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Hall voltage and current distributions in an ideal two-dimensional system

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The redistribution of charge, which generates the Hall voltage, cannot occur only at the edges in a two-dimensional system but must occur in the bulk. Using a Hartree approximation we derive a self-consistent equation which describes the charge, current, and Hall voltage distributions in a two-dimensional electron gas with filled Landau levels. These distributions are weighted toward the sample edges with a decay length into the bulk which depends on sample size and magnetic field strength.

The observation by von Klitzing, Dorda, and Pepper[1] that the Hall resistance of a two-dimensional (2D) electron gas with filled Landau levels is simply related to the fine-structure constant and the speed of light has sparked considerable interest in this phenomenon. The interest centers on the fact that the Hall current I is given by

$$I = \bar{n} e^2 \tilde{V}/h \quad , \tag{1}$$

where $\bar{n}$ is to a very high accuracy an integer. Theoretically, this result has been shown to be valid whenever the Fermi energy lies in a region of localized states.[2,3] In addition, semiclassical arguments[4,5] have been used to show that no matter how complex the impurity potential and the resulting field distribution is, the voltage $\tilde{V}$ across the entire sample must satisfy Eq. (1). The general result must therefore be satisfied by any calculation of the electric field distribution for Hall experiments.

The only calculations of actual field distribution in two-dimensional Hall-type geometries are the classical calculations[6] using a local conductivity tensor. Here we show that in the case of filled Landau levels, the field distribution and, consequently, the current are not given by the local result in the simple case of an infinitely long conducting strip. The calculated current is nonuniform even in this idealized geometry.

We consider a 2D electron gas in the x-y plane with a magnetic field H in the $\hat{z}$ direction. We choose to work in the Landau gauge

$$\vec{A} = H(0,x,0) \tag{2}$$

and introduce a potential energy $V(x)$, which depends only on x and is, for the moment, unspecified. The eigenfunctions of the Schrödinger equation are then of the form $L_y^{-1/2}\exp(ik_y y)\psi_n(x-x_0)$, where

$$x_0 = \frac{\hbar c k_y}{eH} \quad . \tag{3}$$

L_y is the length of the gas in the y direction and $\psi_n(x)$ satisfies

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \tfrac{1}{2} m\omega_c^2 x^2 + V(x_0+x)\right]\psi_n(x) = \epsilon_n \psi_n(x) \quad . \tag{4}$$

Provided that $V(x)$ is slowly varying, i.e., $V''(x_0) << m\omega_c^2$, we can replace $V(x_0+x)$ by $V(x_0)+xV'(x_0)$ and obtain

$$\psi_n(x-x_0) = \phi_n(x-x_0') \quad , \tag{5}$$

where (ϕ_n) are harmonic-oscillator wave functions,

$$\epsilon_n = \hbar\omega_c(n+\tfrac{1}{2}) + V(x_0) + \frac{m}{2}\left[\frac{cV'(x_0)}{eH}\right]^2 \tag{6}$$

and

$$x_0' = x_0 - \frac{mc^2 V'(x_0)}{H^2 e^2} \quad . \tag{7}$$

The allowed values of k_y are fixed by adopting periodic boundary conditions in the y direction or, equivalently, by assuming the Corbino ring geometry:

$$k_y L_y = 2\pi p \quad , \tag{8}$$

where p is an integer. Defining $x_p = \delta x_0 p$, where

$$\delta x_0 = \frac{2\pi c\hbar}{L_y eH} = \frac{2\pi a_L^2}{L_y} \tag{9}$$

and $a_L = (\hbar c/eH)^{1/2}$ is the magnetic length, we find that the density of the current in the y direction is

$$j_y(x) = -\frac{e\omega_c}{L_y}\sum_{p,n}(x-x_p)\phi_n^2(x-x_p') \quad , \tag{10}$$

where $x_p' = x_p - V'(x_p)/m\omega_c^2$. The sum in Eq. (10) is over all occupied states and we shall assume that no Landau levels are partially occupied. {This requires, from Eq. (5), that $V << \hbar\omega_c$ and that $[V'(x_0)]^2/m\omega_c^2 << \hbar\omega_c$. These conditions are normally satisfied in the high-field limit.} Since $\delta x_0/a_L \sim a_L/L << 1$ we can convert the sum in Eq. (10) to an integral and obtain

$$j_y(x) = (\bar{n}e/h)V'(x) \ , \tag{11}$$

where we have again assumed that $V''(x_0) << m\omega_c^2$. Equation (11) essentially expresses the fact that the current density is such that the Lorentz force is locally canceled by the electrostatic force. On integrating over x we obtain Eq. (1).

The potential $V(x)$ must be generated by a redistribution of the electron density in the 2D gas, $\delta\sigma(x)$:

$$V(x) = -2e^2 \int_{-L_x/2}^{L_x/2} \ln|x-x'|\delta\sigma(x')\,dx' \ . \tag{12}$$

We see immediately that, unlike the 3D case where a linearly varying potential is produced by a surface charge at the sample edges, any finite linear charge density at the edge of the 2D gas would produce an infinite Hall voltage. We must therefore allow the density redistribution to occur in the bulk of the 2D gas. On the other hand, the density change required to produce a linearly varying potential,

$$\delta\sigma(x) \propto \frac{x}{[(L_x/2)^2 - x^2]^{1/2}} \tag{13}$$

cannot be self-consistently generated. To see this we evaluate the surface charge density of the electrons, in the same manner as we evaluated the current density in Eq. (10). The result is

$$\delta\sigma(x) = \frac{1}{L_y} \sum_{p,n} \phi_n^2(x - x_p') - \phi_n^2(x - x_p)$$
$$= \frac{\bar{n}V''(x)}{h\omega_c} \tag{14}$$

in the limit $\delta x_0 / a_L << 1$.

When Eq. (14) is inserted into Eq. (12) a homogeneous integrodifferential equation is obtained, which, when lengths are expressed in units of $L_x/2$, becomes[7]

$$V(x) = -\frac{4e^2\bar{n}}{h\omega_c L_x} \int_{-1}^{1} dx' \ln|x-x'|\,V''(x') \ . \tag{15}$$

We have used the charge-neutrality requirement that

$$\int_{-1}^{1} dx\,\delta\sigma(x) = \frac{\bar{n}}{h\omega_c}[V'(1) - V'(-1)] = 0 \ , \tag{16}$$

which restricts us to odd solutions of Eq. (15). Equivalently, Eq. (15) may be obtained by minimizing the Coulomb plus kinetic energies in the magnetic field

$$E = -\frac{4e^2}{L_x^2} \int_{-1}^{1} dx \int_{-1}^{1} dx' \,\delta\sigma(x) \ln|x-x'|\delta\sigma(x') + \frac{1}{\delta x_0 m \omega_c^2 L_x} \int_{-1}^{1} dx\,[V'(x)]^2 \ , \tag{17}$$

with $\delta\sigma(x)$ and $V(x)$ related by Eq. (14).

The solutions to Eq. (15) give us the profile of the Hall potential drop in 2D gas and via Eqs. (11) and (14) the current-density and charge-density distributions. The magnitude of the prefactor in Eq. (15) is typically much smaller than 1 in the high-field limit and decreases $\propto H^{-2}$ as the magnetic field is increased. (For example, if $L_x = 5\times10^5$ a.u., $h\omega_c = 4\times10^{-3}$ Ry, and $\bar{n}=1$, the prefactor is 10^{-3}). If, as a rough approximation, we replace $\ln|x-x'|$ by $-\delta(x-x')$, then we see that $V(x)$ and $V''(x)$ are proportional to each other and the Hall potential drop is confined to a distance $W = (L_x e^2 \bar{n}/h\omega_c)^{1/2}$ from the sample edges.

We have not succeeded in solving Eq. (15) analytically but approximate solutions may be obtained numerically by discretizing the integral, using finite difference approximations for $V''(x)$, and assigning definite values for $V(x)$ at the end points of the interval. The resulting matrix equations may be solved by standard methods. [We actually use $V''(x) = -V''(-x)$ to reduce the integral to the interval from 0 to 1 and set $V(0)=0$.] Equation (15) may be viewed as an eigenvalue equation and our success in finding approximate solutions indicates that the eigenvalue spectrum is continuous. Correspondingly, our solutions, which may be regarded as eigenfunctions of a linear operator, may be multiplied by an arbitrary constant. Therefore a change in the current through the sample does not change the shape of the distribution. The Hall voltage obtained at $W/L_x = 1/\sqrt{40}$ is illustrated in Fig. 1. Note that, unlike the 3D case, the electric field produced by the charge redistribution is nonzero outside the metal. The corresponding current and charge distributions are illustrated in Fig. 2 and we see that these are localized somewhat more strongly at the edge compared with the potential drop.

We have solved Eq. (15) numerically for prefactors $(W/L_x)^2$ varying in magnitude from $\sqrt{0.1}$ to $\sqrt{0.00001}$. The characterization of the solution by the length W mentioned above, while certainly qualitative, describes reasonably well the dependence on its prefactor. For example, when the prefactor decreases by an order of magnitude from

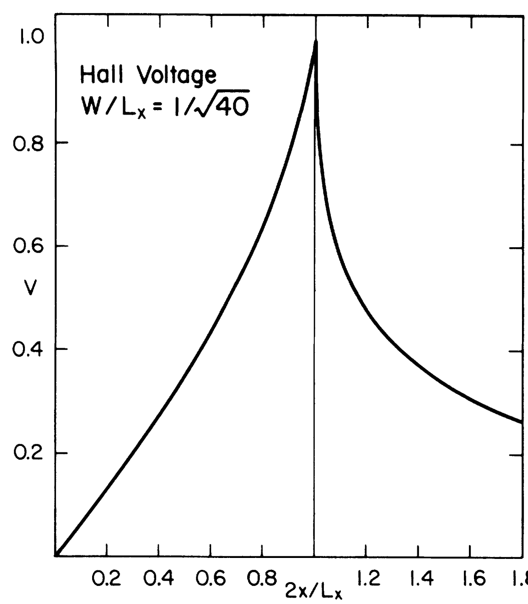

FIG. 1. Hall potential for the current-carrying state of a 2D electron gas in a magnetic field. These results were obtained by dividing the interval from 0 to 1 into 100 equal-length subintervals, replacing $V''(x')$ by a finite-difference approximation in each subinterval and doing the logarithmic integral in each subinterval analytically. The points just outside the interval near 0 and 1 were set to the values $V(x)=0$ and 1, respectively. The voltage is in units of $(h/2\bar{n}e^2)I$, half the quantized value from Eq. (1).

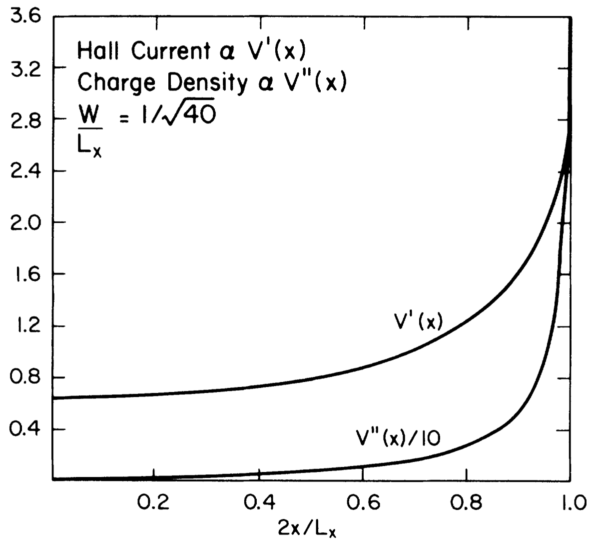

FIG. 2. As in Fig. 1 but for the Hall current [see Eq. (11)] and the electron density change [see Eq. (14)].

$\sqrt{0.1}$ to $\sqrt{0.001}$, W/L_x is reduced by a factor of $\sqrt{10}=3.16$. For the same prefactor the length measured from the edge at which $V(x)=0.5V(L_x/2)$ changes from $0.422L_x$ to $0.208L_x$, i.e., a factor of 2.0. For $V(x)=0.6V(L_x/2)$ the corresponding change is from 0.327 to 0.137, a factor of 2.4. Generally, we find that measures of the potential drop length as a fraction of L_x decrease toward zero somewhat more slowly than W/L_x, at least in the range of this parameter which we have surveyed. Another important feature of our numerical results is that $V''(x)$ seems to diverge near the edge with an exponent $\approx -\frac{1}{2}$. This feature is expected from Eq. (13) and because the integral equation clearly allows no solution which is analytic at the edges.

In conclusion, we have shown that the current-carrying state of an ideal 2D interacting electron gas in a magnetic field, treated in the Hartree approximation, is substantially different from that of a noninteracting gas where the Hall potential must be supplied by an external electric field. At this stage we have made no attempt to include edge effects,[8] but do not believe these to be important since even in the high-field limit W should be much larger than a_L. The result of this investigation emphasizes the importance of electron-electron interactions in theories of the quantized Hall effect.

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