect is stronger with increasing anisotropy. Less intersections are needed then, for percolation, and thus the dependence on the particular seed becomes stronger. This means that for high $P_\|/P_\perp$ values the sample is statistically insufficient and that the values of $L_{c\|}$ and $L_{c\perp}$ are not quantitatively reliable. On the other hand, the qualitative features (a) and (b) mentioned above are clearly reproduced and the values of $P_\|/P_\perp = 1$ all lie between $4r_s$ and $5r_s$, in good agreement with Ref. 8 ($L_c = 4.2r_s$).

Following the results shown in Fig. 10 we expect that a larger ensemble will be a better approximation for the more interesting infinite ensemble. The dependence of the results on the ensemble size is hoped to indicate how to extrapolate the results of the infinite ensemble from the results obtained for the finite ensembles. In addition, one expects that in the larger system the "fine structures" associated with the particular seed used, will be smoothed out. Indeed, these expectations are fulfilled as we found by using five samples of 1000 sticks. This can be seen by comparing the results of Fig. 10 with the results obtained for any of the $N = 1000$ stick samples shown in Fig. 11. First, we note that the $L_{c\|}$ curve is higher in Fig. 11 than in Fig. 10 and that the $L_{c\perp}$ curve is lower in Fig. 11 than in Fig. 10. Hence, there is a closing up of the "gap" between the $L_{c\|}$ and the $L_{c\perp}$ curves. We have determined an $L_{c\|}$ and an $L_{c\perp}$ uncertainty regions by using the data of the five seeds. These are shown by the dashed lines in Fig. 11. For $P_\|/P_\perp < 4$ we may say that $L_{c\|} = L_{c\perp}$ since each computed point falls in the uncertainty region of both $L_{c\|}$ and $L_{c\perp}$. Second, we see that the effect of the seed variation for the $P_\|/P_\perp$ values investigated is much smaller here than in Fig. 10. The third observation is that for the isotropic case we find that $L_{c\|} = L_{c\perp} = (4.2 \pm 0.3) r_s$. Since this value was statistically established in Ref. (8) [to be

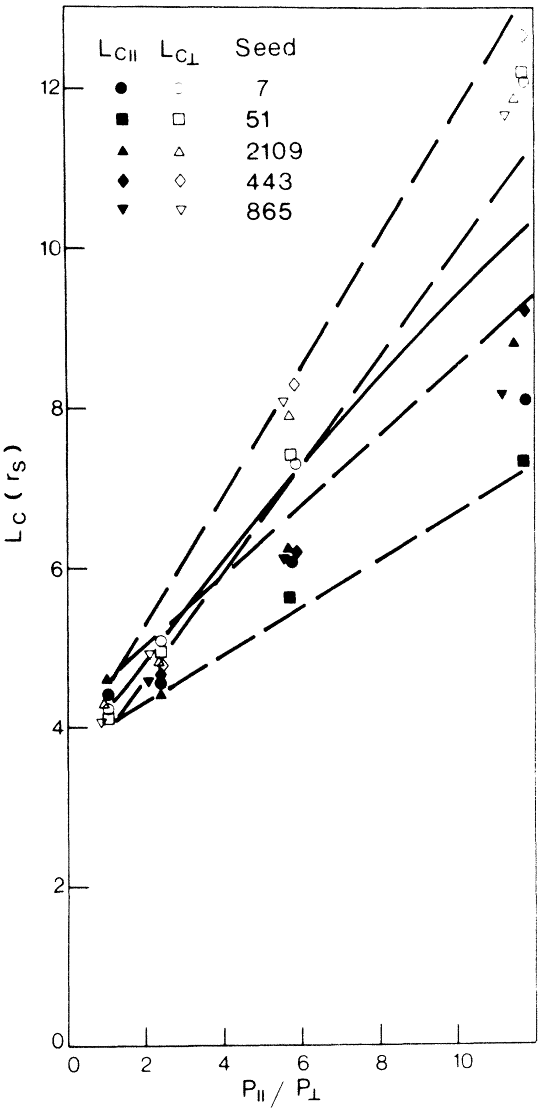

FIG. 11. Results of the percolation threshold computations for five samples of 1000 equal-length sticks randomly aligned within a given interval of alignments. Solid curve is the expected $P_{\parallel}/P_{\perp}$ dependence of both $L_{c\parallel}$ and $L_{c\perp}$ as $N\rightarrow\infty$. Expected dependence is given in Eq. (12).

$(4.2\pm0.1)r_s$] the result indicates a fast convergence of the L_c values when N increases from 100 to 1000. An explanation for this convergence is apparent when one considers a sample of intermediate anisotropy and a unit length, in which the fixed stick length is $L=10r_s=10/\sqrt{\pi N}$. For $N=100$, $L>\frac{1}{2}$ so that an intersection of two sticks may yield a longitudinal percolation. For $N=1000$, $L<0.2$, and thus at least five intersections are needed for percolation. Hence, in the $N=1000$ case, for a given degree of anisotropy, a higher connectivity is required for percolation. In the smaller ensembles the role of the stick-length component (e.g., $\sum_{i=1}^{N}|\cos\theta_i|$ in the longitudinal direction) dominates the role of the connectivity while in the infinite ensemble it is only the connectivity which determines the onset of percolation.

The most important observation in Fig. 11 is the agreement between the analytic result [Eq. (12)] shown by the solid curve and the computed results for the five seeds. With the use of the one point adjustment, $f_0=4.2$, the analytic result derived is clearly seen to be within the two uncertainty regions and thus the simple prediction of Eq. (12) is confirmed. On the other hand, the fact that the analytic result lies intermediately between the $L_{c\parallel}$ and the $L_{c\perp}$ values of each seed indicates that a 1000-stick sample sets reliable limits on the expected common percolation threshold. We shall use this conclusion below in assuming that the results of the computations are reliable limits to the threshold dependence for all the fiber-orientation and fiber-length distributions which are considered in the present work.

In the FOD discussed above the same weights were given to the various alignments within a given interval. We have chosen to represent the FOD's where different weights are assigned to different alignments by using a normal distribution of the angles θ_i. One notes that such distributions are more pertinent to plastic composites in which anisotropy is introduced by melt flow or shear action.[5–7] In variance with the previous FOD's this distribution is always anisotropic (i.e., the case $P_{\parallel}/P_{\perp}=1$ cannot be obtained). The degrees of anisotropy associated with this FOD were determined, as above, by using Eq. (2). The computations yielded the results shown in Fig. 12. It is seen that the two main features of the functions $f_{\parallel}$ and $f_{\perp}$ are found again for this normal FOD and that

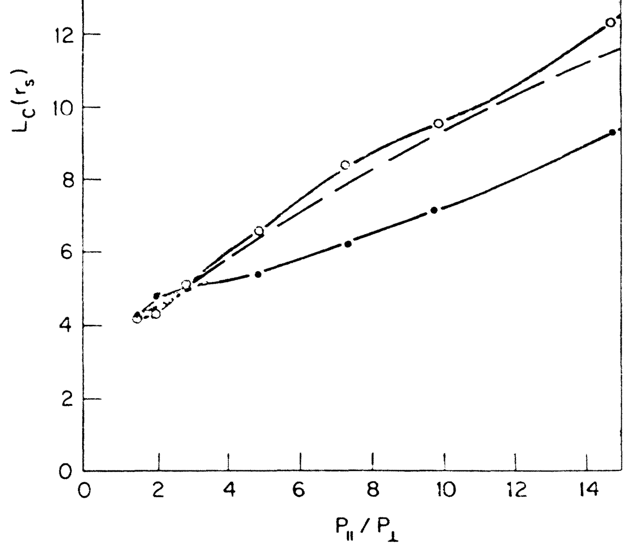

FIG. 12. Results of the percolation threshold computations for a sample of 1000 equal-length sticks with alignments which are normally distributed around $\theta=0$ with a variable standard deviation $2\sigma=\theta_\sigma$. Dashed curve was derived from Eq. (12).

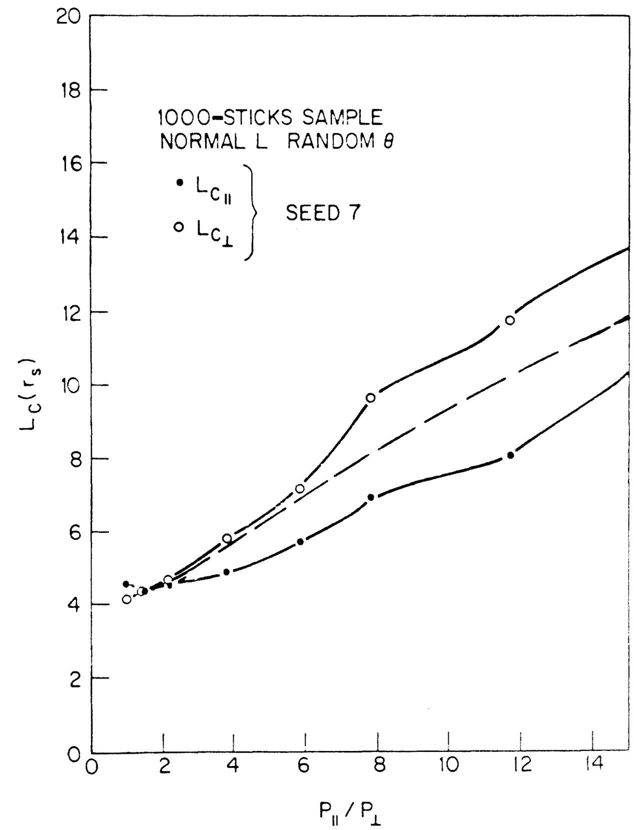

FIG. 13. Results of the percolation threshold computations for a sample of 1000 sticks, the lengths of which are normally distributed and the alignments of which are randomly distributed within given intervals. Standard deviation of the stick lengths was $2\sigma=2r_s$. Here $L_{c\parallel}$ and $L_{c\perp}$ represent the means of the stick-length distribution. Dashed curve was derived from Eq. (12) with the use of the f_0 value computed for $P_\parallel/P_\perp=1$.

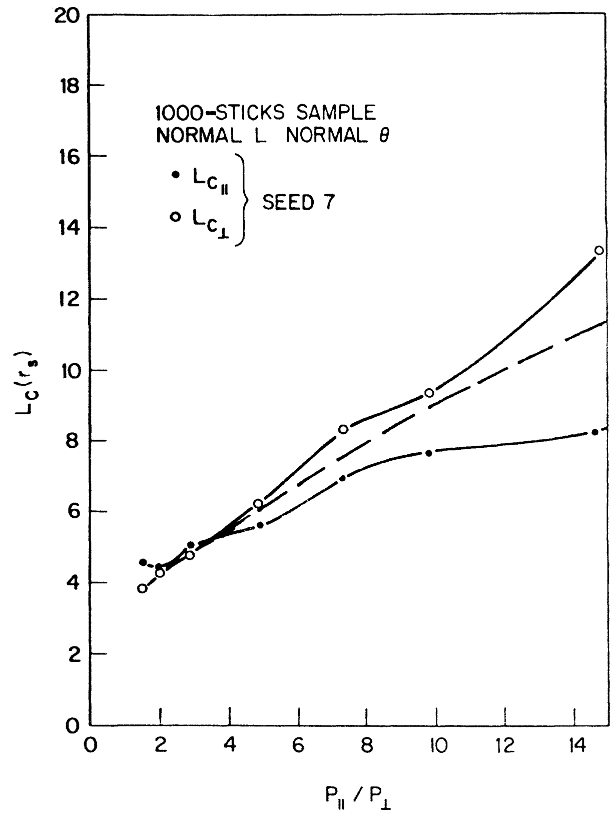

FIG. 14. Results of the percolation threshold computations for a sample of 1000 sticks, the lengths of which are normally distributed ($2\sigma=2r_s$) and the alignments of which are also normally distributed ($2\sigma=\theta_\sigma$). Dashed curve is the predicted dependence [Eq. (12)].

a good agreement is obtained between the results and the analytic expression Eq. (12). Again, the only adjustment is made by setting $f_0=4.2$.

Let us turn now to the computation of the effect of various FLD's on $f_\parallel$ and $f_\perp$. We consider first the case of normally distributed stick lengths l_i and randomly distributed angles θ_i. $P_\parallel/P_\perp$ is determined by Eq. (13). The results of these computations, with $L_{c\parallel}$ and $L_{c\perp}$ being now the means of the normal distribution, are shown in Fig. 13. The results are similar to those obtained for the same seed with equal-length sticks (Fig. 11). Within the accuracy of the computations it is impossible to tell whether the slightly lower $L_{c\parallel}$ and $L_{c\perp}$ values obtained (in comparison with those of Fig. 11) are a "real" effect. We shall see below that this may be a "real" effect, since the longer sticks dominate the percolation process. We shall further see that this lowering effect becomes more apparent the broader the stick-length distribution. Taking a normal distribution of the angles as well as a normal distribution of the stick lengths yields the results shown in Fig. 14. The findings show that this combination does not cause any significant changes in $f_\parallel$ and $f_\perp$ in comparison with the results of Fig. 13. Also apparent, as in the previous cases, is the agreement with the analytic result given by Eq. (12).

The effect of a much broader FLD becomes apparent when one uses the log-normal FLD instead of the normal FLD. As can be expected from the stick clusters of a sample with a log-normal FLD (Fig. 8), the few very long sticks determine the percolation threshold. Indeed, a significant reduction in the thresholds of the mean values of the distribution, $L_{c\parallel}$ and $L_{c\perp}$, takes place when comparison is made with the fixed length or the normal FLD cases. The corresponding results [for which the degree of anisotropy was determined by using Eq. (13)] are shown in Fig. 15. Again the two features (a) and (b) are reproduced. As in the previous cases we also plot the analytic result (12) by adjusting it at the isotropic case. In the present case, we take f_0 from the data shown using the mean of the isotropic sample which is 0.8. It is clearly seen that the agreement between the analytic result and the computed results is as good as in the previous case. One may note that we show here a shorter range of $P_\parallel/P_\perp$ than in the previous cases. The reason is that since in the present system the longer sticks are the ones participating in the percolation, there are effectively less sticks in the sample and for the higher anisotropies the transverse percolation may be determined by very few (sometimes one) sticks. Nonetheless, in all cases the analytic result lies between the $L_{c\parallel}$ and $L_{c\perp}$ values obtained in the computations. The

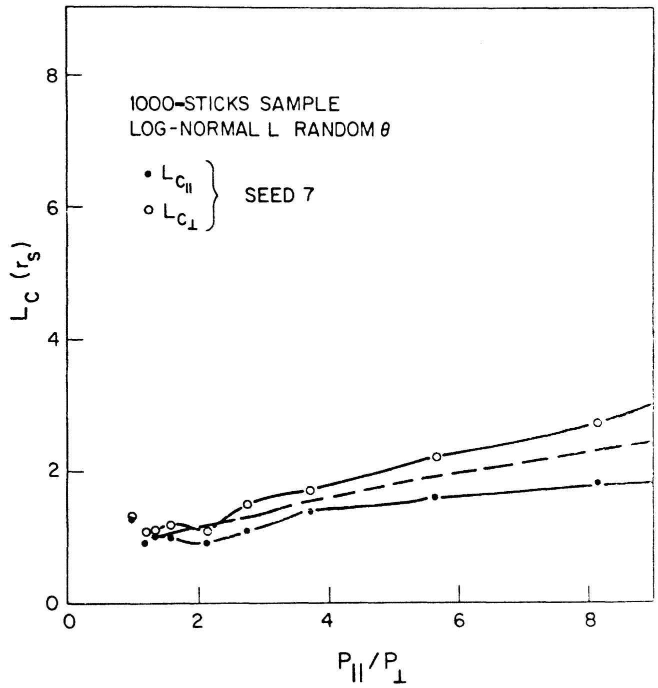

FIG. 15. Results of the percolation threshold computations for a sample of 1000 sticks, the lengths of which are log-normally distributed, with a standard deviation of $2\sigma=1$, and the alignments of which are random within given intervals. Dashed curve is a dependence expected [with the use of Eq. (12) and $f_0=0.8$] for both $L_{c\parallel}$ and $L_{c\perp}$ as $N\rightarrow\infty$.

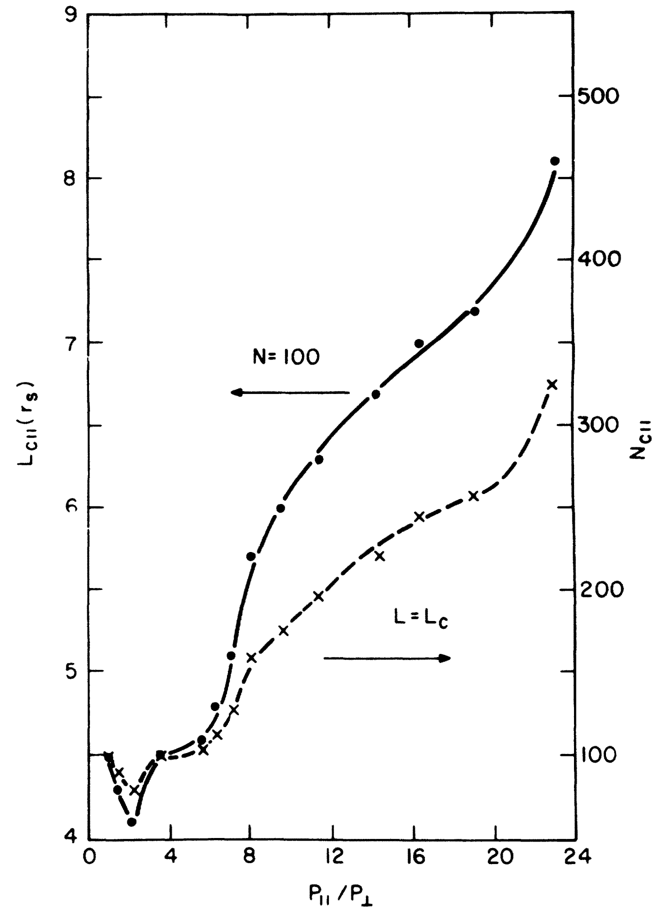

FIG. 16. Results of Fig. 10 for $L_{c\parallel}$ vs $P_\parallel/P_\perp$, obtained for a sample of 100 sticks (solid curve). Same results are presented as an $N_{c\parallel}$-vs-$P_\parallel/P_\perp$ dependence for a stick ensemble in which all the sticks have the same fixed length, $L_c=4.5r_s$ (dashed curve).

agreement between the analytic results and the computed results further show that, as expected, the FOD and the FLD affect the functions $f_\parallel$ and $f_\perp$ independently when the θ_i's and the l_i's are determined independently.

So far we have presented the results of the percolation thresholds in terms of the critical stick length required for the onset of percolation in the two-dimensional anisotropic samples. We have pointed out in Sec. III that for studied composites it is the fiber lengths[1,2] or their distribution[3,7] which are usually given, while the experimentally variable parameter is the critical fiber concentration N_c or its equivalents (e.g., vol % or wt. %). All our results can be expressed in terms of N and $N_c(P_\parallel/P_\perp)$ by using relations (14) and (15) and the above-computed $f_\parallel(P_\parallel/P_\perp)$ and $f_\perp(P_\parallel/P_\perp)$ values. In the cases where the sticks are not all of the same length, $N_{c\parallel}$ and $N_{c\perp}$ are the critical stick concentrations associated with the mean length L of the FLD. Hence we can present all the results shown above by the dependence of $N_{c\parallel}$ and $N_{c\perp}$ on $P_\parallel/P_\perp$. For an illustration of such a presentation of the data we use the results shown (by the solid curve) in Fig. 10 for the longitudinal percolation in a sample of $N=100$ randomly oriented sticks. The results are reproduced (with an extended $P_\parallel/P_\perp$ scale) by the solid curve in Fig. 16. For the $N_{c\parallel}$ representation we have chosen the fixed stick length L to be $L_c=4.5r_s$ (which is the value borne out by the isotropic case when $N=100$) and have computed $N_{c\parallel}$ by using the corresponding $f_\parallel$ results and Eq. (14). The results obtained are shown by the dashed curve in Fig. 16. This curve indicates by how much one must increase the stick concentration (in comparison with the isotropic $N=100$ case) in order to obtain percolation under a given degree of anisotropy. As is expected from the percolation criteria [Eqs. (14) and (15)], the dependence of $N_{c\parallel}$ on the anisotropy is stronger than that of $L_{c\parallel}$. The local decrease of both $L_{c\parallel}$ and $N_{c\parallel}$ around $P_\parallel/P_\perp=2$ as well as other "wiggles" in the curves are due, as explained above, to the dependence of the results on the particular seed used in this rather small sample.

V. DISCUSSION

In the present study we have found the dependence of the percolation threshold on the anisotropy of a two-dimensional random system. A quantitative relation has been derived for this dependence [Eq. (12)] and computer simulations have indicated that it is applicable to all kinds of FOD's and FLD's as long as the two distributions are not correlated. We note that although intuitively expected, previous experimental studies[11,12] (on other two-dimensional random anisotropic systems) could not even establish qualitatively a dependence of the threshold on the macroscopic anisotropy. These expectations, which are confirmed in the present study, are that the percolation threshold ($L_{c\parallel}$ and $L_{c\perp}$ or $N_{c\parallel}$ and $N_{c\perp}$) increases with increasing macroscopic anisotropy, $P_{\|}/P_{\perp}$, and that the threshold is isotropic in the infinite ensemble. As was shown in Sec. II for the random stick system and as will be shown for lattices in the Appendix, these two observations are quite general.

The present results have shown that the details of the FOD are unimportant and that the threshold is determined by the macroscopic (or overall) anisotropy. The agreement of this finding with the expectation from universality (that the onset of percolation is determined only by the overall connectivity of the sample) indicates that the larger ($N=1000$) samples used in this study are statistically reliable and that the definition used for the macroscopic anisotropy [Eqs. (2) and (13)] is a good one. The effects of the FOD and the FLD on the percolation threshold were found to be independent.

The above conclusion regarding an isotropic percolation threshold was based on our finding that increasing the stick ensembles from $N=100$ to 1000 has yielded a decrease in the difference between the longitudinal and the transverse percolation thresholds. In fact, for our larger samples we have shown that for $P_{\|}/P_{\perp}\leq 4$ (which is already a considerable anisotropy) the thresholds are practically the same. Since rescaling the length of the samples does not change the connectivity (though it changes the conductance), we expect such an isotropic behavior for the infinite ensemble. We can also turn the argument and say that as long as $L_{c\|}\approx L_{c\perp}$, the results correspond to those of an effectively infinite sample. A deviation from this behavior indicates that the sample is too small for the deduction of exact asymptotic values. On the other hand, even in the smaller samples we can estimate the threshold since the computed $L_{c\|}$ and $L_{c\perp}$ values can serve as bounds for the exact values. Our conclusion is in accord with the expected behavior of the coherence length ξ. This length (in intersite unit length $2/\sqrt{\pi N}$) must be smaller than the sample size in order for the latter to be effectively infinite.[13] In fact, one approaches the effectively infinite sample as $N^{1/2\nu}$. This is borne out by the argument[13] that the above threshold region for which the infinite-ensemble results are not represented by the results of the finite ensemble shrinks as $N^{-1/2\nu}$, where ν is the correlation-length exponent. The fact that by just increasing $P_{\|}/P_{\perp}$ we decrease the density of the sticks (Sec. II), as well as the deviation from the $L_{c\|}=L_{c\perp}$ condition, is added support for the suggestion[11] (and the corresponding calculations[13]) that the extent of the asymptotic region is reduced with increasing anisotropy.

The principal result of the present work, i.e., the increase of the percolation threshold with the increase of the macroscopic orientation (and hence the anisotropy) of the stick ensemble, may have been expected by viewing the increase of the threshold with $P_{\|}/P_{\perp}$ as an increase which is due to a continuous lowering of the dimensionality of the system [the all-parallel stick sample (Fig. 2) is essentially a one-dimensional system]. Using this approach, we can use the present results to predict the percolation criterion for two- and three-dimensional stick systems.

We have seen above [Eqs. (14) and (15)] that the percolation criterion of the effectively infinite two-dimensional sticks system can be written as

$$N_c L^2 = f \,, \tag{16}$$

where f is a function of the anisotropy. As pointed out in Sec. I, if the sticks in the two-dimensional system have a width D, the nature of the problem does not change and f is a weak function of D. This conclusion breaks down when we approach the one-dimensional-like case of parallel sticks. In this system for a finite L and $D=0$, there is no percolation. On the other hand, we do know that in two-dimensional systems[8–11] there is a critical area above which the system percolates. Hence for a finite D the criterion for the all-parallel stick system will be

$$N_c L D = f \,. \tag{17}$$

Following this argument and knowing that a three-dimensional stick system can percolate only for sticks with a finite diameter, we conclude that for an all-parallel stick system percolation will be obtained for a critical volume fraction of conducting material. Hence the corresponding percolation criterion will be

$$N_c L D^2 = f \,. \tag{18}$$

From Eqs. (9)–(11) one expects then that in the general (nonparallel) three-dimensional system of sticks the percolation criterion will be

$$N_c L^2 D = f \,, \tag{19}$$

where again f is a function of the macroscopic anisotropy. This result can be derived more rigorously using Onsager's method of excluded volumes.[14,15]

The percolation criteria given by Eqs. (16) and (19) indicate how to prepare a system with a low percolation threshold from a given amount of "stick material." For example, in two dimensions one can cut a fiber LN long into $2N$ sticks each having a length of $L/2$ or into $N/2$ sticks each having a length of $2L$. There may be an N and L such that the second system will be above percolation threshold while the first system will be below the threshold. This argument, however, cannot be carried too far since the sample loses its effectively infinite nature at a rate of $N^{-1/2\nu}$ (see above). The criterion given by Eq. (16) explains also the dominant role of the very long sticks in a given FLD for the percolation onset.

We have seen above that the percolation threshold depends on the anisotropy of the conducting stick system. Let us examine now the effect of this dependence on the conductivity of the system. This examination is done in order to evaluate the importance of this dependence in comparison with other factors[11,13] which determine the conductance dependence on anisotropy, $G(P_{\|}/P_{\perp})$. The conclusions of this evaluation can explain the qualitative features of $G(P_{\|}/P_{\perp})$ which were found experimentally in a three-dimensional composite.[7,16]

We start by assuming a critical behavior of the following type well known to exist in lattices[4]:

$$G \propto (p-p_c)^t \,. \tag{20}$$

Here p is the probability for finding a bond or a site, p_c is the critical value of p, and t is the conductivity critical exponent. Let us assume that t is a constant independent of anisotropy (which is expected[13] to be the case when p is very close to p_c). The result (20) can be generalized to lattices composed of conducting spheres since the volume of the percolating component is known[17] to be proportional to p. Hence

$$G \propto (V - V_c)^t \,, \tag{21}$$

where V is the three-dimensional (or two-dimensional) conducting volume (area) fraction and V_c is its critical value. Equation (21) can also be expressed by the weight fraction of the conducting material,[7] ω, i.e.,

$$G \propto (\omega - \omega_c)^t \,. \tag{22}$$

Indeed, such a dependence, with the expected universal t, has been found experimentally[3] for a three-dimensional system of conducting fibers (see below). Considering the predicted universal values for isotropic systems[4] ($t = 1.3$ for two dimensions and 1.7 for three dimensions) and the L_c dependence on $P_{\parallel}/P_{\perp}$ (for example, by use of the dashed curves of Figs. 11 and 15), one concludes that both the longitudinal resistivity $\rho_{\parallel}$ and the transverse resistivity $\rho_{\perp}$ increase with anisotropy. The prediction for an isotropic L_c in the two-dimensional infinite sample means that if t is independent of $P_{\parallel}/P_{\perp}$ and if one retains dependences of the form of Eqs. (20) and (21), the macroscopic anisotropy will not yield an anisotropic resistivity, i.e., $\rho = \rho_{\parallel} = \rho_{\perp}$. Hence, the observation of anisotropy in the resistivity of some systems, as well as the fact that the conductivity at the threshold is expected to be isotropic[18,19] (see the Appendix), indicates that t becomes anisotropic upon deviation from the threshold. These conclusions are consistent with the results obtained[13] for an anisotropic two-dimensional lattice.

The above conclusions may also explain experimental results[7,16] that were obtained on a three-dimensional composite for which an anisotropic resistivity has been obtained. These results have shown that while there is a strong dependence of the resistivity on the anisotropy of the system, the resistivity anisotropy ($\rho_{\parallel}/\rho_{\perp}$) is relatively small and its dependence on the system's anisotropy is relatively weak. It seems then, in view of the above discussion, that the strong dependence of the resistivity on the anisotropy is due primarily to the effect of macroscopic anisotropy on the percolation threshold. The anisotropy of the latter quantity as well as that of the critical exponent appear to depend rather weakly on the macroscopic anisotropy in this composite.

In conclusion, we have shown analytically and confirmed "experimentally" that the percolation threshold of a two-dimensional conducting stick system increases with the macroscopic anisotropy of the system and that the threshold of such a system is expected to be isotropic. The dependence of the resistivity of a two-dimensional percolating system on its anisotropy is expected to be associated with both the dependence of the threshold on the anisotropy and the dependence of the critical exponent on the anisotropy. Only the latter dependence is expected to yield an anisotropy in the resistivity.

ACKNOWLEDGMENTS

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APPENDIX

In Sec. IV we argued that for an anisotropic two-dimensional system one should expect an isotropic percolation threshold but an anisotropic conductance, which disappears as the threshold is approached. Here we show that in the case of a two-dimensional square lattice such a behavior can be obtained by applying simple renormalization-group arguments. From universality and the assumption that the random system can be divided into square blocks, one may expect such a behavior for random systems in general, and for the present stick system in particular.

Let us consider an anisotropic square lattice in which the nearest neighbors to a site may be connected to the site by resistors. The anisotropy of the system can be introduced by either assuming two different resistors in the two principal directions $1/g_x$ and $1/g_y$, or by assuming the same resistors but two different probabilities for the presence of the resistors p_x and p_y. The more general anisotropy can be obtained by the combination of the two anisotropies.

The first case has been considered previously.[13,19] If the probability for the resistors presence is the same in the two perpendicular directions, there will be the same probability of having a percolation path in the two directions and thus the same percolation threshold. The conductivity, on the other hand, will be anisotropic as can be appreciated by noticing that the fully occupied square lattice will have a conductance anisotropy which is exactly g_y/g_x. As one approaches the percolation threshold from above, the percolation path becomes tortuous and the conductance becomes isotropic.

The other case mentioned above is that in which all the resistors have the same resistance $1/g$ but the probabilities of occupation are different in the two directions.[10,20] The rescaling transformation of the occupation probabilities[21] are the same for the two directions (since they are obtained by a rotation of 90°) and thus the rescaled probabilities are

$$p_x' = R(p_x, p_y) \,, \quad p_y' = R(p_y, p_x) \,. \tag{A1}$$

The percolation threshold [the unstable fixed point (p_x^*, p_y^*)] is obtained by simultaneously solving Eqs. (A1) for $p_x' = p_x$ and $p_y' = p_y$. This yields symmetric equations for p_x^* and p_y^*. Hence, $p_x^* = p_y^*$ and the percolation threshold is isotropic.

Turning to the conductance of the system, let us examine the conductance distribution function $P_n(\sigma)$. This function gives the probability of finding a conductance σ between two nearest-neighbor sites in the lattice (which is obtained after the nth rescaling transformation). This function has the form[21]

$$P_n(\sigma) = \sum_i f_i(p_x, p_y)\delta(\sigma - \sigma_i) \,, \tag{A2}$$

where $\sigma_i=\sigma_i(g)$ are the conductances obtained for the various paths between the two sites after the $n-1$ transformation. The function $f_i(p_x,p_y)$ is the probability of finding the conductance σ_i in the latter lattice. Since the conductance of the entire lattice is given by[21]

$$\langle\sigma\rangle_n=\int P_n(\sigma)\sigma\,d\sigma\ , \tag{A3}$$

it is clear that if $P_n(\sigma)$ is anisotropic then $\langle\sigma\rangle_n$ will be anisotropic. Now, because of the possible 90° rotation and the single value of the resistors, the values of σ_i are the same in both the x and y directions. If $p_x\neq p_y$, the distribution functions for the two directions $P_{xn}(\sigma)$ and $P_{yn}(\sigma)$ will be different. Hence the conductances $\langle\sigma_x\rangle_n$ and $\langle\sigma_y\rangle_n$ will be different. However, at the percolation threshold we have already seen that $p_x^*=p_y^*$ and thus $P_x(\sigma)=P_y(\sigma)$ at this threshold. This means that the conductance is isotropic.

The most general case (disregarding the essentially one-dimensional problems of p_x or $g_x=0$) can be deduced from the above results. Since the result $p_x^*=p_y^*$ is independent of the initial resistor values, it appears that close to the percolation threshold the general case can be reduced to the anisotropic resistors (g_x and g_y) case. Hence, in the most general case, one may expect an isotropic threshold and an anisotropic conductance which disappears at the threshold.

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