Solutions to the Schrödinger equation on some fractal lattices

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The Schrödinger equation is solved on a variety of fractal lattices using a recursive technique. In this method, the energy levels and wave functions on a lattice with N_n sites is calculated in terms of the corresponding quantities on a smaller lattice with N_{n-1} sites, via a kind of decimation process. As $n \to \infty$ the resulting energy levels are discrete, very closely spaced, and highly degenerate. Smoothed densities of states have a wide variety of singularities.

I. INTRODUCTION

A. Motivation

Recently, statistical physicists have begun[1-3] a systematic study of critical phenomena and statistical mechanics on a set of lattices of noninteger fractal dimensionality.[4] In general, these lattices have self-similar properties but do not obey full translational invariance in any ordinary (integral dimensional) space. One of the many reasons for the interest in these lattices is that they serve as models for the backbone of percolating systems.

In this paper we follow Ref. 3 in the consideration of solutions to the Schrödinger equation on such lattices. We are particularly interested in seeing how the density of states behaves in such self-similar but nontranslationally invariant systems. We hope that our results may be useful for gaining some understanding of localization problems on more random lattices.

Since localization problems are rather intricate and have a structure which is still not fully elucidated, we shall seek out a class of particularly simple problems which can be exactly solved. These all involve special situations in which the lattice has a finite ramification number, i.e., the lattice will fall apart if a finite number of sites are removed. We should not be surprised if such situations produce rather localized wave functions. However, we did find one rather surprising result: The density of states for the problems considered here are dominated by very highly degenerate states.[5]

An additional surprising result is the form of the density of states. The energy levels are discrete and degenerate. However, as the number of sites goes to infinity, the density of states per site $\rho(E)$ approaches a limit with an infinite number of different kinds of singularities (all with different critical indices) and—in two of our cases—an infinite number of band gaps.

B. Two problem types

Given a fractal lattice, we develop two separate and different kinds of Schrödinger equations. Look at a typical lattice site numbered 0 connected to other sites labeled $1,2,3,\ldots,m$ by bonds. In one kind of Schrödinger equation, we consider hopping along the links to the sites in question. The hopping amplitude is given by t_j $(j=1,2,\ldots,m)$ and the Schrödinger equation at the site has the typical tight-binding form

$$E\psi_0 = -\sum_j t_j \psi_j \ . \tag{1.1}$$

Typically, we then set all (or a subset of) the t_j's equal. The alternative formulation is due to Alexander.[3] Set a coordinate x_j on each wire which runs from $x_j=0$ (the joining point) to $x_j=a$ (the neighboring site). Along each wire there is a wave function $\psi_j(x)$, with $\psi_j(0)=\psi_0$ and $\psi_j(a)=\psi_j$. If

$$\frac{\partial^2}{\partial x^2}\psi_j(x) = -2mE\psi_j(x) \ ,$$

then we have a wave function of the form $\psi_j(x)=\psi_{+j}e^{qx}+\psi_{-j}e^{-qx}$ with $q=\sqrt{2mE}$. Of course, ψ_{+j} and ψ_{-j} are determined by ψ_j and ψ_0. The current conservation condition

$$\frac{\partial}{\partial x}\sum_j \psi_j(x) = 0$$

then translates into the statement

$$\sum_{j=1}^{m}(\psi_0 \cos qa - \psi_j) = 0 \ , \tag{1.2a}$$

$$q = \sqrt{2mE} \ . \tag{1.2b}$$

Equations of the form (1.2a) also arise when one tries to describe the result of "decorating" on a tight-binding model Hamiltonian, such as that in Eq. (1.1).

It is tempting to say that formulations (1.1) and (1.2) are essentially equivalent. Indeed if all the t_j in Eq. (1.1) are equal to other, one can relate (1.1) and (1.2) by writing

$$m \cos qa = -E/t \ . \tag{1.3}$$

The right-hand side of Eq. (1.3) describes the tight-binding problem; the left-hand side, the Alexander "wire" formulation.[3]

The equivalence is indeed perfect if the couplings t_j are

all equal and if m, the number of bonds per site, is the same for all sites. In two of our model problems these equivalence conditions will be satisfied; in another they will not.

C. Recursive method

In each of our problems we consider a succession of lattices defined by an integer n. The level n lattice has N_n sites and hence N_n energies $E_{n,\alpha}$. As n increases, N_n increases geometrically. We use a decimation method to eliminate sites from the level-n lattice and thereby generate the level-$(n-1)$ lattice.

At each level, and for each energy we shall characterize the energy by a dimensionless parameter $\epsilon_{n,\alpha}$. For the wire problem

$$\epsilon_{n,\alpha}=\cos(2mE_{n,\alpha}a^2)^{1/2}\,, \tag{1.4}$$

while in the tight-binding problems

$$\epsilon_{n,\alpha}=E_{n,\alpha}/t\,. \tag{1.5}$$

In each case, our decimation will imply a relationship between energies on the level-n lattice and those on the level-$(n-1)$ lattice of the form

$$\epsilon_{n-1,\alpha}=b_0+b_1\epsilon_{n,\beta}+b_2\epsilon_{n,\beta}^2\,, \tag{1.6}$$

where the b's will depend only upon the lattice in question. Since the relationship is quadratic for each energy in the level-$(n-1)$ system, there are two in the level-n system. (In addition there will be some special energies not covered by the recursion relation—this will be discussed later.) Thus by knowing the energies in the level-$(n-1)$ case, we know them on the larger lattice. In this way, it becomes easy to calculate all the energies.

Determination of the spectrum is now quite simple, owing to the fact that recursion relations of the form (1.6) have been subjected to very considerable analysis.[6,7] By making a linear change of variables $\epsilon_{n,\beta}=cx+d$, $\epsilon_{n-1,\alpha}=cx'+d$, one can convert the recursion relation (1.6) into the standard form

$$x'=rx(1-x) \tag{1.7}$$

with

$$r=1+[(b_1-1)^2-4b_0b_2]^{1/2}\,. \tag{1.8}$$

The three cases we shall study correspond to the standard form (1.7) with $r=4$, 5, and 6. Much of our knowledge of the final answers will emerge from previous work on the recursion relation (1.7).

II. DERIVATIONS OF SINGLE-PARAMETER RECURSION RELATIONS

Consider a set of nearest-neighbor hopping Hamiltonians of the form

$$\mathscr{H}=\sum_i u_i\,|\,i\rangle\langle i\,|-\sum_{\langle ij\rangle}(t_{ij}\,|\,i\rangle\langle j\,|+t_{ji}\,|\,j\rangle\langle i\,|)\,. \tag{2.1}$$

Here we imagine that we are working on some particular lattice, e.g., the Sierpinski gasket of Fig. 1. This Hamiltonian operates in a Hilbert space spanned by local site functions $|\,i\rangle$; by $\langle ij\rangle$ we denote a pair of nearest-neighbor sites. The eigenvalue equation has the form

$$H\,|\,\psi\rangle=E\,|\,\psi\rangle \tag{2.2a}$$

in the tight-binding model, and

$$H\,|\,\psi\rangle=0 \tag{2.2b}$$

in the wire model. We put these together by writing

$$H\,|\,\psi\rangle=\mathscr{E}\,|\,\psi\rangle \tag{2.2c}$$

with $\mathscr{E}$ being E in the tight-binding case and 0 in the wire model. We now divide our lattice (and our Hilbert space) into two sublattices (subspaces), denoted by 1 and 2. Denoting the projection of $|\,\psi\rangle$ onto these as $|\,\psi_1\rangle$ and $|\,\psi_2\rangle$, the eigenvalue equation (2.2c) can be written as

$$\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}|\,\psi_1\rangle\\ |\,\psi_2\rangle\end{bmatrix}=\mathscr{E}\begin{bmatrix}|\,\psi_1\rangle\\ |\,\psi_2\rangle\end{bmatrix}. \tag{2.3}$$

If now, using the second line of Eq. (2.3), we express

$$|\,\psi_2\rangle=(\mathscr{E}-H_{22})^{-1}H_{21}\,|\,\psi_1\rangle \tag{2.4a}$$

and substitute into the first line, we get

$$\begin{aligned}\mathscr{H}_{\text{eff}}\,|\,\psi_1\rangle&=[H_{11}+H_{12}(\mathscr{E}-H_{22})^{-1}H_{21}]\,|\,\psi_1\rangle\\&=\mathscr{E}\,|\,\psi_1\rangle\,.\end{aligned} \tag{2.4b}$$

Thus, instead of solving an eigenvalue equation that involves all sites of our lattice, it is sufficient to study a problem with only a subset of the original sites. It should be noted, however, that in general, the problem, as defined in the subspace 1, is more complicated (i.e., it may involve longer-range hopping) than the one defined on the original lattice. Here we will concentrate on cases where the new eigenvalue equation (2.4b) can be brought into a similar form as the original Hamiltonian (2.1). In such cases various properties of interest, such as the spectrum of eigenvalues, the eigenfunctions, and Green's functions of a (practically) infinite lattice can be calculated on the basis of very simple, single-parameter nonlinear recursion relations. We now turn to derive recursive solutions for some lattices.

A. Linear chain

The simplest such example is the linear chain. In the tight-binding model, we set $u_i=0$, $t_{i,i+1}=t_{i-1,i}=t$, and $\mathscr{E}=E$ to find

$$\mathscr{H}=-t\sum_i(\,|\,i\rangle\langle i+1\,|+|\,i+1\rangle\langle i\,|\,) \tag{2.5a}$$

and the eigenvalue equation

$$E\psi_i=-(\psi_{i+1}+\psi_{i-1})t\,, \tag{2.5b}$$

while in the wire model we set $\epsilon=0$, $u_i=2\cos qa$, and have instead of Eq. (2.5b) the relation [see Eq. (1.2a)]

$$(2\cos qa)\psi_i=\psi_{i+1}+\psi_{i-1}\,. \tag{2.6}$$

Of course, Eqs. (2.5b) and (2.6) are really identical. The mapping between the two is given by Eq. (1.3).

Use the tight-binding picture to analyze this situation. Work on a lattice of $N_n=2^{n+1}$ sites with periodic boundary conditions $\psi_0=\psi_{N_n}$. Then define the set 1 to include

all even-numbered lattice sites $i=2J$ and 2 to be the odd-numbered sites. From Eq. (2.5b) we find

$$\psi_{2j+1}=-\frac{t}{E}(\psi_{2j}+\psi_{2j+2})$$

so that we can write a Schrödinger equation for the even-numbered sites alone[8] as

$$\left[E-\frac{2t^2}{E}\right]\psi_{2j}=\frac{t^2}{E}(\psi_{2j+2}+\psi_{2j-2})\ . \tag{2.7}$$

Equation (2.7) is exactly of the same form as (2.5b), except that the parameters have been changed. In place of t there appears

$$t'=-t^2/E \tag{2.8a}$$

and in place of E, we have

$$E'=E-2t^2/E\ . \tag{2.8b}$$

One dimensionless parameter

$$\epsilon\equiv\epsilon_{n,\alpha}=E_{n,\alpha}/t\equiv E/t \tag{2.9}$$

describes the solution in the lattice of size $N_n=2^{n+1}$. On the decimated lattice, the corresponding parameter is $\epsilon_{n-1,\beta}=E'/t'\equiv\epsilon'$. From Eqs. (2.8), we find

$$\epsilon'=2-\epsilon^2 \tag{2.10a}$$

or

$$\epsilon_{n-1,\beta}=2-\epsilon_{n,\alpha}^2\ . \tag{2.10b}$$

This recursion relation may be interpreted in the following ways. Given an eigenstate

$$|\psi\rangle=\sum_i \psi_i^n\,|i\rangle$$

of the system of size $N_n=2^{n+1}$ with $\epsilon=\epsilon_{n,\alpha}$, there must be a corresponding solution

$$|\psi\rangle'=\sum_i \psi_{2i}^n\,|i\rangle$$

with parameter ϵ' on the lattice with N_{n-1} sites. Alternatively, one can work backward. For every solution

$$|\psi\rangle'=\sum_{j=1} \psi_j^{n-1}\,|j\rangle$$

on the lattice of N_{n-1} sites, there exist two corresponding eigenstates

$$|\psi,\pm\rangle=\sum_{i=1}^{N_n}\psi_i^{n,\pm}\,|i\rangle \tag{2.11a}$$

with energy parameters

$$\epsilon_{n,\pm}=\pm(2-\epsilon_{n-1,\beta})^{1/2} \tag{2.11b}$$

and wave functions

$$\psi_{2j}^{n,\pm}=\psi_j^{n-1}\ , \tag{2.11c}$$

$$\psi_{2j+1}^{n,\pm}=-\frac{\psi_j^{n-1}+\psi_{j-1}^{n-1}}{\epsilon_{n,\pm}}\ .$$

Note that we have assumed $\epsilon_{n,\pm}\neq 0$. Whenever this condition holds, i.e., whenever $\epsilon_{n-1,\beta}\neq 2$, there exist precisely two energy eigenstates in the level-n system corresponding to the given eigenstate in the level-$(n-1)$ system. When this condition fails, a special analysis will be necessary (see Sec. III).

The recursion relation (2.10b) is most easily understood in terms of the wire model. This linear chain is one wire and the N_n sites i are just equally spaced observation stations along the wire. From Eq. (2.6), $\epsilon=-2\cos qa$ so that the recursion relation (2.10b) is $2\cos q'a=-2+4\cos^2 qa$ or $q'=2q$. Thus by doubling the lattice constant, we have—as expected—doubled the observed wave vector. Finally notice that by making the change of variables $\epsilon=4x-2$, we achieve a recursion relation of the form (1.7) with $r=4$.

B. Triangular Sierpinski gasket

We consider a sequence of fractal lattices, as shown in Fig. 1. The boundary conditions used identify the corners of two triangles on the largest scale; the lattice is generated by inserting sites (that form a downward-pointing triangle) onto each upward-pointing triangle. Stopping after any finite number n of such insertions, we consider the tight-binding eigenvalue equation associated with the Hamiltonian (2.1) for the special case in which $u_i=0$ and all the t_{ij} are equal to t. Once again we use the abbreviation $\epsilon=E/t$ for the dimensionless energy parameter. Because each site has four neighbors, ϵ is bounded in the region

$$-4<\epsilon<4\ . \tag{2.12}$$

To obtain the recursion equations, we employ the general method of Eq. (2.4). The set of one sites includes all those that were present prior to inserting the nth generation; the set 2 consists of these sites of the smallest scale.

In order to obtain the analog of Eq. (2.4a), i.e., express ψ_2 in terms of ψ_1, one needs to consider the 6×6 matrix associated with nearest-neighbor hopping on the basic triangle of Fig. 1(b). We have $(i,j=1,2,3)$ $(\underline{H}_{11})_{ij}=0$, $(\underline{H}_{12})_{ij}=(\underline{H}_{22})_{ij}=-1+\delta_{ij}$: hence, using (2.4b), we get

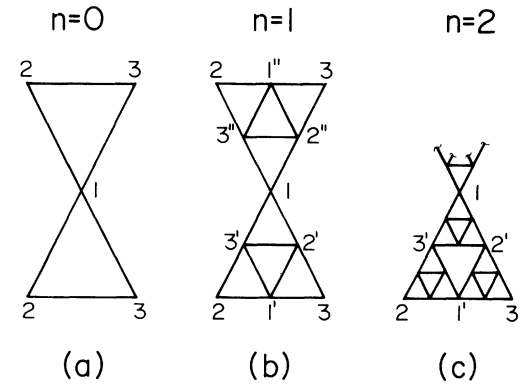

FIG. 1. Construction of a Sierpinski gasket with (a) three sites on the largest scale ($n=0$). (b) By inserting three sites into each of the triangles in (a), the $n=1$ gasket is obtained. (c) Continuing the insertion procedure yields the $n=2$ gasket, from which $n=3,4,\ldots$ can be constructed. The lower (upward-pointing) triangle of (b) is a prototype of the "basic structure" from which sites 1′,2′,3′ can be eliminated by decimation.

for a *single* (upward-pointing) triangle of one sites the Hamiltonian

$$H_\Delta=[\underline{H}_{11}+\underline{H}_{12}(E\underline{1}-\underline{H}_{22})^{-1}\underline{H}_{21}]_{ij}$$
$$=t\frac{\epsilon-2+(\epsilon+2)\delta_{ij}}{(\epsilon+2)(\epsilon-1)} . \tag{2.13}$$

In order to construct the Hamiltonian for the entire (decimated) lattice of one sites, note that each one site appears on the corner of *two* basic triangles; therefore the diagonal element appearing in the full $H_{\rm eff}$ is *twice* that of H_Δ. Therefore, for the level-$(n-1)$ system the effective Hamiltonian will be of the form (2.1) with on-site energy

$$u_i=u'=\frac{4t\epsilon}{(\epsilon+2)(\epsilon-1)} \tag{2.14a}$$

and hopping interaction

$$t_{ij}=t'=t\frac{2-\epsilon}{(\epsilon+2)(\epsilon-1)} . \tag{2.14b}$$

The level-n system had $u=0$. To force the level-$(n-1)$ system into this same form we notice that u' only appears in the combination $E-u'$. Hence if we replace E by $E-u'$, and u' by 0 we have changed nothing. The level-$(n-1)$ system has its properties determined by an energy parameter

$$\epsilon'=\frac{E-u'}{t'} . \tag{2.14c}$$

By combining Eqs. (2.14) we obtain a recursion relation for the energy parameter

$$\epsilon'=-\epsilon(\epsilon+3) . \tag{2.15}$$

Note that if we make the change of variables $\epsilon=5x-4$, Eq. (2.15) converts into the form (1.7) with $r=5$.

Thus given an energy eigenstate of the level-$(n-1)$ system with energy parameter $\epsilon_{n-1,\alpha}$ we can predict that there will be two corresponding eigenstates of the level-n system with parameters

$$\epsilon_{n,\pm}=\frac{-3\pm(9-4\epsilon_{n-1,\alpha})^{1/2}}{2} . \tag{2.16}$$

In regard to the eigenfunctions, Eq. (2.4a) in the present case will give the amplitudes ψ_i' on the three internal sites of an upward-pointing triangle as a function of the amplitudes ψ_j on the corner sites, through the following relation [see Fig. 1(b) for the appropriate indexing of sites]:

$$\psi_i'=\frac{1}{(\epsilon+2)(\epsilon-1)}\sum_{j=1}^{3}[\epsilon-(\epsilon+2)\delta_{ij}]\psi_j . \tag{2.17}$$

The derivation, and thus Eq. (2.16), holds only for $\epsilon\neq-2, +1, +2$. For the first two of these special energies the matrix $(\epsilon\underline{1}-\underline{H}_{22})$ is singular; for $\epsilon=+2$ the effective Hamiltonian has vanishing hopping amplitude ($t'=0$), and thus these cases have to be considered separately.

C. Three-dimensional Sierpinski gasket

This object is obtained by the construction indicated by Fig. 2. Recursion relations are obtained in a manner similar to Sec. II B, with a few differences. The basic object (tetrahedron) that has to be considered now has 10 sites, of which the four (corner) sites belong to set 1, while the six internal ones belong to set 2 (see Fig. 2).

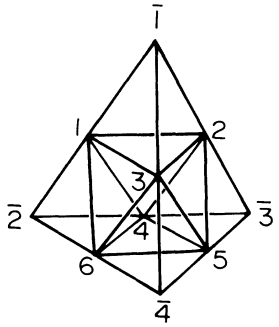

FIG. 2. Basic unit of a Sierpinski gasket embedded in three-dimensional space. The next generation of sites is obtained by inserting six sites into each of the four tetrahedra (such as 3456) in the same manner as sites 1,2,. . .,6 were inserted into the tetrahedron $\bar{1}\bar{2}\bar{3}\bar{4}$.

We now construct a single-tetrahedron Hamiltonian as was done in Eq. (2.13); note that each site from one is shared by two basic tetrahedra, and by the same manipulation as used before we get

$$\epsilon'=-(\epsilon^2+6\epsilon+6) . \tag{2.18}$$

This recursion relation is equivalent to

$$x'=rx(1-x), \quad r=6 . \tag{2.19}$$

As before, Eq. (2.18) does not hold for some special energies, which are $\epsilon=0, -4, +2$.

D. Berker lattice: Tight-binding version

This lattice is generated in the manner indicated in Fig. 3; to insert a new level or generation of sites, each bond of the lattice is replaced by two bonds with a site centered on each. The Migdal-Kadanoff recursion relations provide an exact solution to spin models defined on this lattice.[2]

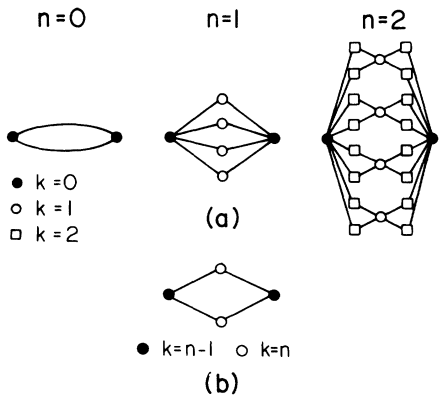

FIG. 3. (a) Berker lattice: The level-n lattice is obtained by replacing each bond of the level-$(n-1)$ lattice by two bonds, with one site of the nth generation on each. For lattices with $n=0,1,2$ the sites that belong to generations $k=0,1,2$ are indicated. (b) The basic structure from which the nth-generation sites (open circles) can be eliminated by decimation.

Here we consider solutions of the nearest-neighbor hopping tight-binding Hamiltonian. Again we consider a lattice obtained by inserting n generations. It should be noted that now the number of bonds incident on a given site depends on the generation to which the site belongs. For this reason, in order to derive simple recursive formulas, we must assume that either the site energies or the hopping depend on the location of the respective site on the lattice. For this lattice a number of different problems can be defined and considered.

Variable hopping. Consider a lattice of n generations of sites,

$$H=-\sum_{k,\alpha,\alpha'} t_k^{(n)}(|k,\alpha\rangle\langle n,\alpha'|+\text{H.c.}) , \tag{2.20}$$

where $|k,\alpha\rangle$ denotes a particular site that belongs to the kth generation; the nearest neighbors of any such site belong to the last, nth generation and are denoted $|n,\alpha'\rangle$. Site $|k,\alpha\rangle$ has 2^{n-k} units such as those shown in Fig. 3(b) attached to it; note also, that the next-nearest neighbors of $|k,\alpha>$ belong to the $(n-1)$th generation. In writing the Hamiltonian (2.20) we have assumed that the hopping element depends on the indices k,n only. By eliminating in each basic structure [Fig. 3(b)] the two n sites, we obtain

$$\begin{aligned} H_{k,n-1}=\frac{2}{E}[&(t_k^{(n)})^2|k,\alpha\rangle\langle k,\alpha| \\ &+(t_{n-1}^{(n)})^2|n-1,\alpha'\rangle\langle n-1,\alpha'| \\ &+t_k^{(n)}t_{n-1}^{(n)}(|k,\alpha\rangle\langle n-1,\alpha'|+\text{H.c.})] . \end{aligned} \tag{2.21}$$

When the Hamiltonian for the full decimated lattice is considered, we get for site k a diagonal element

$$u_k'=\frac{2}{E}(t_k^{(n)})^2 2^{n-k} . \tag{2.22}$$

In order to be able to write the new Hamiltonian in a form similar to (2.20), u_k' must be independent of k. Choosing

$$t_k^{(n)}=2^{-(n-k)/2}t , \tag{2.23}$$

we get a recursion relation for $\epsilon=E/t$ and $\epsilon'=(E-u')/t'$

$$\epsilon'=-\epsilon^2+2 , \tag{2.24}$$

which is identical to (2.10b), and therefore also corresponds to $r=4$.

E. Berker lattice: Wire version

The eigenvalue problem just considered can be written as the determination of eigenstates with eigenvalue 0 for the Hamiltonian

$$\begin{aligned} H_1=&\sum_{k,\alpha}\epsilon|k,\alpha\rangle\langle k,\alpha| \\ &+\sum_{k,\alpha,\alpha'}2^{-(n-k)/2}(|k,\alpha\rangle\langle n,\alpha'|+\text{H.c.}) . \end{aligned} \tag{2.25}$$

The problem is to adjust ϵ until the eigenvalue 0 becomes possible. If we multiply Eq. (2.25) on the left and right by operators X such that

$$X|k,\alpha\rangle=2^{(n-k)/2}|k,\alpha\rangle ,$$

we find that we have a new problem: the determination of the zero eigenstates of

$$\begin{aligned} H_2=&XH_1X \\ =&\sum_{k,\alpha}\epsilon 2^{n-k}|k,\alpha\rangle\langle k,\alpha| \\ &+\sum_{k,\alpha,\alpha'}(|k,\alpha\rangle\langle n,\alpha'|+\text{H.c.}) . \end{aligned} \tag{2.26}$$

Since a kth-generation site has $2\times 2^{n-k}$ nearest neighbors, one can rewrite the Hamiltonian H_2 in terms of a set of wave-function equations of the form (1.2)—i.e., the wire form—as

$$\begin{aligned} 0=&\sum_{n,\alpha'}\left[\psi_{k,\alpha}\frac{\epsilon}{2}-\psi_{n,\alpha'}\right] \quad \text{for } k<n \\ =&\sum_{k,\alpha}\left[\frac{\epsilon}{2}\psi_{n,\alpha'}-\psi_{k,\alpha}\right] , \end{aligned}$$

where in each case n,α' and k,α are nearest neighbors. Thus the recursion relations (2.24) apply equally well to the wire case with the identification

$$\epsilon=-2\cos qa . \tag{2.27}$$

With the identification (2.27), the recursion relation (2.24) can also be written in the very simple form

$$q'=2q . \tag{2.28}$$

III. ENERGY LEVELS AND EIGENSTATES

In this section we show how the recursion relations, derived in the preceding section, can be used in order to obtain the spectrum of eigenvalues and the eigenstates. All lattices considered will be built up by starting with a basic "structure" of a small number of sites, into which levels of new sites, each defined on a scale smaller than the previous one, are inserted. The eigenvalues of the system with n such insertions are related to those of a system with $n-1$ levels by relations (2.10), (2.16), etc., which can be written generically as

$$\epsilon_{n-1}=f(\epsilon_n) . \tag{3.1}$$

We shall not make use of Eq. (3.1) directly. Instead we go backwards. Since f is a quadratic function, there exist two inverse functions $g_\pm(\epsilon)$, such that

$$\epsilon=f(g_\pm(\epsilon)) . \tag{3.2}$$

Given an allowed parameter of the level-n system, $\epsilon_{n,\alpha}$, we derive two allowed values of the level-$(n+1)$ system $\epsilon_{n+1,\alpha,\pm}$ as

$$\epsilon_{n+1,\alpha,\pm}=g_\pm(\epsilon_{n,\alpha}) . \tag{3.3}$$

Equation (3.3) does not work for exceptional values of the energy parameter ϵ. Let us define S as the set of all ϵ_{n+1} which are not properly generated by the recursion relation (3.3).

Leaving these exceptional energies aside, it is very easy to calculate the degeneracies of the various eigenstates. Let $N_n(\epsilon')$ be the number of states in the level-n system

with energy parameter ϵ'. Let $\epsilon_\pm$ be $g_\pm(\epsilon')$. Then if $\epsilon_\pm$ is not an element of S

$$N_{n+1}(\epsilon_\pm)=N_n(\epsilon') . \tag{3.4}$$

Notice that if $N_{n+1}>2N_n$, Eq. (3.4) cannot exhaust all the possible values—and degeneracies—of the energy parameter at level $n+1$. Hence, in these cases, in order to generate the entire spectrum at level $n+1$, we must have contributions from the exceptional energies.

A. Linear chain (ring)

The recursion relation

$$\epsilon'=2-\epsilon^2=f(\epsilon) \tag{3.5}$$

maps the interval $[-2,2]$ onto itself. Energies outside this interval map to $\epsilon\to-\infty$; since for a chain of any number of sites the spectrum is bounded from below, this means that no eigenstate with $|\epsilon|>2$ may exist. A ring with 2^{n+1} sites is constructed as follows: The basic structure of two sites connected by two bonds ($n=0$) of Fig. 4(a) is "decorated" by introducing a site on each bond, to yield the $N_1=4=2^2$ site ring of Fig. 4(b), from which the $N_2=2^3$ site ring is constructed, etc. Given the energy levels $\epsilon_{n,\alpha}$, $\alpha=1,2,\ldots,N_n$ of the system with n levels, those of a ring with $N_{n+1}=2^{n+2}$ sites, i.e., $\epsilon_{n+1,\beta}$, can be obtained by inverting the recursion relation (3.5). We have

$$g_\pm(\epsilon)=\pm\sqrt{2-\epsilon} . \tag{3.6}$$

We encounter difficulty only when the higher-level ϵ equals zero. Hence S contains one element, 0.

Now let us find the energy parameters. At level 1, we have a two-site system with $\epsilon_{1,1}=-2$, $\epsilon_{1,2}=+2$. From the former energy parameter we find from Eq. (3.6), elements $\epsilon_{2,1}=-2$, $\epsilon_{2,2}=2$. Since the latter element gives rise to $\epsilon_{2,\beta}=0$, it is an exceptional case. But, we know that this four-site system must have four separate eigenstates. Hence $\epsilon_{2,3}=0$ must exist and be doubly degenerate. In this way we find that the level-2 system has energy parameters $(-2,2,0)$ and degeneracies $N_2(-2)=N_2=1$ and $N_2(0)=2$. This procedure is now repeated; after $n=10$ steps the density of states obtained is shown in Fig. 5. For $\epsilon_n\neq2$, the eigenstates $\psi^{(\beta)}_{n+1}$ are obtained by using (2.11c) to construct the amplitudes $\psi^{(\beta)}_{n+1}$ from $\psi^{(\alpha)}_n$. For $\epsilon_{n+1}=0$ the known two eigenstates, $\psi_{n+1}(l)=\exp(\pm\frac{1}{2}i\pi l)$ appear.

B. Sierpinski gasket

The recursion relation for the energy, Eqs. (2.15), has no stable fixed point or cycle. No interval of ϵ maps onto itself under repeated applications of the recursion relation.

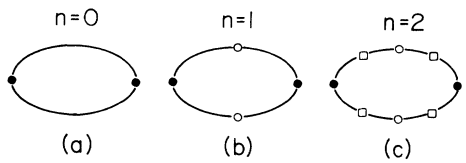

FIG. 4. Construction of a linear chain (ring). The $n=0$ lattice of (a) transforms by decoration to the $n=1$ lattice of (b), from which, in turn, the lattice (c) with $n=3$ is obtained.

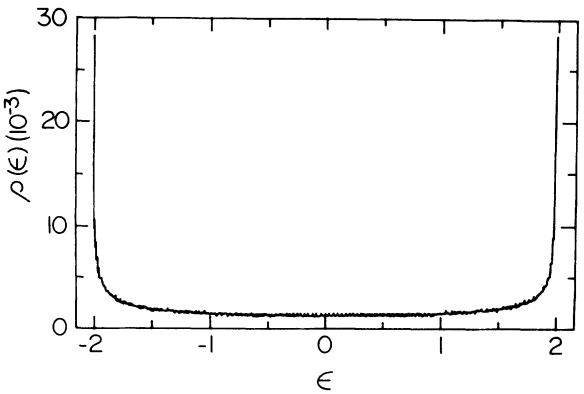

FIG. 5. Density of states (number of states in interval $\Delta\epsilon$, divided by total number of states) for the linear chain (ring) as obtained by the algorithm of Sec. III after $n=10$ iterations.

In terms of the eigenvalues, this means that any value of ϵ other than a set of measure zero (points that belong to some unstable cycle) will map, eventually to $\epsilon\to=\infty$.

However, since the spectrum of eigenvalues is bounded from below, this means that there exists no interval (i.e., band) of allowed energy levels for the infinite Sierpinski gasket. If one disregards, for the time being, the special values of $\epsilon^*=\pm2,+1$ for which the recursion relation (2.16) is inapplicable, one can generate a sequence of intervals I_k that constitute gaps in the spectrum, as shown in Fig. 6. The interval I_0, defined by $\epsilon>1$ *or* $\epsilon<-4$, maps towards $\epsilon=-\infty$ and therefore cannot contain allowed eigenvalues. The interval $I_1=[-(3+\sqrt{5})/2, -(3-\sqrt{5})/2]$ maps under one iteration onto a subset of I_0, and therefore also constitutes a gap. Continuing in this manner, we note that I_2, the union of two intervals as shown in Fig. 6, maps onto I_1; we find that the sequence of forbidden intervals generated in this manner, I_k, converges geometrically to cover the entire range $-4<\epsilon<1$ [or $0<x<1$, in terms of the variable of Eq. (1.7)]. We found, numerically, that for large k

$$I_k\sim A\beta^{-k},\quad \beta=1.738\,052\,3\ldots,\ A=0.711\,437\,4 .$$

Thus the set (of measure zero) of allowed values of ϵ must be either related to those values ϵ^* for which the recursion is not applicable, or belong to (unstable) cycles of the recursion relation.

As will be shown, the allowed energy values can be divided into two classes, with respect to the gap intervals I_k, defined above. One class consists of values lying *inside* the gaps; another class is values that constitute the gap edges.

Our basic, largest scale ($n=0$) structure of $N_0=3$ sites is shown in Fig. 1(a). The gasket generated after n inser-

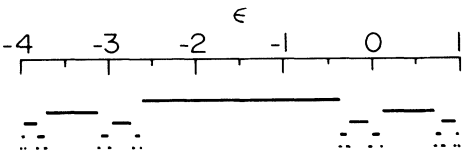

FIG. 6. Gap intervals I_k of the spectrum obtained from the recursion relation (2.16); $k=1,2,\ldots,5$ are shown.

tions, in the manner indicated in the figure, contains $N_n = 3^{n+1}$ sites

We now proceed to construct an algorithm that will provide the energy levels ϵ_n and their multiplicities $N_n(\epsilon_n)$, calculated from those of the system at level $n-1$. The parameters of the system with N_n sites are $\epsilon_n^{(\alpha)}$, $\alpha = 1, \ldots, N_n$. These are related to the parameters of the system with N_{n-1} sites via (2.16):

$$\epsilon_n = g_{\pm}(\epsilon_{n-1}) = [-3 \pm (9 - 4\epsilon_{n-1})^{1/2}]/2 \tag{3.7}$$

provided $\epsilon_n \neq \pm 2, 1$. These cases are now treated in detail. Consider first an eigenvalue $\epsilon_{n-1} = -4$, which gives rise, via (3.7), to one eigenvalue with $\epsilon_n = -4$; the other solution of (3.7), namely $\epsilon_n = 1$, is discarded. Therefore, if the level-$(n-1)$ system has $N_{n-1}(-4)$ states with energy $\epsilon_{n-1} = -4$, these give rise to

$$N_n(-4) = N_{N-1}(-4)$$

states, with $\epsilon_n = -4$, at level n. The second "problematic" value is $\epsilon_{n-1} = 2$. If its multiplicity is $N_{n-1}(2)$, one generates from (3.4) only

$$N_n(-1) = N_{n-1}(2) \tag{3.8}$$

states with energy $\epsilon_n = -1$. The remaining N'_{n-1} states have energies ϵ_{n-1} for which (3.7) can be used, and these generate $2N'_{n-1}$ states of energies ϵ_n given by (3.7). Thus, so far, knowing the spectrum at level $n-1$, we know the energies (and multiplicities) of

$$2N'_{n-1} + N_{n-1}(-4) + N_{n-1}(2)$$
$$= 2N_{n-1} - N_{n-1}(2) - N_{n-1}(-4)$$

states. Thus we have

$$N_n - [2N_{n-1} - N_{n-1}(2) - N_{n-1}(-4)]$$
$$= N_{n-1} + N_{n-1}(2) + N_{n-1}(-4)$$

states at level n with unknown energies. These energies must be $\pm 2, 1$ [otherwise they would be related to $\epsilon_{(n-1)}$ by (3.7)], and thus we must have

$$N_n(2) + N_n(-2) + N_n(1) = N_{n-1} + N_{n-1}(2) N_{n-1}(-4) . \tag{3.9}$$

We will immediately show that

$$N_n(2) = N_{n-1} , \tag{3.10}$$

and therefore Eq. (3.9) becomes

$$N_n(-2) + N_n(1) = N_{n-1}(2) + N_{n-1}(-4) . \tag{3.11}$$

To prove Eq. (3.10), we assume that there exists a solution of the eigenvalue equation at level n, with $\epsilon_n = 2$ with amplitudes $\psi_1(\alpha)$ on the N_{n-1} sites α present at the $n-1$ level and amplitudes $\psi_2(\alpha')$ on the sites α' that are introduced at the nth level. The latter can be eliminated from the eigenvalue equation and the remaining N_{n-1} amplitudes $\psi_1(\alpha)$ satisfy an eigenvalue equation which for $\epsilon = 2$ is parametrized by $u' = 2t$ and $t' = 0$, and therefore reads $2\psi_1(\alpha) = 2\psi_i(\alpha)$, which has N_{n-1} "solutions" $(\beta = 1, \ldots, N_{n-1})$

$$\psi_1^{\beta}(\alpha) = \delta(\alpha, \beta) . \tag{3.12}$$

From these N_{n-1}-independent functions the amplitudes $\psi_2(\alpha')$, defined on the level-n sites, can be calculated, using (2.17). Thus Eq. (3.10), and therefore (3.11), are proved.

In order to obtain no more relation between $N_n(-2)$, $N_n(1)$, and multiplicities at level $n-1$, we use the trace condition

$$\sum_{\alpha=1}^{N_n} E_{n,\alpha} = \mathrm{Tr} H = 0 . \tag{3.13}$$

The $2N'_{n-1}$ states generated by (3.7) contribute $2\,(-\frac{3}{2})N'_{n-1} = -3N'_{n-1}$ to the sum (3.13). This does not include the contribution of states with eigenvalues $\epsilon_n = -4, -1$ which has to be considered separately [only *one* of the roots of (3.4) was taken]; these contribute $(-1)N_n(-1) + (-4)N_n(-4)$. In addition, the special values $\epsilon_n = 2, +1$ must be added, thus obtaining

$$-3N'_{n-1} - N_n(-1) - 4N_n(-4) + 2N_n(2)$$
$$-2N_n(-2) + N_n(1) = 0 .$$

Using now (3.8), (3.10), and (3.11), we get

$$N_n(1) - 2N_n(-2) = N_{n-1} - 2N_{n-1}(2) + N_{n-1}(-4) . \tag{3.14}$$

Equations (3.11) and (3.14) have the solution

$$N_n(1) = N_{n-1}/3 + N_{n-1}(-4) , \tag{3.15}$$

$$N_n(-2) = N_{n-1}(2) - N_{n-1}/3 . \tag{3.16}$$

For all $N > 0$ we have $N_n(2) = N_n/3$ and thus $N_n(-2) = 0$; only for $n = 0$ one has $N_0(2) = 2$ and hence $N_1(-2) = 1$. It is also trivial to show that the state with lowest energy $E_n = -4$, is nondegenerate; $N_n(-4) = 1$. The eigenfunction for this level has a constant amplitude on all sites.

The spectrum and multiplicity of a system with n levels is thus generated from that of the system with $n-1$ levels by the algorithm outlined above, and summarized below.

(1) For all n $N_n(-4) = 1$.
(2) $N_n(2) = N_{n-1}$.
(3) $N_n(1) = N_{n-1}/3 + 1$.
(4) $N_n(-2) = \delta(n, 1)$.
(5) For all other energies $N_n(\epsilon) = N_{n-1}\epsilon')$; $\epsilon' = -\epsilon(\epsilon+3)$.

The density of states obtained after ten iterations (i.e., that of a gasket with 3^{11} sites) is shown in Fig. 7. In this histogram the fraction of states within an interval $\Delta\epsilon = 0.01$ is plotted versus ϵ.

First, note that the state with energy parameter -4 and the states resulting from the $(n=1)$ state with $\epsilon = -2$ are nondegenerate, and in effect do not show up on the histogram. Also note, that the highest peak (with $\frac{1}{3}$ weight) at $\epsilon = +2$ is *not* shown.

Second, we can now identify the state with $\epsilon = 2$ and its "descendents" at all generations, i.e., $\epsilon = -1$, $(-3 \pm \sqrt{13})/2, \ldots$ as the set of "midgap states" mentioned above; they occur *inside* the gap intervals I_k. On the other hand, the states with $\epsilon = +1$ and its "descendants," i.e., $\epsilon = (-3 \pm \sqrt{5})/2, \ldots$ form the edges of the gap intervals.

Figures 8 and 9 display a set of histograms that reveal a strikingly self-similar structure. In what follows, we ex-

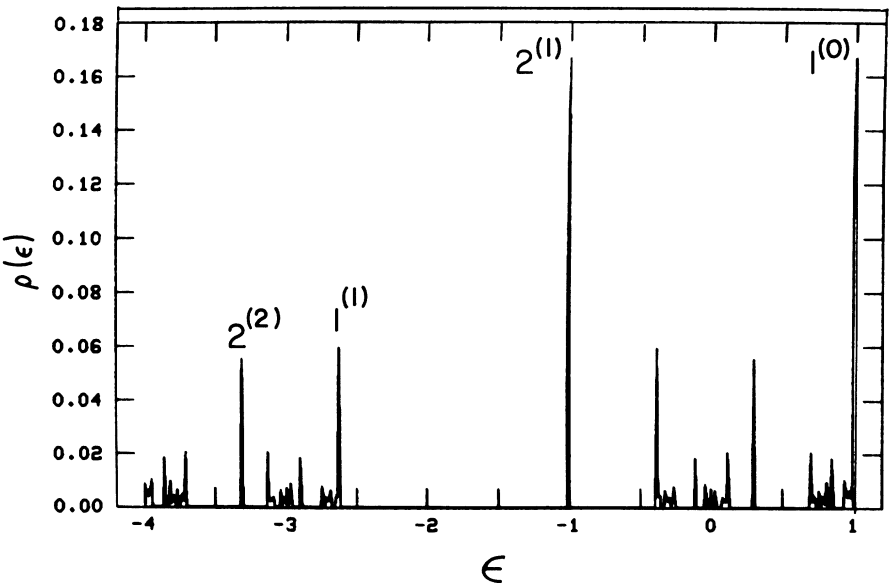

FIG. 7. Fraction of states $\rho(\epsilon)$ within intervals $\Delta\epsilon=0.01$ vs ϵ for the Sierpinski gasket with $n=10$ (3^{11} sites). The notation $2^{(k)}$ ($1^{(k)}$) indicates a state obtained after k iterations of Eq. (3.7) from an $\epsilon=2$ ($\epsilon=1$) state.

plain the origin of this self-similar (or scaling) behavior, and characterize it quantitatively.

Figures 8(a)–8(d) present scaling behavior near the fixed point $\epsilon^*=-4$. Denote $\Delta\epsilon=\epsilon+4$, and $N(\Delta\epsilon)$ the number of states at $\Delta\epsilon$. We find

$$N(\Delta\epsilon)=|\alpha|^{\nu}N(\Delta\epsilon/\alpha) \tag{3.17}$$

with eigenvalue $\alpha=5$ and exponent $\nu=\ln 3/\ln 5$. To understand this result note that the states with nonvanishing weight at level n are descendents of either the $\epsilon=+2$ or the $\epsilon=+1$ eigenvalues that resulted at some earlier stage. We denote by $2^{(k)}$ the eigenvalue obtained using (3.4) with the minus sign, k times, with $\epsilon=2$ as the initial value. Similar notation, i.e., $1^{(k)}$, is used for states that originate at $\epsilon=1$.

Thus for the n-generation gasket $2^{(k)}$ stands for the kth descendant of the $\epsilon=2$ state that resulted at the $(n-k)$th generation. This sequence of states is identified in Figs. 8(a)–8(d).

For any initial energy [other than one that belongs to some cycle of (2.16)], a sequence generated by repeated application of (3.4) can be characterized as $\eta_1,\eta_2,\eta_3,\ldots$, with $\eta_i=\pm 1$ corresponding to the sign chosen when (3.7) is used for the ith time. All sequences that satisfy $\eta_j=-1$ for $j>j_0$ converge towards $\epsilon=-4$ (or $\Delta\epsilon=0$). For small enough $\Delta\epsilon$ this convergence is governed by the eigenvalue of (2.15) at the fixed point;

$$\alpha=\left.\frac{d\epsilon'}{d\epsilon}\right|_{\epsilon=-4}=5\ . \tag{3.18}$$

Thus if after n iterations we observe a state with (small) $\Delta\epsilon$ and multiplicity $N_n(\Delta\epsilon)$, it is a direct descendant of a state with $\Delta\epsilon'=5\Delta\epsilon$, whose multiplicity, after $n-1$ iterations, $N_{n-1}(\Delta\epsilon')$, was exactly $N_n(\Delta\epsilon)$. Now, say, $\Delta\epsilon'$ was reached *at level* $n-1$ by applying (3.7) l times on the eigenvalue $\epsilon=2$ that resulted at level k, with multiplicity $N_k(2)=3^k$, and therefore $N_{n-1}(\Delta\epsilon')=3^k$. Obviously this same $\Delta\epsilon'$ is reached *at level* n by applying (3.7) l times on the same initial eigenvalue, but evolved at level $k+1$; therefore $N_n(\Delta\epsilon')=3^{k+1}$. Thus we have

$$N_n(\Delta\epsilon)=N_{n-1}(\Delta\epsilon')=N_n(\Delta\epsilon')/3\ . \tag{3.19}$$

If we now substitute $\Delta\epsilon=\Delta\epsilon'/5$ we get the scaling relation (3.17). Indeed, when we scale the energy axis by 5 and the multiplicity by 3, the sequence shown in Figs. 8(a)–8(d) is generated and displays the expected self-similar structure.

The statements made above concerning the neighborhood of $\epsilon^*=-4$ are valid for any member $\tilde{\epsilon}$ of any cycle, with eigenvalues determined by the derivative of the recursion relation at $\tilde{\epsilon}$. For example, the sequence $\eta_1,\eta_2,\eta_3,\ldots$ with $\eta_i=+1$ for $i>i_0$ converges to the fixed point $\tilde{\epsilon}=0$, giving rise to scaling behavior there. Figures 9(a) and 9(b) show the scaling behavior near $\tilde{\epsilon}_1=-1-\sqrt{3}=-2.732\,05\ldots$, which belongs to the two cycle ($\tilde{\epsilon}_2=-1+\sqrt{3}$). The sequence that tails with $\eta_i,\ldots=+1,-1,+1,-1,\ldots$, converges to the two cycle via the pattern $L_1R_2R_1L_2$. This means that a point to the left of $\tilde{\epsilon}_1$ iterates to the right of $\tilde{\epsilon}_2$, then to the right of $\tilde{\epsilon}_1$, to the left of $\tilde{\epsilon}_2$, and only then back to the left of $\tilde{\epsilon}_1$ where it started. The reason for this is the negative sign of the eigenvalue that corresponds to the two cycle,

$$\alpha_2=\left.\frac{d\epsilon'}{d\epsilon}\right|_{\epsilon=\tilde{\epsilon}_2}\left.\frac{d\epsilon'}{d\epsilon}\right|_{\epsilon=\tilde{\epsilon}_2}=-11\ . \tag{3.20}$$

The scaling relation (3.17) now takes the form

$$N(\Delta\epsilon)=3^2N(-\Delta\epsilon/11) \tag{3.21}$$

and therefore $\nu=2\ln 3/\ln 11$.

We now turn to describe the eigenfunctions generated by our iterative procedure. There is a single state with $\epsilon=-4$ at each level. At level $n=1$, there is one state with

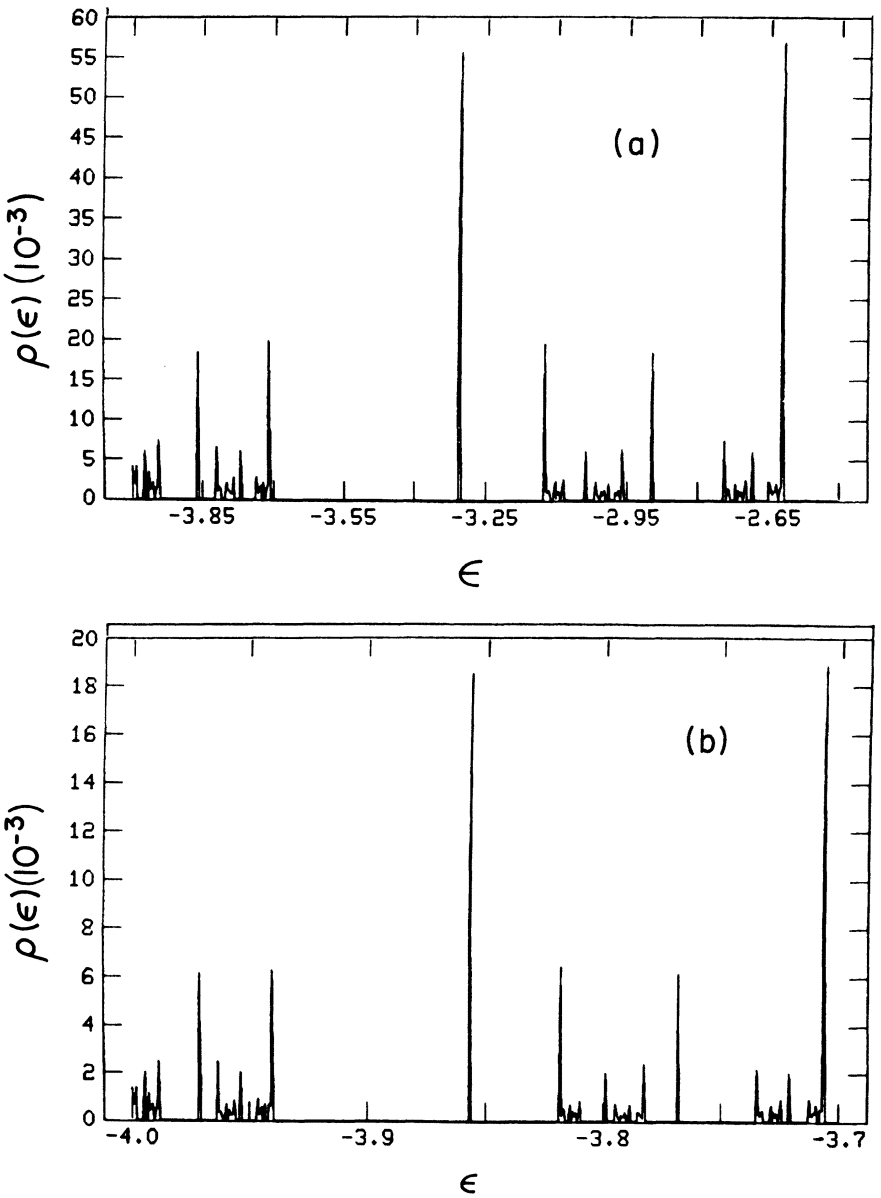

FIG. 8. Increasing "magnification" of Fig. 7 near $\epsilon=-4$, obtained by successively expanding the ϵ scale (by a factor of 5) and the $\rho(\epsilon)$ scale (by factor of 3.) A fixed (self-similar) distribution is approached.

$E=-2$; at level n, 2^{n-1} descendants of this state appear. All the remaining states at level n are descendants of states that resulted (with energy 2 or 1) at levels $k \leq n$. If $\epsilon_n \neq +2,1$ we can use Eq. (2.17) to construct eigenfunctions from those of level $n-1$. The amplitudes on the sites of the $(n-1)$th- generation gasket are those of the eigenfunction $\psi_{n-1}(\alpha)$, and the amplitudes on the sites that belong to the nth generation are given by (2.17). If there are two or more *independent* eigenfunctions $\psi^{\beta}_{n-1}(\alpha)$, $\beta=1,\ldots,N_{n-1}(\epsilon_{n-1})$, we generate $N_{n-1}(\epsilon_{n-1})$ eigenfunctions with energy $\epsilon_n^{(+)}$ and the same number with energy $\epsilon_n^{(-)}$ [the $(\pm)$ sign corresponds to the sign used in Eq. (3.7)]. Although it is obvious that the $N_{n-1}(\epsilon_{n-1})$ functions with eigenvalue $\epsilon_n^{(-)}$, for example, are also independent, it should be noted that they are not orthogonal. Thus within each degenerate subspace an orthogonalization procedure should be used. This has to be done at each level n; orthogonality at level $n-1$ does not ensure orthogonality of the functions [obtained by (2.17)] at level n.

At each level new states result [i.e., *not* via (3.7)] with energies $\epsilon_n=2$ and $\epsilon_n=1$. A particular choice of the $N_n/3$ states with energy $+2$ was identified above [see (3.12)]; they are completely localized with respect to the $(n-1)$-stage gasket sites, with amplitude 1 on one of this set of sites and zero on all others. The amplitudes on the sites that belong to the nth generation are then calculated using (2.17), giving rise to functions such as that of Fig. 10.

In this form the functions are not orthogonal. They all have the same normalization. Moreover, only functions belonging to nearest-neighbor sites on the $(k-1)$-stage gasket overlap. Thus the orthogonalization only involves one overlap integral which depends on the "age" of the

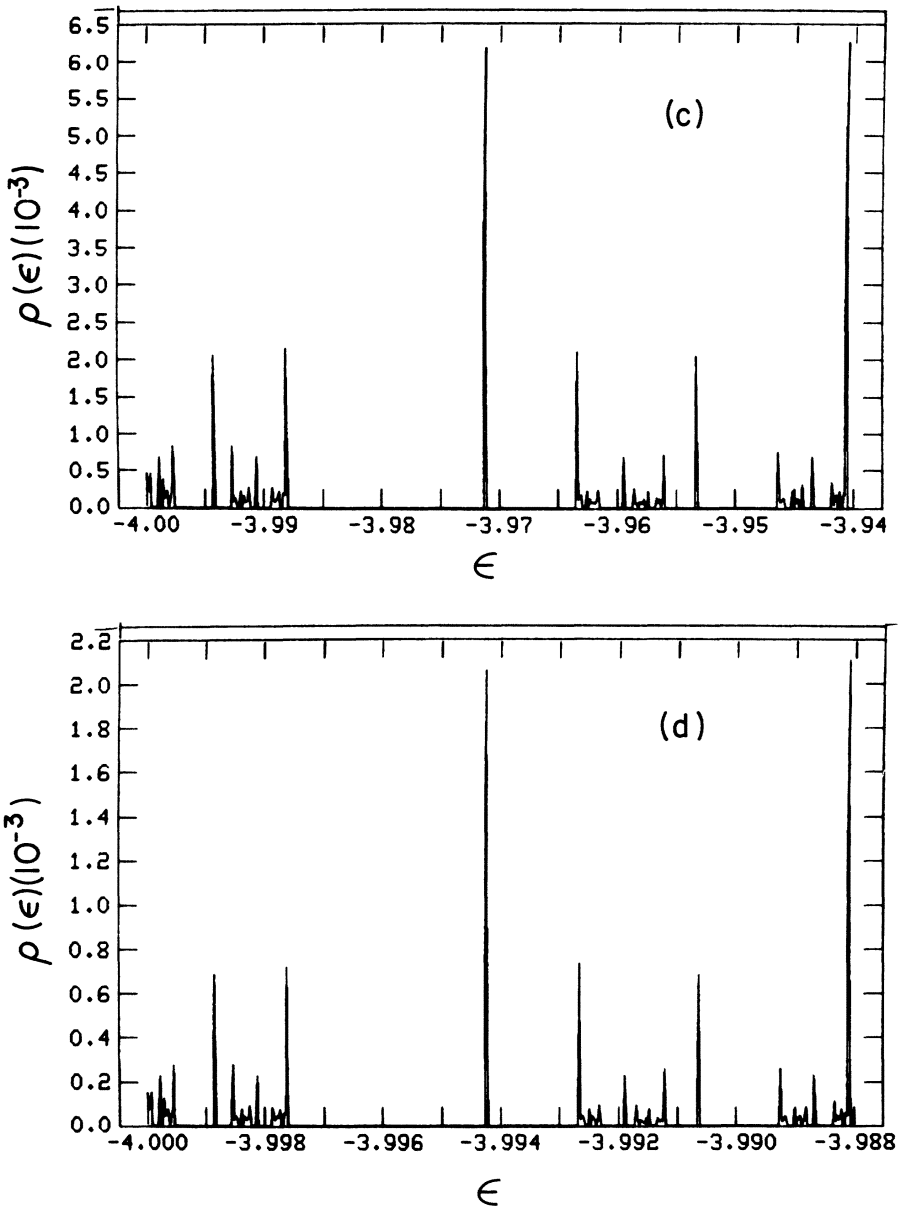

FIG. 8. (*Continued.*)

state $a=n-k$ (but not on n or k separately). This suggests a straightforward orthogonalization procedure.

The functions are labeled on the $(k-1)$-state gasket. We first orthogonalize the functions for the new sites on this gasket. This gives

$$\bar{\varphi}_i=\varphi_i/\sqrt{1-\alpha}-\frac{1}{3}\left[\frac{1}{\sqrt{1-\alpha}}-\frac{1}{\sqrt{1+2\alpha}}\right]\sum_{j=1}^{3}\varphi_j$$

for each triangle. The remaining functions are labeled on the $(k-2)$-stage gasket. We orthogonalize them to the $\bar{\phi}_i$ and calculate the overlap of the new functions [nearest neighbors on the $(k-2)$-stage gasket]. One finds

$$\alpha'=\frac{\alpha^2(1-2\alpha)}{(1+\alpha)^2}\ .$$

One can now proceed as before with α' replacing α. Thus the functions are all localized on the $(k-1)$-stage gasket decay exponentially outside their $(k-1)$-stage triangles with a cutoff depending on the smallest $(k-l)$-stage gasket to which they belong.

The eigenfunctions with $\epsilon_n=1$ [with multiplicity $N_n(1)=N_n/9+1$] can be characterized as follows. The amplitudes are nonzero only on sites that belong to the nth generation. In this set of sites we consider loops that surround "holes," i.e., triangles whose corners belong to generation $1\le k\le n-1$. On these loops we place amplitudes ±1, with alternating signs. Two such states, that loop around the $(n-1)$- and $(n-2)$-generation holes are shown in Fig. 11. Since the number of holes that belong to generation k is $2\times3^{k-1}$, the total number of states generated in this manner is given by

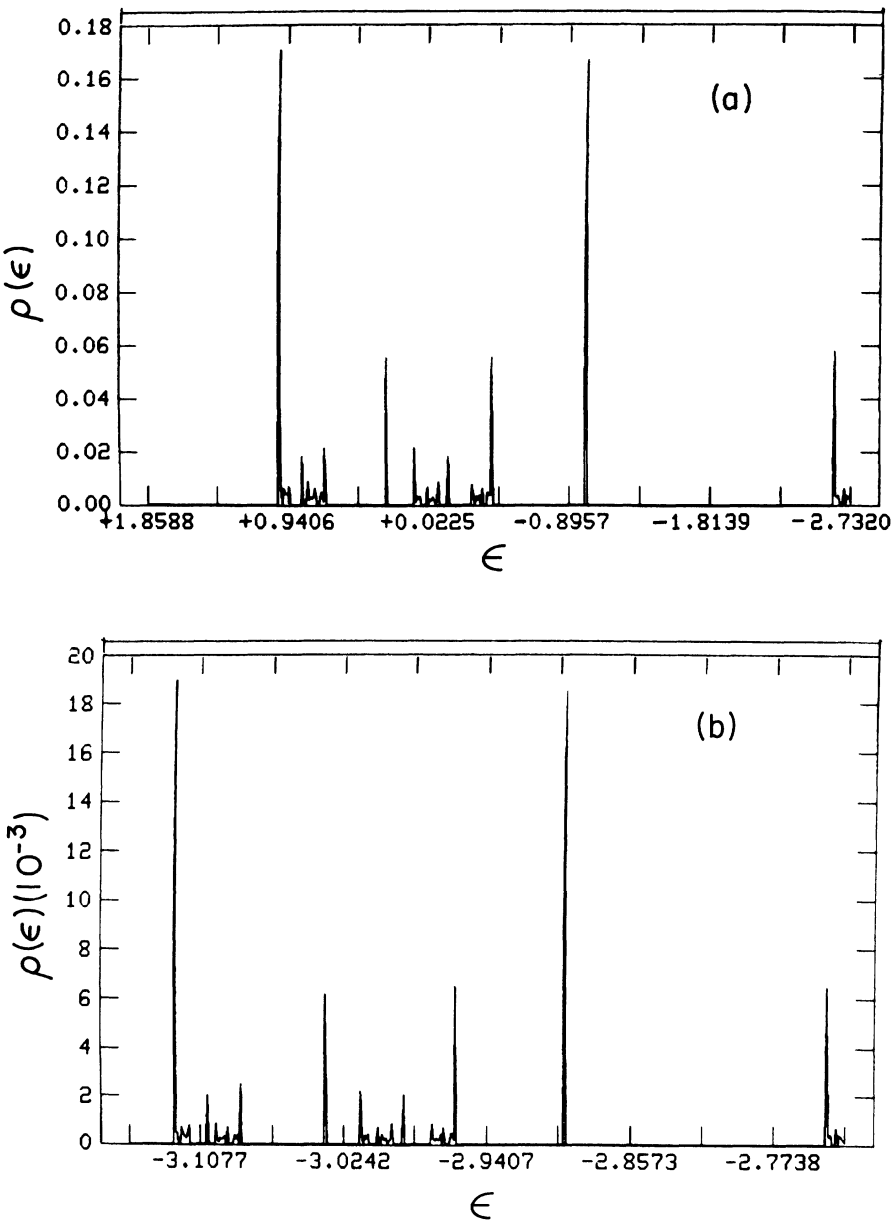

FIG. 9. Increasing magnification of Fig. 7 near $\epsilon=-1-\sqrt{3}=-2.73205\ldots$, obtained by successively expanding the ϵ scale (by -11) and the ρ scale by 9. A self-similar distribution is approached.

$$2\sum_{k=1}^{n-1}3^{k-1}=3^{n-1}-1=N_n(1)-2\ .$$

The remaining two independent states loop around two of the holes that were created by our choice of boundary conditions.

We can obtain orthogonal functions by a Schmidt procedure always orthogonalizing a function belonging to a k hole to all those with larger values of k (smaller holes) which it intersects. This never generates an overlap between functions belonging to the same generation.

C. A lattice with a continuum spectrum

In our analysis we used the convenient boundary conditions of Fig. 1. We found, however, that almost all eigenfunctions are generated in later generations (as $\epsilon_k=2$ or $\epsilon_k=1$) and are not even exponentially sensitive to the boundaries. It is interesting to see how this affects a problem with more general boundary conditions. We consider the lattice of gaskets (i.e., each upward-pointing triangle of an infinite triangular lattice contains an nth-generation gasket). Thus instead of the three discrete solutions at stage zero we now have the continuum of eigenstates of the triangular lattice:

$$\varphi_i=e^{i\vec{q}\cdot\vec{r}_i}\ ,$$

$$\epsilon_q=-2\cos q_x-4\cos(q_x/2)\cos(\sqrt{3}q_y/2)\ ,$$

$$-6\le\epsilon_q\le+3\ .$$

We exclude the special values $\epsilon_q=2$, -4. There are also

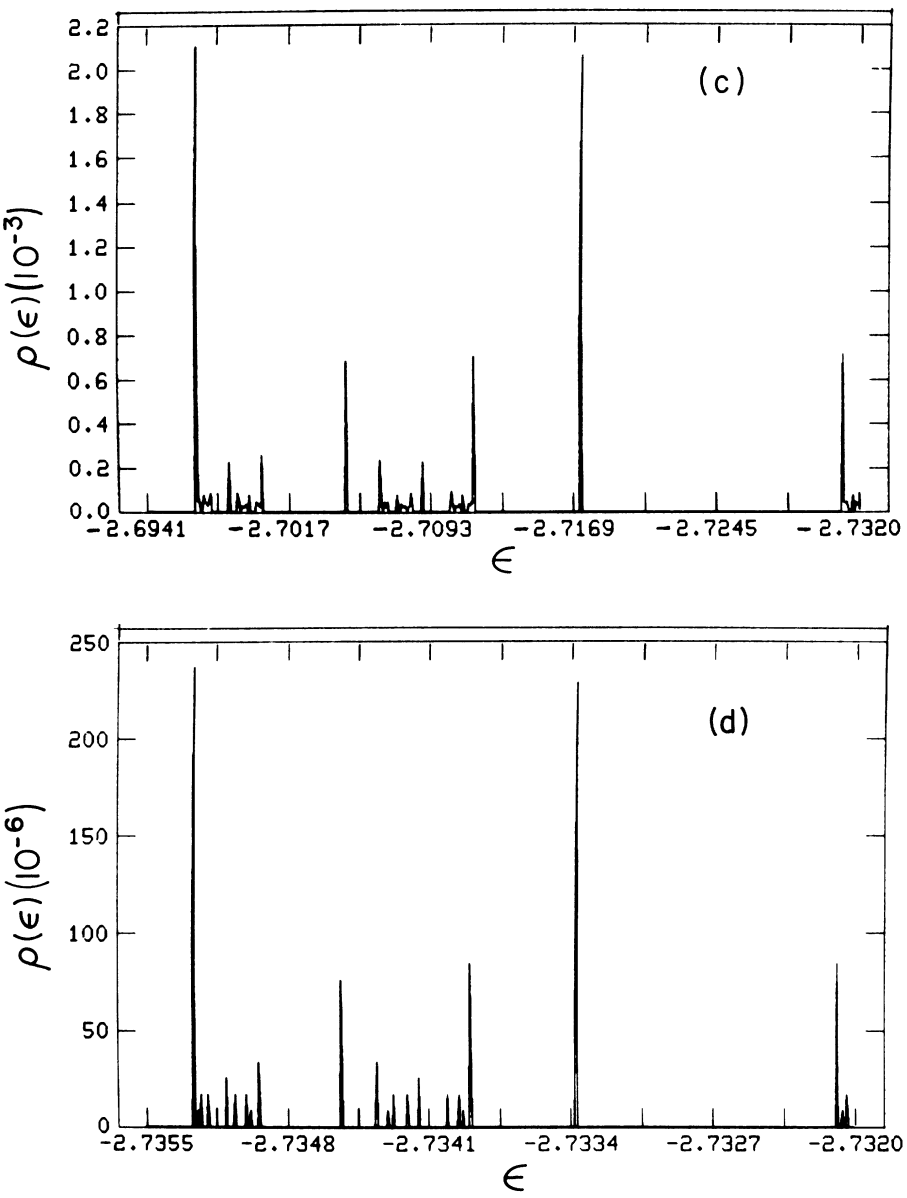

FIG. 9. (*Continued.*)

no solutions on the gasket for $\epsilon_q > \frac{9}{4}$. All other states map on 2^n bands. The lowest of these extends on *both* sides of the $\epsilon = -4$ fixed point. These extended state bands have a very strange density of states. Their edges are the mappings of the points $\epsilon_0 = -6, +\frac{9}{4}$, and the density of states there reflects that of the triangular lattice and is constant. Superimposed on this is the singular structure imposed by all the cycles of the iteration process. In addition one does, of course, have the "new" states of the gasket which show up essentially in the same way as on the single gasket and are localized. Altogether the extended states represent only a fraction $(\sim\frac{2}{3})^n$ of the total number of states. The solutions at or originating from $\epsilon = 1$ are embedded in the continuum. On the other hand, $\epsilon = 2$ and its descendants always sit in the gaps of the continuum spectrum.

D. Berker lattice

The recursion relations for the energy parameters in the Berker lattice are identical with those of the linear chain. For both cases, we can write the energies as

$$\epsilon_{n,p} = -2\cos 2\pi q_{n,p} , \tag{3.22}$$

where the $q_{n,p}$ are given by

$$q_{n,p} = \frac{p}{2^{n+1}}, \quad p = 0,1,\ldots,2^n . \tag{3.23}$$

The states with the largest and smallest p's are nondegenerate. The others are doubly degenerate.

Since the level-0 energies ($\epsilon_0 = \pm 2$) and recursion relations are the same for the Berker lattice as the linear

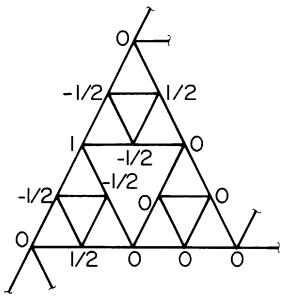

FIG. 10. A typical (unnormalized) eigenfunction with $\epsilon=2$; on sites not shown the amplitude vanishes.

chain, the spectrum (3.22) and (3.23) holds in both cases. However, for the Berker lattice, the number of sites is given by

$$N_n=\frac{4^{n+1}+2}{3} \tag{3.24}$$

which is, for $n>0$, larger than that for the linear chain. Hence the degeneracies of the levels must also be higher for the Berker lattice.

This higher degeneracy arises from states with $\epsilon=0$ or $p=2^{n-1}$ which are born at level n. To find the degeneracies, $N_n(\epsilon_{n,p})$, notice that all energies at level $n-1$ except the energy $\epsilon_{n-1}=2$ give rise to precisely two energies at the next level. Hence

$$N_n=N_n(0)+2[N_{n-1}-N_{n-1}(2)] .$$

Since $N_{n-1}(2)=1$, we find

$$N_n(0)=\frac{2\times 4^n+4}{3} . \tag{3.25}$$

This state with $\epsilon_k=0$ has $p=2^{k-1}$ so that $p2^{1-k}=1$. This level and all its higher-n "daughters" will have a degeneracy $N_k(0)$. They can all be recognized because they obey

$$p2^{1-k}=1 \ (\text{mod}2) \tag{3.26}$$

and hence have degeneracy

$$N_n(\epsilon_{n,p})=N_k(0)=\frac{2\times 4^k+4}{3} . \tag{3.27}$$

This degeneracy statement holds for all ϵ values except $\epsilon=\pm 2$, which do not arise as "daughters" of the $\epsilon=0$ states. In these two cases we have

$$N_n(+2)=N_n(-2)=1 . \tag{3.28}$$

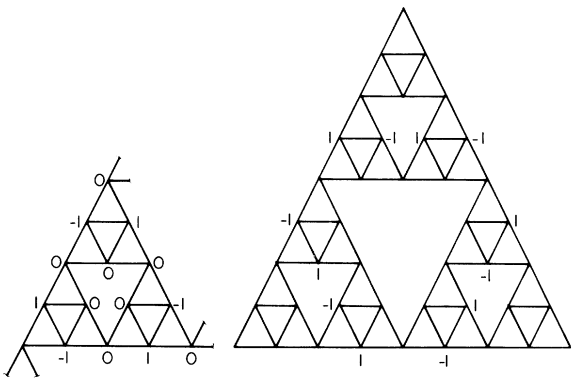

FIG. 11. Two states with $\epsilon=1$, one loops around the smallest scale "hole," the other around a "hole" one scale larger.

E. Phonon interpretation

Additional physical information can be derived from the spectra we have calculated. For example, the Sierpinski gasket which was just solved in terms of electronic wave function can be interpreted in terms of transversal phonons, i.e., a two-dimensional lattice of point masses m linked by strings with an elastic constant k and allowed to move perpendicularly to the plane of the lattice. It is then straightforward to show that the eigenfrequencies are related to the energies previously found by

$$\omega_{n,\alpha}=\omega_0(\epsilon_{n,\alpha}+4)^{1/2} , \tag{3.29}$$

where $\omega_0=\sqrt{k/m}$. The phonon heat capacity is given by

$$C_v=k\int d\omega \left[\frac{\beta\omega}{2}\right]^2\left[\sinh\frac{\beta\omega}{2}\right]^{-2} \times\sum_\alpha N_n(\epsilon_{n,\alpha})\delta(\omega-\omega_{n,\alpha}) . \tag{3.30}$$

Here β is the inverse temperature in energy units.

Consider the low-temperature limit $\beta\omega_0=T^{-1}\to\infty$. Then C_v is dominated by the part of the spectrum near $\epsilon=-4$. In this limit we have $N(\Delta\epsilon)\sim(\Delta\epsilon)^\nu$ and hence

$$C_v\sim T^{2\nu}=T^{1.365} . \tag{3.31}$$

Figure 12 shows the low-temperature specific heat derived from Eq. (3.30). Apparently the convergence to the result (3.31) is rather slow since our observed specific heat is better fit by $C_v\sim T^{1.388}$.

F. Green's functions

Further analysis of the localization problem on these lattices would be considerably helped if we knew the Green's function

$$G_{ij}(E)=[(H-E)^{-1}]_{ij} . \tag{3.32}$$

In general, the evaluation of this function is quite difficult. However, the analysis of Sec. II shows that if i and j lie on sublattice 1, then[9]

$$G_{ij}(E)=[H_{\text{eff}}(E)-E]^{-1}{}_{ij} \tag{3.33}$$

with H_{eff} given by (2.4a). This rather powerful result enables us to calculate G_{ij} for a lattice with N_n sites in terms of one with N_{n-1} sites. For the case of interest, the Sierpinski gasket, we get, using (2.14),

$$G_{ij}^{(n)}\epsilon=f\epsilon G_{ij}^{(n-1)}(\epsilon')$$

where

$$f(\epsilon)=(\epsilon+2)(\epsilon-1)/(2-\epsilon) ,$$
$$\epsilon'=-\epsilon(\epsilon+3) . \tag{3.34}$$

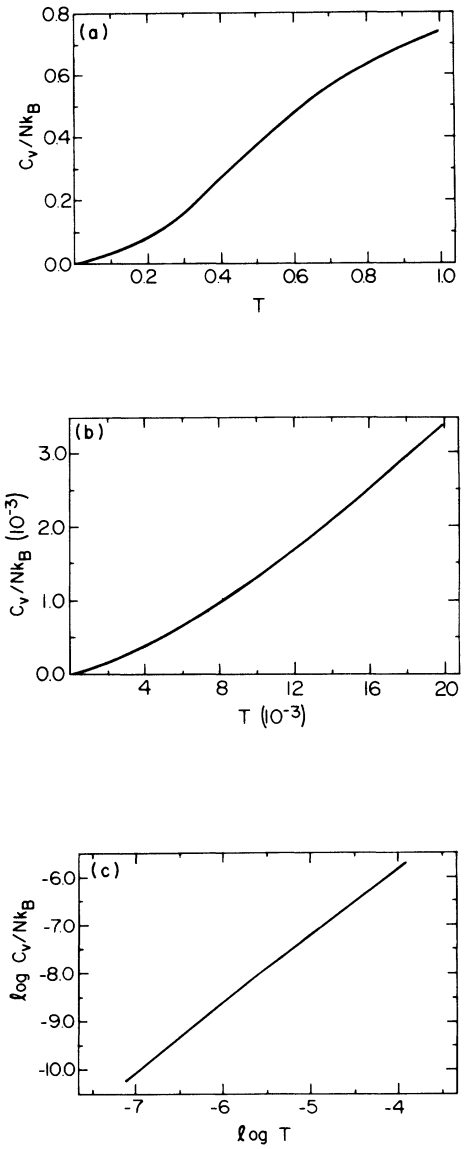

FIG. 12. Specific heat per site C_v/Nk_B vs (a),(b) temperature T and (c) $\log_{10}T$.

We emphasize again that this relationship holds only when both i and j belong to the subspace of N_{n-1} sites that were not eliminated. To demonstrate the manner in which (3.33) can be used, consider the case where i,j are two sites of level 0 (i.e., the largest scale). For a lattice of N_n sites, we get

$$G_{ij}E=\prod_{k=1}^{n}[f(\epsilon_k)]G_{ij}^0(\epsilon_0)\frac{1}{t}\ , \tag{3.35}$$

where

$$\begin{aligned}&\epsilon_n=E/t\ ,\\ &\epsilon_{k-1}=-\epsilon_k(\epsilon_k+3)\ ,\end{aligned} \tag{3.36}$$

and G_{ij}^0 is the Green's function of the level-0 problem (with three sites),

$$G_{12}^0(\epsilon)=2/(\epsilon+4)(\epsilon-2)\ . \tag{3.37}$$

Thus we see that $G_{ij}(E)$ will have poles at those values of E which after k iterations of (3.36) map onto $\epsilon_{n-k}=2$. It is straightfoward to show that for large $n-k$ the residue from the pole, $\epsilon_{n-k}=2$, is $\sim e^{-L_{n-k}/\xi}$ where $L_n=2^n$ is the distance, in terms of the smallest scale lattice spacings, between the two sites i,j. This result shows plausibly that the orthogonalized eigenstates with $\epsilon_{n-k}=2$ are exponentially localized.

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