Dispersion relations and sum rules in the Faraday effect

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(Received 7 April 1983)

Dispersion relations and sum rules are obtained for a gaseous insulating medium, under the influence of a weak constant external magnetic field. The application of the causality principle to the constitutive equation of the medium together with the behavior of free electrons in the high-frequency limit give the analytic behavior of the optical constants in the upper half complex plane, new dispersion relations, and new sum rules. The fact that the average refractive index, as well as the difference of the complex refractive indices of right and left circularly polarized waves, satisfies dispersion relations is justified by the application of relativistic causality.

I. INTRODUCTION

The application of the principle of causality to the birefringence phenomena to derive dispersion relations and sum rules has been shown to be successful in the case of natural optical activity.[1] The Faraday effect[2] exhibits also a birefringence phenomena, but its source is different from natural optical activity. Birefringence effects in the Faraday effect come from the presence of an external constant magnetic field while natural optical activity is a consequence of spatial dispersion.[1]

We will describe a gaseous, isotropic, insulating medium in the presence of an external weak constant magnetic field through its constitutive equations. Imposing the condition that the reaction of the medium to the presence of an external field be causal, and assuming that the free-electron behavior in the high-frequency limit will give us the analytical behavior of the optical constants and dispersion relations, we derive new dispersion relations and sum rules, as well as the known ones.

Unlike Smith[3] we show that the complex refractive index $N_{\pm}(\omega)$ of each type of circularly polarized wave satisfies a dispersion relation even though in the sum rules it is necessary to consider a mixture of both types of complex refractive index. The sum rules in the Faraday effect that involve only the average complex refractive index, $\bar{N}(\omega)$, are the same as in the case of natural optical activity. However, the sum rules that come directly from the principle of causality mix the average complex refractive index $\bar{N}(\omega)$ with the complex rotatory angle $\Theta(\omega)$; for example,

$$\int_0^\infty \bar{n}(\omega)\phi_F(\omega)d\omega=\int_0^\infty \bar{\kappa}(\omega)\theta(\omega)d\omega \ , \qquad (1.1)$$

where $\bar{n}(\omega)$ and $\bar{\kappa}(\omega)$ are the real and imaginary parts of $\bar{N}(\omega)$, respectively, and $\phi_F(\omega)$ and $\theta(\omega)$ are real and imaginary parts of $\Theta(\omega)$.

In Sec. II we present the constitutive equation of the medium, and the relations among the complex refractive index, the complex rotatory angle per unit length, and the optical constants that appear in the constitutive equation. We show that those constants that appear in the constitutive equation are causal. In Sec. III we derive the asymptotic behavior of the optical constants assuming that the reaction of the medium at $\omega\to\infty$ is described by a free-electron behavior. In Sec. IV we obtain the crossing relations for the fields and the optical constants. In Sec. V we derive the new sum rules for the optical constants through the superconvergence theorem.[4,5] All the sum rules are rewritten in terms of $\bar{N}(\omega)$ and $\Theta(\omega)$ which are associated with measured quantities. Finally, in Sec. VI we justify the known dispersion relations and sum rules through the principle of relativistic causality. Using its analytic behavior, we show that $N_{\pm}(\omega)-1$ satisfies the dispersion relation. In Appendixes A and B we quote Titchmarsh's theorem and the superconvergence theorem, respectively, and in Appendix C there is a more careful derivation of the crossing relations for the components of electric field.

II. CONSTITUTIVE EQUATIONS AND CAUSALITY

Let us consider a gas of low density in the presence of an external constant magnetic field $\vec{B}_0$. We will give a macroscopic description to the medium through its constitutive equations. The medium in the absence of the external magnetic field has the following constitutive equations:

$$\vec{D}(\omega)=\epsilon(\omega)\vec{E}(\omega) \qquad (2.1)$$

and

$$\vec{H}(\omega)=\vec{B}(\omega) \ , \qquad (2.2)$$

for the Fourier components of the fields of frequency ω.

We are considering the case when the external magnetic field $\vec{B}_0$ is not strong so that the constitutive equation of the medium is given by[2]

$$\vec{D}(\omega)=\epsilon(\omega)\vec{E}(\omega)+i\beta(\omega)\vec{B}_0\times\vec{E}(\omega) \ , \qquad (2.3)$$

$$\vec{H}(\omega)=\vec{B}(\omega) \ , \qquad (2.4)$$

and we note that $\epsilon(\omega)$ and $\beta(\omega)$ are the optical constants that describe the reaction of the medium to the presence of the external fields.

The constitutive equations (2.3) and (2.4) will give rise to birefringence phenomena, in this case called the Faraday effect. Thus the right- and left-hand circularly polarized waves have different complex refractive indices (this means that they have different phase velocity and in the resonant region are attenuated in different ways). But the optical constants $\epsilon(\omega)$ and $\beta(\omega)$ are independent of the type of polarization of the propagating field.

Using the Fourier components of Maxwell's equations and the constitutive equation (2.3) and (2.4), we get the complex refractive index in the limit of the weak field[6] $\vec{B}_0^{(2)}$, for waves propagating parallel to the direction of the external field,

$$N_\pm(\omega)=[\epsilon(\omega)]^{1/2}\mp\frac{1}{2}\frac{\beta(\omega)B_0}{[\epsilon(\omega)]^{1/2}} \tag{2.5}$$

and

$$N_\pm(\omega)=n_\pm(\omega)+i\kappa_\pm(\omega)\ , \tag{2.6}$$

where N_+ (N_-) is the complex refractive index of a left- (right-) handed circularly polarized wave relative to the direction of $\vec{B}_0$ (chosen in the z direction) and n_+ (n_-) and κ_+ (κ_-) are the real and imaginary parts of the complex refractive index, respectively.

Since left and right circularly polarized waves have different complex refractive indices, when a linearly polarized wave is incident upon the medium, the emergent wave will be, in general, an elliptically polarized wave, and the direction of the principal axis of the ellipse will be rotated relative to the direction of oscillation of the incident linearly polarized wave. But unlike the case of natural optical activity,[2] if the field $\vec{E}(\omega)$ is reflected back to the gaseous medium by a mirror, the rotated angle will not vanish; this occurs because the presence of the external constant magnetic field $\vec{B}_0$ breaks the isotropy of the medium.

The complex rotatory angle per unit length is given by[7]

$$\Theta(\omega)=\frac{\omega}{2c}[N_+(\omega)-N_-(\omega)] \tag{2.7}$$

and

$$\Theta(\omega)=\phi_F(\omega)+i\theta(\omega)\ , \tag{2.8}$$

where $\phi_F(\omega)$ is the Faraday rotation per unit length and $\theta(\omega)$ is ellipticity per unit length.

Rewriting Eq. (2.3) as a function of time, we find

$$\vec{D}(t)-\vec{E}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'\epsilon(t-t')\vec{E}(t') + \frac{i}{2\pi}\vec{B}_0\times\int_{-\infty}^{\infty}\beta(t-t')\vec{E}(t')dt'\ , \tag{2.9}$$

where $\epsilon(t)$ and $\beta(t)$ are the Fourier transformations of $[\epsilon(\omega)-1]$ and $\beta(\omega)$, respectively.

The right-hand side (rhs) of (2.9) represents the reaction of the medium to the presence of the external fields. Assuming that the medium does not generate energy by itself, the rhs of (2.9) must be a causal response to the external excitation. As $\vec{E}(t)$ and $\vec{B}_0$ are independent fields, so in the absence of $\vec{B}_0$ the linear functional of $\vec{E}$ in (2.9) must be causal by itself, so this implies that

$$\epsilon(t-t')=0 \quad \text{for } t'>t\ . \tag{2.10}$$

When we have an external magnetic field $\vec{B}_0$, the second term on the rhs of (2.9) contributes, and the only way that the entire rhs of (2.9) can be causal, for arbitrary external fields, is if

$$\beta(t-t')=0 \quad \text{for } t'>t\ . \tag{2.11}$$

Our aim is to obtain dispersion relations à la Kramers-Krönig for the optical constants that characterize the medium. For this we will use the causal behavior of $\epsilon(t)$ and $\beta(t)$, and through Titchmarsh's theorem (Appendix A) we can get the analytic properties of the Fourier transformation of those functions. However, to apply this theorem we need information about the asymptotic behavior of the optical constants for $\omega\to\infty$.

III. BEHAVIOR OF THE OPTICAL CONSTANTS IN THE HIGH-FREQUENCY LIMIT

Let us discuss what is meant by the high-frequency limit and whether we can still describe the medium through its constitutive (macroscopic) equations.[8] We mean by high-frequency limit when the movement of the electron can be described by the motion in the presence of the external fields, neglecting the binding energy of the electron to the other particles of the molecule. However, the wave vector $\vec{k}$ of the external oscillating field must satisfy $kd \ll 1$, where d is a typical dimension of the molecule and k is the modulus of the wave vector of the external field.

So, to get the leading-order behavior of the optical constants as a function of ω, we solve the classical equations of motion for each electron in the gas as a free particle of charge e and mass m in the presence of a constant external magnetic field $\vec{B}_0$ and an electric field $\vec{E}(t)$, that is,

$$m\ddot{\vec{x}}=e\vec{E}(\omega)e^{-i\omega t}+\frac{e}{c}\dot{\vec{x}}\times\vec{B}_0\ . \tag{3.1}$$

From the solution of the previous equation we can obtain the induced electric polarization vector in the limit $\omega\to\infty$,

$$\vec{P}(\omega)=-\frac{e^2\eta}{m}\frac{\vec{E}(\omega)}{\omega^2}+\frac{ie^3\eta}{m^2c}\frac{\vec{B}_0\times\vec{E}(\omega)}{\omega^3}\ , \tag{3.2}$$

where η is the density of electrons in the gas. We should note, however, that we did not take into account spin effects; these could change the factor that multiplies the second term on the rhs of (3.2).[3] Using the fact that

$$\vec{D}(\omega)=\vec{E}(\omega)+4\pi\vec{P}(\omega)\ , \tag{3.3}$$

and comparing this expression with (2.3), we conclude that for $\omega\to\infty$, the optical constants have the following asymptotic forms:

$$\lim_{\omega\to\infty}\epsilon(\omega)-1=-\frac{4\pi e^2\eta}{m}\frac{1}{\omega^2}=-\frac{\omega_p^2}{\omega^2} \tag{3.4}$$

and

$$\lim_{\omega\to\infty} \beta(\omega)=\frac{4\pi e^3}{m^2 c}\frac{\eta}{\omega^3}\ , \tag{3.5}$$

where ω_p is the plasma frequency.

By equations (3.4) and (3.5) the functions[9] $[\epsilon(\omega)-1]$ and $\beta(\omega)$ are square integrable and[10]

$$\mathrm{Re}[\epsilon(\omega)-1]=O(\omega^{-2})\ , \tag{3.6}$$

$$\mathrm{Im}[\epsilon(\omega)-1]=o(\omega^{-2})\ , \tag{3.7}$$

$$\mathrm{Re}\beta(\omega)=O(\omega^{-3})\ , \tag{3.8}$$

$$\mathrm{Im}\beta(\omega)=o(\omega^{-3})\ . \tag{3.9}$$

By the previous asymptotic conditions, we can apply Titchmarsh's theorem to the functions $[\epsilon(\omega)-1]$ and $\beta(\omega)$ (see Appendix A,[11] and as they satisfy the first statement of Titchmarsh's theorem [cf. (2.10) and (2.11)] they also satisfy the two Plemelj formulas, that is,

$$\mathrm{Re}[\epsilon(\omega)-1]=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}[\epsilon(\omega')-1]}{\omega'-\omega}d\omega'\ , \tag{3.10}$$

$$\mathrm{Im}[\epsilon(\omega)-1]=-\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}[\epsilon(\omega')-1]}{\omega'-\omega}d\omega'\ , \tag{3.11}$$

$$\mathrm{Re}\beta(\omega)=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\beta(\omega')}{\omega'-\omega}d\omega'\ , \tag{3.12}$$

and

$$\mathrm{Im}\beta(\omega)=-\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\beta(\omega')}{\omega'-\omega}d\omega'\ . \tag{3.13}$$

Our next aim is to derive sum rules for the optical constants through the superconvergence theorem[4,5] (see Appendix B); however, to apply this theorem the integrals that appear in (3.10)–(3.13) must be restricted to the interval of integration $[0,\infty)$. To restrict the interval of integration of the mentioned integral we use the crossing relations of the optical constants to relate the positive and negative components of the Fourier transformation in the ω space.

IV. CROSSING RELATIONS

Using the fact that the fields that are propagating through the medium are real, we obtain a relation between the positive- and negative-frequency Fourier components of the fields. Let us consider the case of the electric field (see Appendix C)

$$\vec{\mathrm{E}}(t)=\int_{-\infty}^{\infty}\vec{\mathrm{E}}(\omega)e^{-i\omega t}d\omega\ . \tag{4.1}$$

Taking the complex conjugate of both sides of this equation, and using the fact that $\vec{\mathrm{E}}(t)$ is a real quantity, we obtain

$$\vec{\mathrm{E}}(-\omega)=\vec{\mathrm{E}}^*(\omega)\ . \tag{4.2}$$

This relation between the Fourier components of the electric field for positive and negative frequencies is valid also for the other fields.

With these facts, we will obtain the relation between the positive and negative components of the Fourier transformation of $\beta(t)$ and $\epsilon(t)$. For this we will use the constitutive equation for $\vec{\mathrm{D}}(t)$ [cf. (2.3) and (2.9)], that is,

$$\vec{\mathrm{D}}(t)-\vec{\mathrm{E}}(t)=\int_{-\infty}^{\infty}[\epsilon(\omega)-1]\vec{\mathrm{E}}(\omega)e^{-i\omega t}d\omega + i\vec{\mathrm{B}}_0\times\int_{-\infty}^{\infty}\beta(\omega)\vec{\mathrm{E}}(\omega)e^{-i\omega t}d\omega\ . \tag{4.3}$$

Taking the complex conjugate of both sides of this expression, using the fact that the left-hand side (lhs) of (4.3) is real, and that each term on the rhs must be real,[12] we obtain

$$\epsilon(-\omega)=\epsilon^*(\omega) \tag{4.4}$$

and

$$\beta(-\omega)=-\beta^*(\omega)\ . \tag{4.5}$$

Now, using (4.4) and (4.5) we rewrite (3.10)–(3.13) restricted to the positive frequencies and using the asymptotic behaviors (3.6)–(3.9) we can apply the superconvergence theorem to get the sum rules.

V. SUM RULES

A. Sum rules in the high-frequency limit

We will now obtain through the superconvergence theorem the sum rules for the optical constants that come from the dispersion relations (3.10)–(3.13) and have the asymptotic behaviors (3.6)–(3.9). We remark that, as opposed to Smith,[3] we do not need to guess the asymptotic behavior of any optical constant because the information that we get from the behavior of free electrons for $\omega\to\infty$ and the superconvergence theorem is enough.

By (4.4) and (4.5) we can rewrite (3.10)–(3.13) as

$$\mathrm{Re}[\epsilon(\omega)-1]=\frac{2}{\pi}\mathrm{P}\int_{0}^{\infty}\frac{\omega'\mathrm{Im}\epsilon(\omega')}{\omega'^2-\omega^2}d\omega'\ , \tag{5.1}$$

$$\mathrm{Im}\epsilon(\omega)=-\frac{2\omega}{\pi}\mathrm{P}\int_{0}^{\infty}\frac{\mathrm{Re}[\epsilon(\omega')-1]}{\omega'^2-\omega^2}d\omega'\ , \tag{5.2}$$

$$\mathrm{Re}\beta(\omega)=\frac{2\omega}{\pi}\mathrm{P}\int_{0}^{\infty}\frac{\mathrm{Im}\beta(\omega')}{\omega'^2-\omega^2}d\omega'\ , \tag{5.3}$$

and

$$\mathrm{Im}\beta(\omega)=-\frac{2}{\pi}\mathrm{P}\int_{0}^{\infty}\frac{\omega'\mathrm{Re}\beta(\omega')}{\omega'^2-\omega^2}d\omega'\ . \tag{5.4}$$

By (3.6) we have that

$$\mathrm{Re}[\epsilon(\omega)-1]=O(\omega^{-2})$$

for $\omega\to\infty$, so applying the superconvergence theorem to (5.2) and making the change in variable $y=\omega^2$, by (B2) we get for $\omega\to\infty$ that

$$\mathrm{Im}\epsilon(\omega)=\frac{2}{\pi}\frac{1}{\omega}\int_{0}^{\infty}\mathrm{Re}[\epsilon(\omega')-1]d\omega'+O(\omega^{-3})\ , \tag{5.5}$$

so for (5.5) to be compatible with (3.7) we need to have that

$$\int_{0}^{\infty}\mathrm{Re}[\epsilon(\omega)-1]d\omega=0\ . \tag{5.6}$$

As a consequence of (5.5) and (5.6), we obtain for $\omega\to\infty$

$$\mathrm{Im}\epsilon(\omega)=O(\omega^{-3})\ . \tag{5.7}$$

But if $\mathrm{Im}\epsilon(\omega)$ satisfies (5.7) for $\omega\to\infty$, we have

$$\mathrm{Im}\epsilon(\omega)=O(\omega^{-2}\ln^{-2\alpha}\omega)\ ,\quad \alpha>1$$

as $\omega\to\infty$. Making the change in variable $y=\omega^2$ in (5.1) by (B1) we find that for $\omega\to\infty$

$$\mathrm{Re}[\epsilon(\omega)-1]=-\frac{1}{\pi}\frac{2}{\omega^2}\int_0^\infty \mathrm{Im}\epsilon(\omega')d\omega'+O(\omega^{-2}\ln^{1-\alpha}\omega)\ . \tag{5.8}$$

Thus by (3.4) we have that

$$\int_0^\infty \omega\,\mathrm{Im}\epsilon(\omega)d\omega=\frac{2\pi^2\eta e^2}{m}\ , \tag{5.9}$$

which is the Thomas-Reiche-Kuhn sum rule.

The sum rules (5.6) and (5.9) are the same as those for natural optical activity and we will see that they are related to the average refractive index. Using $\mathrm{Re}\beta(\omega)=O(\omega^{-3})$ for $\omega\to\infty$ [cf. (3.8)], we apply the superconvergence theorem to (5.4) in the form (B2); after making $y=\omega^2$ we find that

$$\mathrm{Im}\beta(\omega)=\frac{1}{\pi}\frac{1}{\omega^2}\int_0^\infty \omega'\mathrm{Re}\beta(\omega')d\omega'+O(\omega^{-3})\ . \tag{5.10}$$

However, for this expression to be compatible with (3.9) we need to have

$$\int_0^\infty \omega'\mathrm{Re}\beta(\omega')d\omega'=0\ . \tag{5.11}$$

From (5.10) and (5.11) it follows that for $\omega\to\infty$

$$\mathrm{Im}\beta(\omega)=O(\omega^{-3})\ . \tag{5.12}$$

Finally, with the use of (5.12) and the superconvergence theorem in the form (B2), (5.3) can be written for $\omega\to\infty$ as

$$\mathrm{Re}\beta(\omega)=-\frac{1}{\pi}\frac{1}{\omega}\int_0^\infty \mathrm{Im}\beta(\omega')d\omega'+O(\omega^{-3})\ , \tag{5.13}$$

and comparing this expression with (3.8) we get

$$\int_0^\infty \mathrm{Im}\beta(\omega)d\omega=0\ . \tag{5.14}$$

So, using the superconvergence theorem and the high-frequency behavior we were able to derive some sum rules, but we can also obtain other sum rules in the limit of $\omega=0$.

B. Sum rules in the limit of $\omega=0$

Let us go back to (5.1)—(5.4) and analyze these relations in the limit of $\omega=0$. From (5.2) and (5.3) in the limit of $\omega=0$ we have

$$\mathrm{Im}\epsilon(\omega)=0 \tag{5.15}$$

and

$$\mathrm{Re}\beta(\omega)=0\ . \tag{5.16}$$

And from (5.1) for $\omega=0$ we have

$$\mathrm{Re}[\epsilon(0)-1]=\frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{\mathrm{Im}\epsilon(\omega')}{\omega'}d\omega'\ , \tag{5.17}$$

and in the limit of $\omega=0$, (5.4) gives

$$\mathrm{Im}\beta(0)=-\frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{\mathrm{Re}\beta(\omega')}{\omega'}d\omega'\ . \tag{5.18}$$

Now we are going to rewrite the sums that we got in terms of the real and imaginary parts of the average refractive index, and of the real and imaginary parts of the complex rotatory angle per unit length.

C. Sum rules involving only the average refractive index

From (2.5) and (2.6) we have

$$\epsilon(\omega)=\bar{n}^2(\omega)-\bar{\kappa}^2(\omega)+2i\bar{\kappa}(\omega)\bar{n}(\omega)\ , \tag{5.19}$$

where

$$\bar{n}(\omega)=\tfrac{1}{2}[n_+(\omega)+n_-(\omega)] \tag{5.20}$$

and

$$\bar{\kappa}(\omega)=\tfrac{1}{2}[\kappa_+(\omega)+\kappa_-(\omega)] \tag{5.21}$$

are the average real refractive index and average extinction coefficient, respectively.

Using the previous expressions relating $\epsilon(\omega)$ and the average refractive index, we rewrite the relations (5.6), (5.9), and (5.17) as

$$\int_0^\infty [\bar{n}^2(\omega)-1]d\omega=\int_0^\infty \bar{\kappa}^2(\omega)d\omega\ , \tag{5.22}$$

$$\int_0^\infty \omega\bar{n}(\omega)\bar{\kappa}(\omega)d\omega=\frac{\eta\pi^2e^2}{m}\ , \tag{5.23}$$

and

$$\mathrm{P}\int_0^\infty \frac{\bar{n}(\omega)\bar{\kappa}(\omega)}{\omega}d\omega=\frac{\pi}{4}[\bar{n}^2(0)-1]\ , \tag{5.24}$$

where in (5.24) we used (5.15) and (5.19) such that for $\omega=0$ we have

$$\bar{\kappa}(0)=0\ . \tag{5.25}$$

As in the case of natural optical activity,[1] it is the average refractive index that satisfies the same sum rules for a medium without birefringence phenomena.[5] It is important to remember that we are considering insulating media, so that $\epsilon(\omega)$ does not have a singularity at $\omega=0$.

D. Sum rules involving Faraday rotation

Let us rewrite the sum rules for $\beta(\omega)$ in terms of the average refractive index and the Faraday rotation. From (2.5) to (2.8) we have

$$\beta(\omega)=\frac{-2c}{B_0\omega}\{[\bar{n}(\omega)\phi_F(\omega)-\bar{\kappa}(\omega)\theta(\omega)] +i[\bar{n}(\omega)\theta(\omega)+\bar{\kappa}(\omega)\phi_F(\omega)]\}\ , \tag{5.26}$$

where $\bar{n}(\omega)$ and $\bar{\kappa}(\omega)$ are given by (5.20) and (5.21), respectively.

From (5.26) we see that $\beta(\omega)$ is not solely a function of complex rotation angle per unit length as in the case of natural optical activity.[1] In the Faraday effect $\beta(\omega)$ is proportional to the product of average refractive index with the rotation angle (including ellipticity).

Thus (5.11), (5.14), and (5.18) can be rewritten as

$$\int_0^\infty \bar{n}(\omega)\phi_F(\omega)d\omega=\int_0^\infty \bar{\kappa}(\omega)\theta(\omega)d\omega \ , \tag{5.27}$$

$$\int_0^\infty \frac{\bar{\kappa}(\omega)\phi_F(\omega)}{\omega}d\omega=-\int_0^\infty \frac{\bar{n}(\omega)\theta(\omega)}{\omega}d\omega \ , \tag{5.28}$$

and

$$\lim_{\omega\to 0}\frac{\bar{n}(\omega)\theta(\omega)+\bar{\kappa}(\omega)\phi_F(\omega)}{\omega}=-\frac{2}{\pi}\int_0^\infty \frac{\bar{n}(\omega')\phi_F(\omega')-\bar{\kappa}(\omega')\theta(\omega')}{\omega'^2}d\omega' \ . \tag{5.29}$$

These sum rules are new even though they could be obtained by multiplying the functions that Smith[3] used in obtaining his sum rules. However, the main point is that Smith *assumed* that dispersion relations are satisfied by a mixture of right- and left-handed polarized complex refractive indices, and we *showed* that this mixture is a consequence of causality principle and the reality of the fields (see Appendix C).

As Smith pointed out, we can derive new sum rules for powers of the functions $\beta(\omega)$ and $\epsilon(\omega)$ by multiplying them by polynomials of ω. However, to continue being able to apply Titchmarsh's theorem we need to be sure that the new functions of $\epsilon(\omega)$ and $\beta(\omega)$ are analytic in the upper half complex plane and square integrable. Once we have proved that $\beta(\omega)$ and $[\epsilon(\omega)-1]$ are analytic functions in the upper half plane, the products will be analytic functions also; we must only be careful to not introduce singularities.

VI. JUSTIFICATION FOR THE KNOWN SUM RULES

We have proved that $\epsilon(\omega)$ is an analytic function in the upper complex plane, but the relation between this function and the complex refractive index is given by [cf. (2.5)]

$$\bar{N}(\omega)=[\epsilon(\omega)]^{1/2} \ , \tag{6.1}$$

where

$$\bar{N}(\omega)=\tfrac{1}{2}[N_+(\omega)+N_-(\omega)] \ . \tag{6.2}$$

From (6.1) we see that $\bar{N}(\omega)$ can have a branch point if $\epsilon(\omega)$ vanishes for any value of ω. Our treatment up to now has not given any information on whether $\epsilon(\omega)$ has zeros or not.

Thomaz and Nussenzveig[1] showed in the case of natural optical activity, using relativistic causality[13] that the functions $N_\pm(\omega)-1$ are analytic functions in the upper half plane. In the proof, only the fact that different circular polarizations have different complex refractive indices was used. As this is the case in the Faraday effect, the proof of analyticity of the functions $N_\pm(\omega)-1$ is also valid for the Faraday effect.

So by relativistic causality $N_\pm-1$ is analytic in the upper half complex plane. By (6.1) and (3.6) we have that $N_\pm(\omega)-1$ is a square integrable function, and so it satisfies Titchmarsh's theorem, and the same is true for the average complex refractive index $\bar{N}(\omega)-1$. Then the real and imaginary parts satisfy the two Plemelj formulas (see Appendix A). This is the justification for the dispersion relations that Smith used.

By what was said before, $N_\pm(\omega)-1$ also satisfies the two Plemelj formulas, and we can write the first Plemelj formula for $N_+(\omega)-1$ as

$$n_+(\omega)-1=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\kappa_+(\omega')}{\omega'-\omega}d\omega' \ . \tag{6.3}$$

When we restrict the integral to the positive region of integration, we need the crossing relation for $\kappa_+(\omega)$, and from (C5) we have

$$\kappa_+(-\omega)=-\kappa_-^*(\omega) \ , \tag{6.4}$$

so in the sum rules that we will obtain both types of extinction coefficient will appear.

Since $\bar{N}(\omega)$ is an analytic function, $\epsilon(\omega)$ cannot vanish at any point in the upper half complex plane and as a consequence of this we have that [cf. (2.5) and (2.7)]

$$\Theta(\omega)=-\frac{\omega}{2c}B_0\frac{\beta(\omega)}{[\epsilon(\omega)]^{1/2}} \tag{6.5}$$

is also an analytic function in the upper complex plane, because $\beta(\omega)$ and $[\epsilon(\omega)]^{1/2}$ are analytic functions and $[\epsilon(\omega)]^{1/2}$ does not have any zero in this plane. As $\Theta(\omega)$ is a square integrable function by the asymptotic behavior of $\beta(\omega)$ and $\epsilon(\omega)$, so, by Titchmarsh's theorem, its real and imaginary parts satisfy the two Plemelj formulas.

VII. FINAL REMARKS

As in the case of natural optical activity[1] the application of the principle of causality to the constitutive equations together with the behavior of free electrons in the limit of $\omega\to\infty$ permits us to obtain, through Titchmarsh's theorem, the analytic behavior of the optical constants that characterize the reaction of the medium to the external fields as well as the dispersion relations involving those constants, in our specific case $\beta(\omega)$ and $[\epsilon(\omega)-1]$. Through the relations among $\beta(\omega)$, $[\epsilon(\omega)-1]$, $\bar{N}(\omega)$, and $\Theta(\omega)$, we were able to derive dispersion relations and sum rules involving functions of the real and imaginary parts of the last two. This is really the point that differentiates this work from that of Smith,[3] because we did not need to make any assumptions to derive the dispersion relations for $\bar{N}(\omega)$ and $\Theta(\omega)$ as was done by Smith. Smith started from the assumption that $\bar{N}(\omega)$ and $\Theta(\omega)$ satisfy the dispersion relations directly.

As we saw, the mixture of the two types of complex refractive indices in the sum rules comes from the fact that we cannot write a real field as a superposition of only one type of circularly polarized waves (cf. Appendix C). We would like to mention that in order to derive the sum rules from the superconvergence theorem we did not need any extra assumption about the asymptotic behavior of the $\beta(\omega)$ and $[\epsilon(\omega)-1]$ given in (3.6)–(3.9). Finally, now that we have information about the analytic properties of $\bar{N}(\omega)$, $\Theta(\omega)$, $\beta(\omega)$, and $[\epsilon(\omega)-1]$ and their asymptotic behavior, we can get new functions of these quantities that satisfy the conditions of Titchmarsh's theorem and from them derive new dispersion relations and new sum rules.

ACKNOWLEDGMENTS

I would like to thank M. Ebel, S. Epstein, C. J. Goebel, and R. W. Robinett for useful suggestions and for reading the manuscript, and M. Ebel and the Theory Group at the University of Wisconsin—Madison for their hospitality during my stay. I also thank the Fundação de Amparo a Pesquisa do Estado de São Paulo (FAPESP) for financial support.

APPENDIX A

The following is Titchmarsh's theorem.[11] If a square integrable function $G(\omega)$ fulfills one of the four conditions below, then it fulfills all four of them:

(i) The inverse Fourier transform $g(t)$ of $G(\omega)$ vanishes for $t<0$:

$$g(t)=0 \quad \text{for } t<0 . \tag{A1}$$

(ii) $G(u)$ is, for almost all u, the limit as $v \rightarrow 0+$ of an analytic function $G(u+iv)$ that is holomorphic in the upper half plane and square integrable over any line parallel to the real axis:

$$\int_{-\infty}^{\infty} |G(u+iv)|^2 du < C \text{ for } v>0 . \tag{A2}$$

(iii) $\mathrm{Re}G(\omega)$ and $\mathrm{Im}G(\omega)$ verify the first Plemelj formula:

$$\mathrm{Re}G(\omega)=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}G(\omega')}{\omega'-\omega}d\omega' . \tag{A3}$$

(iv) $\mathrm{Re}G(\omega)$ and $\mathrm{Im}G(\omega)$ verify the second Plemelj formula:

$$\mathrm{Im}G(\omega)=-\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}G(\omega')}{\omega'-\omega}d\omega' . \tag{A4}$$

APPENDIX B

The following is the superconvergence theorem.[4,5] Let $g(y)$ be defined by

$$g(y)=\mathrm{P}\int_0^{\infty}\frac{f(x)}{y-x}dx ,$$

where $f(x)$ is a continuous differentiable function that goes to zero faster than x^{-1} for $x \rightarrow \infty$. Let us consider the following alternative asymptotic behaviors for $f(x)$ and its derivatives:

$$f(x)=O(x^{-1}\ln^{-\alpha}x) , \quad f'(x)=O(x^{-2}\ln^{-\alpha}x) , \quad \alpha>1 \tag{B1}$$

$$f(x)=O(x^{-\beta}) , \quad f'(x)=O(x^{-\beta-1}) , \quad 1<\beta<2 . \tag{B2}$$

So, for $y \rightarrow \infty$ we have first,

$$g(y)=\frac{1}{y}\int_0^{\infty}f(x)dx+O(y^{-1}\ln^{1-\alpha}y) ,$$

if (B1) is valid, and second,

$$g(y)=\frac{1}{y}\int_0^{\infty}f(x)dx+O(y^{-\beta}) ,$$

if (B2) is valid.

APPENDIX C

In the medium in the presence of the weak external magnetic field $\vec{\mathrm{B}}_0$, right- and left-hand circularly polarized waves propagate differently. For any field $\vec{\mathrm{E}}(t)$ in this medium its Fourier transformation can be written in terms of a circularly polarized wave, that is,

$$\begin{aligned}\vec{\mathrm{E}}(\vec{\mathrm{x}},t)=\int_{-\infty}^{\infty}\{&e_-(\omega)(\hat{x}+i\hat{y})\exp[iN_-(\omega)\omega z/c]\\ &+e_+(\omega)(\hat{x}-i\hat{y})\exp[iN_+(\omega)\omega z/c]\}\\ &\times e^{-i\omega t}d\omega ,\end{aligned} \tag{C1}$$

where $\vec{\mathrm{E}}$ is propagating parallel to $\vec{\mathrm{B}}_0$ (which was chosen along the z direction). We have that $e_+(\omega)$ $[e_-(\omega)]$ is the amplitude of frequency ω for left (right) circularly polarized waves, and $N_+(\omega)$ $[N_-(\omega)]$ is the complex refractive index for left (right) circularly polarized waves. Equation (4.1) is shorthand for (C1).

Taking the complex conjugate of (C1), using the fact that $\vec{\mathrm{E}}(\vec{\mathrm{x}},t)$ is a real quantity, and the independence of the Fourier components for x and y components of the previous expression, we get the following relations:

$$\begin{aligned}&e_-(\omega)\exp[iN_-(\omega)\omega z/c]+e_+(\omega)\exp[iN_+(\omega)\omega z/c]\\ &\quad=[e_-(-\omega)]^*\exp[iN_-^*(-\omega)\omega z/c]\\ &\qquad+[e_+(-\omega)]^*\exp[iN_+^*(-\omega)\omega z/c]\end{aligned} \tag{C2}$$

and

$$\begin{aligned}&e_-(\omega)\exp[iN_-(\omega)\omega z/c]-e_+(\omega)\exp[iN_+(\omega)\omega z/c] ,\\ &\quad=-[e_-(-\omega)]^*\exp[iN_-^*(-\omega)\omega z/c]\\ &\qquad+[e_+(-\omega)]^*\exp[iN_+^*(-\omega)\omega z/c] ,\end{aligned} \tag{C3}$$

where z is any point in the medium.

From the above equations we get

$$e_-(-\omega)=[e_+(\omega)]^* \tag{C4}$$

and

$$N_+(-\omega)=N_-^*(\omega) . \tag{C5}$$

From (C4) and (C5) we see that it is not possible to have a real field constructed as superposition of only one type of circularly polarized waves.

[1]M. T. Thomaz and H. M. Nussenzveig, Ann. Phys. (N.Y.) 139, 14 (1982).

[2]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media,* 1st English ed. (Pergamon, New York, 1960), Vol. 8.

[3]D. Y. Smith, Phys. Rev. B 13, 5303 (1976); J. Opt. Soc. Am. 66, 454 (1976).

[4]E. U. Condon, Rev. Mod. Phys. 9, 432 (1937).

[5]M. Altarelli, D. L. Dexter, H. M. Nussenzveig, and D. Y. Smith, Phys. Rev. B 6, 4502 (1972).

[6]The complex refractive index is given by the relation $k=N(\omega)\omega/c$, where k is the complex wave vector, ω is the frequency, and c is the velocity of light.

[7]C. Graham and R. E. Raab, Proc. Phys. Soc. London 90, 417 (1967).

[8]It makes sense to characterize the medium through optical constants only when the wavelength of the field is large comparable to the characteristic dimension of the molecules, but when we have $\omega \rightarrow \infty$ this implies that $\lambda \rightarrow 0$, so we must be careful about what we mean by the high-frequency limit.

[9]We are not considering the case of a metal, so the function $\epsilon(\omega)$ is not singular (Ref. 1) at $\omega=0$.

[10]The notation $g(x)=O(f(x))$ as $x\rightarrow\infty$ means that $|g(x)|/|f(x)|$ remains bounded as $x\rightarrow\infty$, and $g(x)=o(f(x))$ as $x\rightarrow\infty$ means that $\lim_{x\rightarrow\infty}[g(x)/f(x)]=0$.

[11]Quoted from H. M. Nussenzveig, *Causality and Dispersion Relations* (Academic, New York, 1972).

[12]Note that the first term on the rhs is independent of the external magnetic field $\vec{B}_0$, and so itself must be real. As the sum of the rhs of (4.3) must be real, the second term must also be real.

[13]See Ref. 11 on p. 18.