Quantization of the Hall conductance in a two-dimensional electron gas

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We present a microscopic theory of the Hall conductance in a two-dimensional electron gas. Our approach is based on a single-particle picture and explicitly accounts for the effects of a random impurity potential. Within the geometry introduced by Laughlin a general expression is derived from which it is possible to evaluate the Hall conductance in terms of the properties of the electronic spectrum at the Fermi energy for any value of the magnetic field. When the chemical potential lies between the bulk extended states of well-defined neighboring Landau bands, the Hall conductance is quantized in integral multiples of e^2/h, even in the presence of a large density of localized states. Within our model the exactness of this quantization depends on the shape of the confining potential, the thickness of the sample, and the magnetic field.

I. INTRODUCTION

One of the most interesting properties of the two-dimensional electron gas which can occur at a semiconductor interface is the quantization of the Hall conductance.[1] At very low temperature T and high magnetic field strength B, the Hall conductance σ_{xy} as a function of the Landau-level filling factor $\nu=n_S(eB/hc)^{-1}$, n_S being the number of electrons per unit area, is characterized by flat steps[2] at integral multiples of the fundamental value e^2/h. In those regions of concentration n_S in which σ_{xy} has the quantized value, σ_{xx} is essentially equal to zero (it would presumably be zero at zero temperature). This effect was first indicated by Ando, Matsumoto, and Uemura in a study of the effects of impurity centers on the properties of an otherwise noninteracting two-dimensional electron gas.[3]

This result suggests very strongly that if we think of the density of states associated with a particular Landau level as broadened by impurity scattering, extended states exist only very close to the center of the Landau level and localized states exist everywhere else. Establishing the validity of this picture from microscopic theory remains a fundamental unsolved problem. Work in this direction has been recently carried out by several authors.[4–6]

The quantized Hall effect has received particular attention within the single-particle picture in which a prominent role is played by the electronic states which are localized by the impurity random potential.[7–9] Within the same framework Kazarinov and Luryi have presented an argument based on quantum percolation theory.[10] Thouless and co-workers have investigated how the effect is influenced by the presence of a periodic substrate potential.[11]

The observation of a sizable cyclotron resonance shift in Si inversion layers[12] and of additional quantized steps in the Hall resistance at $3h/e^2$ and probably $3h/2e^2$ in GaAs-AlGaAs heterojunctions in the extreme quantum limit[13] casts, however, some doubts on the validity of the single-particle picture. The normal state of a two-dimensional electron gas in the presence of a magnetic field is inherently unstable with respect to a many-body charge-density-wave or Wigner-lattice type of ground state.[14] It is not clear, however, what happens in the presence of an impurity random potential and what the magnetotransport of such an exotic ground state would be. The implication of such many-body effects on the Hall conductance of an ideal two-dimensional electron gas have recently received a great deal of attention.[15–17]

In order to establish the relevance of the many-body effects in the quantized Hall-effect problem a complete and reliable theory based on the single-particle picture must be first at hand which can treat exactly the problem associated with the impurity random potential for any value of the external magnetic field.

Laughlin has presented an elegant argument which attempts to demonstrate that the quantization is due to the long-range phase-rigidity characteristic of a supercurrent, and that it can be derived from gauge invariance and the existence of a mobility gap.[8] He does this by considering the response of a two-dimensional metallic ribbon to a change in the flux threading the ribbon. Because changing the flux threading the ribbon is certainly not a simple gauge transformation, the terminology of Laughlin's argument is inappropriate.[18] Furthermore, his argument contains the implicit assumption that the only consequence of adding an integral number of flux quanta hc/e is to repopulate the current-carrying states. This assumption is obviously valid for the ideal system, but it is not so obvious in the presence of disorder when localized states can exist at the Fermi level.

In this paper we investigate, within a single-particle picture, the eigenfunctions, eigenvalues, and distribution functions of the electrons in a two-dimensional metallic ribbon using a cyclindrical coordinate system appropriate to the geometry of the problem. In Sec. II we consider the ideal system, free of impurities, and establish the notation. There it is stressed that it is strictly necessary to mix orbitals belonging to different Landau levels in order to be able to describe localized states. In Sec. III we introduce an effective random potential and discuss the structure of the electric spectrum. In particular we prove that in a rib-

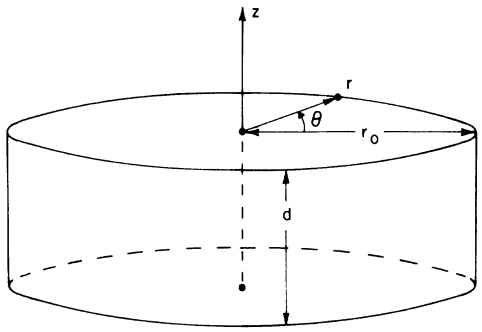

FIG. 1. Geometrical arrangement for the ribbon geometry used throughout the paper.

bon of finite radius the spectrum of the extended states (if any) is countable. This analysis is used in deriving a virtually exact expression for the Hall current which explicitly displays the dependence of this quantity on the temperature and the shape of the confining potential. The conditions for observing a quantization of the Hall conductance are discussed. Finally, Sec. IV contains a discussion of the relevant physical questions and the conclusions.

II. IDEAL METALLIC RIBBON

In this section we consider an idealized model in which the system of electrons is confined to an impurity free ribbon of radius r_0 as shown in Fig. 1. Cylindrical coordinates (r,θ,z) will be used to describe the motion of the electrons. The electrons are restricted to the radius $r=r_0$ by some confining potential, but they are free to move between the edges $(0\le z\le d)$ and around $(0\le\theta\le 2\pi)$ the ribbon. For large values of r_0 any small section of the ribbon will be indistinguishable from a similar small section of a standard Hall bar, in that a current I will flow in the θ direction and a voltage ΔV will be present across the ribbon. The ratio of I to ΔV will define the Hall conductance.

In order to mimic the behavior of a standard Hall bar, we want a magnetic field B which is everywhere perpendicular to the ribbon. We introduce a vector potential $\vec{A}=(A_r,A_\theta,A_z)$ in the cylindrical coordinate system in which $\hat{r}$, $\hat{\theta}$, and $\hat{z}$ are unit vectors, and choose $A_r=A_z=0$ and $A_\theta=-(Bz+A_1)$. Taking the curl of $\vec{A}$ leads to a magnetic field

$$\vec{B}=B\hat{r}-(Bz+A_1)\hat{z}/r\ .$$

Now $B\hat{r}$ is exactly radial magnetic field we want. The other term $r^{-1}(Bz+A_1)\hat{z}$ is extra. It does not bother us because the electrons are strictly confined to $r=r_0$ by a potential of the type

$$V(r)=0\begin{cases}0 & \text{if } r=r_0\\ \infty & \text{otherwise}\ .\end{cases} \tag{1}$$

Therefore, the Lorentz force assoicated with B_z will have no effect on the classical motion of the electrons. Of course, the constant vector potential A_1 just leads to "trapped flux" inside the ribbon which is independent of the magnetic field $B\hat{r}$, but does not affect the classical motion of the electrons.

The flux through the ribbon at a plane z is given by

$$\Phi(z)=\int_{\text{at } z}\vec{B}\cdot d\vec{a}=\oint_{\text{at } z}\vec{A}\cdot d\vec{l}=-2\pi r_0(Bz+A_1)\ . \tag{2}$$

We define $\phi=-2\pi r_0A_1$ as the trapped flux associated with A_1, and we introduce the set z_l of values of z defined by the equation

$$-2\pi r_0Bz_l+\phi=-l\phi_0\ , \tag{3}$$

where $\phi_0=hc/e$ is the flux quantum and l is an integer. Because A_θ depends on z, the flux passing through the circle defined by the intersection of the cylinder $r=r_0$ and the plane $z=$const depends upon z. The plane $z=z_l$ defines the circle through which $-l$ flux quanta pass.

Now let us look at the Hamiltonian. To start we will neglect boundary effects at $z=0$ and d. Because the electrons are confined to the radius $r=r_0$, the radial coordinate does not enter the Hamiltonian. We have

$$\begin{aligned}H_0&=\frac{1}{2m}p_z^2+\frac{1}{2m}\left[\frac{p_\theta}{r_0}+\frac{e}{c}A_\theta\right]^2,\\ &=\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+\frac{1}{2m}\left[\frac{-i\hbar}{r_0}\frac{\partial}{\partial\theta}+\frac{e}{c}A_\theta\right]^2,\end{aligned} \tag{4}$$

where for simplicity only the spatial degrees of freedom have been retained. Because A_θ is independent of θ we can assume that the eigenfunctions of H_0 are of the form

$$\psi(z,\theta)=e^{il\theta}u(z)\ , \tag{5}$$

where l is an integer. Substituting this form into the Schrödinger equation gives the equation

$$\left[\frac{p_z^2}{2m}+\tfrac{1}{2}m\omega_c^2(z-z_l)^2\right]u(z)=Eu(z)\ . \tag{6}$$

Equation (6) is just the Schrödinger equation for a simple harmonic oscillator of frequency $\omega_c=eB/mc$ centered at $z=z_l$; thus, the eigenfunctions and eigenvalues of H_0 can be written

$$\begin{aligned}\psi_{nl}(z,\theta)&=\frac{e^{il\theta}}{\sqrt{2\pi}}u_n(z-z_l)\ ,\\ E_{nl}&=\hbar\omega_c(n+\tfrac{1}{2})\ ,\end{aligned} \tag{7}$$

where $u_n(z)$ is the nth harmonic-oscillator eigenfunction. The quantum numbers n and l have here the following meaning. n labels a set of well-defined Landau levels whose energy spacing is given by $\hbar\omega_c$. l represents the angular momentum of the state. For a given n states with different values of l are degenerate. If the system has unit area the degeneracy of each Landau level is $(2\pi a_B^2)^{-1}$ where $a_B=(c\hbar/eB)^{1/2}$ is the magnetic length. This degeneracy is lifted by any potential term depending on the coordinate z added to the single-particle Hamiltonian (4).

Note that in (7) the "orbit-center" coordinate z_l is exactly the value given by Eq. (3); this value was determined by requiring the circle defined by $r=r_0, z=z_l$ to enclose $-l$ flux quanta. Changing the trapped flux by $\Delta\phi$ simply shifts each orbit center by $\Delta z_l=-\Delta\phi/2\pi r_0 B$. Any change $\Delta\phi$ is allowed; the electron orbits simply adjust themselves by shifting their centers to still enclose an integral number of flux quanta. In the absence of a potential which lifts the degeneracy, the eigenvalue E_{nl}, given by Eq. (7), is unchanged.

If we introduce a constant electric field $\vec{E}$ in the z direction (across the ribbon), there is an additional term eEz in the Hamiltonian. The eigenvalues and eigenfunctions of the ideal system in the presence of $\vec{E}$ are

$$\psi_{nl}(z,\theta)=\frac{e^{il\theta}}{\sqrt{2\pi}}u_n\left[z-z_l+\frac{v_D}{\omega_c}\right],$$
$$E_{nl}=\hbar\omega_c(n+\tfrac{1}{2})+eEz_l-\tfrac{1}{2}mv_D^2, \tag{8}$$

where the drift velocity v_D is defined as $v_D=cE/B$. If $\vec{E}\to 0$ these equations reduce to Eqs. (7). Clearly n and l are still good quantum numbers but the degeneracy with respect to l has been lifted. Notice that in the presence of an homogeneous applied electric field the orbitals $\psi_{nl}(z,\theta)$ are centered at z_l-v_D/ω_c, as compared to z_l in the field-free case.

It is worth mentioning here that the operator $\dot{\theta}=\partial H_0/\partial p_\theta$ can be written

$$(mr_0^2)^{-1}[p_\theta+(er_0/c)A_\theta],$$

where $A_\theta=-(Bz+A_1)$. Because p_θ is a constant of motion with value $\hbar l$, the Hall current I_H^{nl} carried by an electron in state $|nl\rangle$ is

$$I_H^{nl}=-e\langle nl\,|\,\dot{\theta}r_0\,|\,nl\rangle=e\omega_c\langle nl\,|\,(z-z_l)\,|\,nl\rangle=-ev_D\ . \tag{9}$$

It is straightforward to verify that

$$\langle nl\,|\,\dot{z}\,|\,nl\rangle=0 \tag{10}$$

so that no current is carried in the direction of the applied electric field. In (9) and (10), $|nl\rangle$ is the Dirac notation for the state whose wave function is $\psi_{nl}(z,\theta)$. In the absence of an external electric field no current flows in the system. If an electric field is present in the z direction, as in (8), all the electronic states $|nl\rangle$ carry the same Hall current, $-ev_D$, along $\hat{\theta}$. Equations (9) and (10) lead to the values $\sigma_{xx}=0$ for the longitudinal conductivity and

$$\sigma_{xy}=-(e^2N/h)(dr_0/a_B^2)^{-1},$$

where N is the total number of electrons.

It is interesting to realize that exactly the same results are obtained for any wave function $\phi_{nl}(z,\theta)$ of the type

$$\phi_{n\alpha}(z,\theta)=\sum_l c_{nl}^{\alpha}\psi_{nl}(z,\theta)\ . \tag{11}$$

This implies that it is strictly necessary to allow for the mixing of different Landau levels in order to describe a non-current-currying state in a magnetic field.[19]

A different and elegant procedure to evaluate the Hall current in this geometry has been proposed by Laughlin.[8] The approach is based on the following formula:

$$I_H=-c\frac{\Delta E_T}{\Delta\phi}\ , \tag{12}$$

where E_T is the total energy of the system and ΔE_T is the change in E_T caused by a change in flux $\Delta\phi$. This result can be obtained by noting that the current-density operator is

$$j_\theta^{\rm op}=-er_0\dot{\theta}=-er_0\frac{\partial H_0}{\partial p_\theta}=c\frac{\partial H_0}{\partial A_1}\ .$$

Taking the expectation value of $j_\theta^{\rm op}$ and summing over occupied states leads to Eq. (12). We can calculate E_T using perturbation theory. Let us write

$$H=H_0+\Delta H\ , \tag{13}$$

where

$$\Delta H=(e/c)v_\theta\Delta A_1=(e\,\Delta\phi/2\pi c)\dot{\theta}$$

is the change in the Hamiltonian caused by a small change in A_1 (or in the flux $\Delta\phi=-2\pi r_0\Delta A_1$). Let us think of ΔA_1 as varying in time as $\exp(i\omega t)$, where ω is a very low frequency. Then the perturbation ΔH causes a change in the single-particle density matrix $\Delta\rho=\rho-\rho_0$, where ρ_0 is the density matrix in the absence of ΔH. The use of linear-response theory gives

$$\langle nl\,|\,\Delta\rho\,|\,n'l'\rangle=\frac{f_0(\epsilon_{n'l'})-f_0(\epsilon_{nl})}{\epsilon_{n'l'}-\epsilon_{nl}-\hbar\omega}\langle nl\,|\,\Delta H\,|\,n'l'\rangle\ , \tag{14}$$

where $f_0(\epsilon)$ is the usual Fermi-occupation function. The change in the total energy can be written

$$\Delta E=\mathrm{Tr}(\rho H-\rho_0 H_0)\ . \tag{15}$$

Keeping terms linear in the small perturbation gives

$$\Delta E_T=\sum_{n,l}[f_0(\epsilon_{nl})\langle nl\,|\,\Delta H\,|\,nl\rangle+\epsilon_{nl}\langle nl\,|\,\Delta\rho\,|\,nl\rangle]\ . \tag{16}$$

It is clear from Eq. (14) that the second term vanishes; therefore

$$\Delta E_T=+\frac{e\,\Delta\phi}{2\pi c}\sum_{n,l}f_0(\epsilon_{nl})\langle nl\,|\,\dot{\theta}\,|\,nl\rangle=\frac{eNv_D\Delta\phi}{2\pi r_0 c}\ , \tag{17}$$

where we have made use of Eq. (9). Also $\sum_{n,l}f_0(\epsilon_{nl})=N$ since, as noticed above, every state $|nl\rangle$ carries the same current. Another way to obtain this result is to notice that a change $\Delta\phi$ in the threading flux displaces each orbit center by the same amount, $\Delta z_l=-\Delta\phi/B$. From Eqs. (8) this leads to a change $eE\Delta\phi/B$ in the single-particle energy. Summing over all the occupied states the expression (17) for ΔE_T is recovered.[20]

Inserting (17) in (12) we get again

$$I_H=-eNv_D=-\frac{ecN}{B}E\ . \tag{18}$$

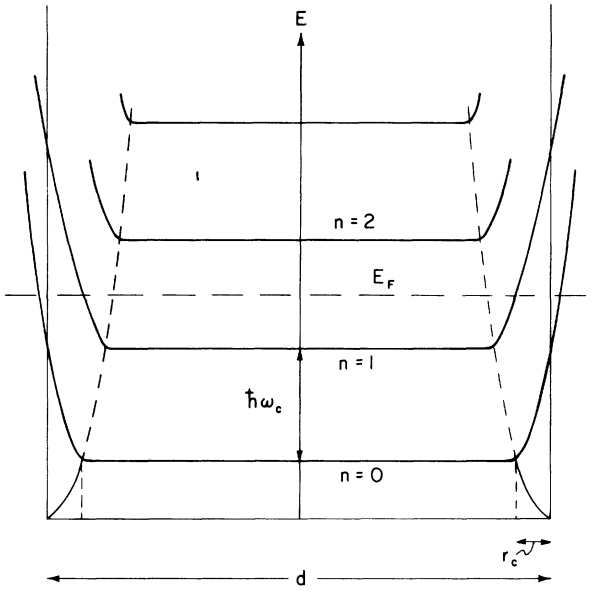

FIG. 2. Schematic energy spectrum for an ideal two-dimensional electron gas confined on a ribbon of width d. In the bulk region the spectrum consists of Landau levels separated by an energy $\hbar\omega_c$. r_c represents the cyclotron orbit radius for the $n=0$ Landau level.

In the case in which an integral number n of Landau levels are fully occupied we can write $N=nr_0d/a_B^2$ and get

$$I_H=-n\frac{e^2}{h}Ed\ , \tag{19}$$

which amounts to the well-known expression for the quantized Hall conductivity

$$\sigma_{xy}=-n\frac{e^2}{h}\ ,\quad n=0,1,2,3,\ldots\ . \tag{20}$$

It is obvious that Eqs. (18)–(20) do not provide a theory for the quantized Hall effect since within this extremely simplified model σ_{xy} is just a monotonic function of both B and the number of electrons N. The quantized expression (20) is recovered only for a set of values of N (or B) of zero measure.

Consider next the problem associated with the edge of the system. The electrons in our metallic ribbon must be confined to the region $0\le z\le d$ by some potential. If we assume that there is an infinite potential barrier at these positions, then the Landau-level energies will increase as the orbit center comes within the cyclotron radius of the wall. The energy as a function of z_l, the position of the orbit center, is sketched in Fig. 2 for the ideal case with $\vec{E}=0$. Note that the $n=0$ level with $z_l=0$ and d has the energy $(\frac{1}{2})\hbar\omega_c$, the value of the $n=1$ level in the bulk. The reason for this is, of course, that for the potential appropriate to these orbit centers, only the odd eigenfunctions of the full harmonic-oscillator potential inside the ribbon are solutions to the Schrödinger equation. Note that for orbit centers outside the ribbon the energy continues to increase approximately quadratically. These states represent "skipping orbits" or edge states. The very presence of the edge states leads to a smooth variation of the Fermi energy between the bulk Landau levels as they provide a nonvanishing density of states in such regions. The role and the physical properties of the edge states of a two-dimensional electron gas in a Corbino disk have been discussed by Halperin.[21] His treatment can be easily repeated for our cylindrical geometry and all his conclusions apply also in the present case.

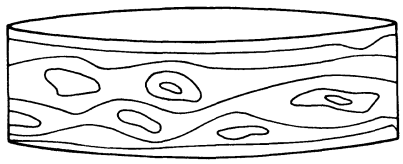

FIG. 3. Schematic of the classical orbits for extended and localized states in the ribbon geometry. Extended states are associated with classical orbits which encircle any flux threading the ribbon.

III. METALLIC RIBBON WITH IMPURITIES

A. Structure of the energy spectrum

Before attempting to evaluate the Hall current a discussion on the nature and the structure of the spectrum of the system in this case is in order. We start with the Hamiltonian

$$H=H_0+ezE+V(z,\theta)\ , \tag{21}$$

where H_0 is given by Eqs. (4) and where $V(z,\theta)$ is the random potential caused by the impurities. As V explicitly depends upon z and θ in general; n and l are not good quantum numbers, which is to say, the set of wave functions ψ_{nl} of Eqs. (8) does not represent a set of eigenstates of H. Accordingly we will introduce a new index (or set of indices) α in order to label the wave functions ψ_α and eigenvalues E_α of the system.

The precise nature of the electronic states of a system in the presence of a random potential has not yet been elucidated. In the absence of an externally applied magnetic field it is believed on the basis of scaling arguments that all the states must be localized.[22] However, when a uniform magnetic field is applied to the system the situation is different and it has been argued by several authors that extended states exist at the center of each Landau level.[4–6,21] We will assume here that this is the case.[23] Accordingly we will divide the states ψ_α into extended and localized states. The particular geometry of our system allows us a quite natural distinction between localized and extended states. Following an analysis similar to that used by Kazarinov and Luryi,[10] we can define the extended states as the ones associated with classical electronic orbits which circle the entire ribbon, whereas the localized states do not, as schematically pictured in Fig. 3. More precisely a state $\psi_{\alpha e}$ is extended if for any value of θ between 0 and

2π there exists a value of z between 0 and d such that $|\psi_{\alpha e}|^2$ goes to zero such as r_0^{-1} as r_0, the radius of the ribbon, goes to infinity. A state $\psi_{\alpha L}$ will instead be localized at z^*,θ^* if $|\psi_{\alpha L}(z^*,\theta^*)|^2$ is of the order r_0^{-1}, whereas for any value of z, $|\psi_{\alpha L}(z,\theta^*+\pi)|^2$ goes exponentially to zero with r_0 as r_0 is made to grow.

We shall now give an argument to show that even in the presence of impurities for a ribbon of finite size the energy eigenvalues of the extended states are isolated and therefore constitute a countable set. Let $\psi_{\alpha e}^{\phi}$ and $E_{\alpha e}^{\phi}$ be the wave function and the energy of a given extended eigenstate αe of H^{ϕ}, which is the Hamiltonian of the system, Eq. (21), where the "trapped flux" associated with A_1 is ϕ [see Eqs. (2) and (3)]. Suppose now that the flux is adiabatically changed to $\phi+\Delta\phi$, with $\Delta\phi<\phi_0$. Since $\psi_{\alpha e}$ is associated with an orbit linked to the flux change an electron in this state will experience an induced emf and will respond to the perturbation. Accordingly $\psi_{\alpha e}^{\phi}$ and $E_{\alpha e}^{\phi}$ will be mapped into $\psi_{\alpha e}^{\phi+\Delta\phi}$ and $E_{\alpha e}^{\phi+\Delta\phi}$ which in principle can be obtained by solving the Schrödinger equation for $H^{\phi+\Delta\phi}$.[24] In the ideal case, where $V=0$, the result of such a flux change is readily established. If we start from a state $\phi_{nl}^{\phi},E_{nl}^{\phi}$ as given in Eqs. (8), it is easy to establish that two possible solutions of the problem are possible,

$$\psi_{n,l}^{\phi},E_{nl}^{\phi}\rightarrow\psi_{nl}^{\phi+\Delta\phi}=e^{i\eta\theta}\psi_{n,l}^{\phi},E_{nl}^{\phi+\Delta\phi}=E_{nl}^{\phi}\ , \tag{22}$$

corresponding to an increment of angular momentum by $\hbar\eta$, or

$$\psi_{nl}^{\phi},E_{nl}^{\phi}\rightarrow\psi_{nl}^{\phi+\Delta\phi}=\tilde{\psi}_{nl}^{\phi},E_{nl}^{\phi+\Delta\phi}=E_{n}^{\phi+\Delta\phi}-\frac{\eta h v_D}{2\pi r_0}\ . \tag{23}$$

Here

$$\tilde{\psi}_{nl}^{\phi}=(2\pi)^{-1/2}\exp(il\theta)u_n\left[\frac{z-z_l+v_D}{\omega_c+\eta a_B^2/r_0}\right]$$

corresponds to a rigid displacement of the orbit center by exactly the amount required to maintain an integral number of flux quanta threading its orbits. In (22) and (23), $\eta=\Delta\phi/\phi_0$. Because the wave function must be single valued, it is clear that for $\eta<1$, the solution (22) is not acceptable and the response of the system will be characterized by the orbit-shifting process of Eq. (23). Notice that this implies that E_{nl}^{ϕ} does not belong to the spectrum of $H^{\phi+\Delta\phi}$. An analogous phenomenon occurs in the system in the presence of impurities. Although $\exp(i\eta\theta)\psi_{\alpha e}^{\phi}$ is an eigenfunction of $H^{\phi+\Delta\phi}$ with eigenvalue $E_{\alpha e}^{\phi}$, such a solution is not acceptable for nonintegral η since it is not a single-valued function of θ. Thus, $E_{\alpha e}^{\phi}$ does not belong to the spectrum of $H^{\phi+\Delta\phi}$.[25] Therefore, the energy of the state αe will change with ϕ. Furthermore, due to the random nature of $V(z,\theta)$, it is reasonable to expect that for very small values of η, $E_{\alpha e}^{\phi+\Delta\phi}$ will be smaller than $E_{\alpha e}^{\phi}$ by an amount linear in η. This process is exemplified in Fig. 4(a). It is obvious that here the change in $\psi_{\alpha e}^{\phi}$ will be much more complicated than a simple rigid shift of the center of the orbit since the potential $V(\theta,z)$ has a complicated local structure. Now, since $\psi_{\alpha e}^{\phi+\Delta\phi}$ in an eigenfunction of $H^{\phi+\Delta\phi}$, the wave function $\exp(-i\eta\theta)\psi_{\alpha e}^{\phi+\Delta\phi}$ is an eigenfunction of H^{ϕ} with eigenenergy $E_{\alpha e}^{\phi+\Delta\phi}$. The fact

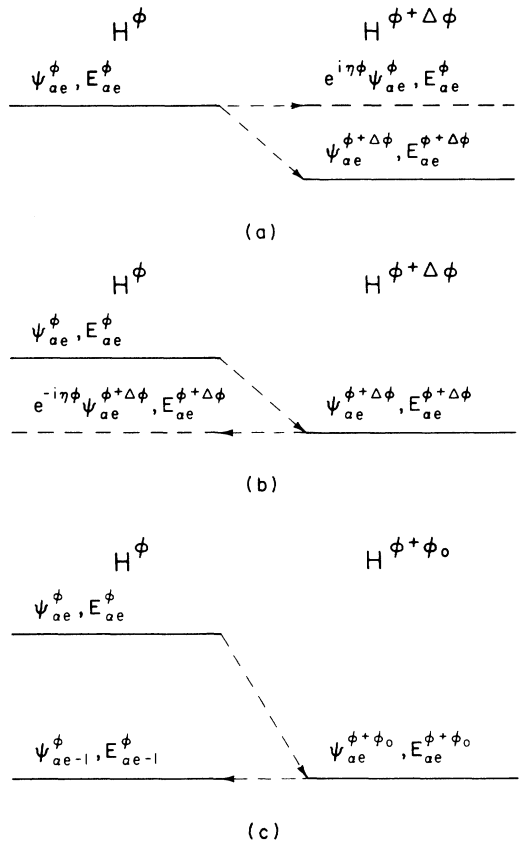

FIG. 4. Evolution of an extended state level $E_{\alpha e}$ (with wave function $\psi_{\alpha e}$) upon an adiabatic change $\Delta\phi$ of the threading flux. $\eta=\Delta\phi/\phi_0$. (a) $\eta<1$, $\exp(i\eta\theta)\psi_{\alpha e}^{\phi}$ is not an acceptable wave function; (b) $\eta<1$, $\exp(-i\eta\theta)\psi_{\alpha e}^{\phi+\Delta\phi}$ is not an acceptable wave function and $E_{\alpha e}$ is isolated; (c) $\eta=1$, $E_{\alpha e}^{\phi+\phi_0}=E_{\alpha e-1}^{\phi}$ as $\psi_{\alpha e}^{\phi+\phi_0}=\exp(i\theta)\psi_{\alpha e-1}^{\phi}$.

that $\psi_{\alpha e}^{\phi+\Delta\phi}$ is an acceptable (i.e., single-valued) solution implies that $\exp(-i\eta\theta)\psi_{\alpha e}^{\phi+\Delta\phi}$ is not and that the eigenvalue $E_{\alpha e}^{\phi+\Delta\phi}$ does not belong to the spectrum of H^{ϕ}. Since the flux change $\Delta\phi$ is completely arbitrary we conclude that for a ribbon of finite size the eigenvalues $E_{\alpha e}^{\phi}$ of H^{ϕ} are isolated and the spectrum of the extended states of the system is countable. This situation is represented in Fig. 4(b).

If $\eta=1$, i.e., $\Delta\phi=\phi_0$, then $\exp(-i\theta)\psi_{\alpha e}^{\phi+\phi_0}$ is an acceptable solution of H^{ϕ} with eigenenergy $E_{\alpha e}^{\phi+\phi_0}$, which by definition is nothing other than $\psi_{\alpha e-1}^{\phi}$ [see Fig. 4(c)]. The spectrum of $H^{\phi+m\phi_0}$, m being any integer, is the same as the one of H^{ϕ}; the wave functions simply differ by the phase factor $\exp(im\theta)$. We have established that by changing the trapped flux ϕ by one flux quantum each extended state $\psi_{\alpha e}$ is mapped into its nearest-neighboring $\psi_{\alpha e-1}$, with

$$E_{\alpha e}^{\phi+\phi_0}=E_{\alpha e-1}^{\phi}\ . \tag{24}$$

This expression will be useful later.

It is obvious that formally the same argument can be

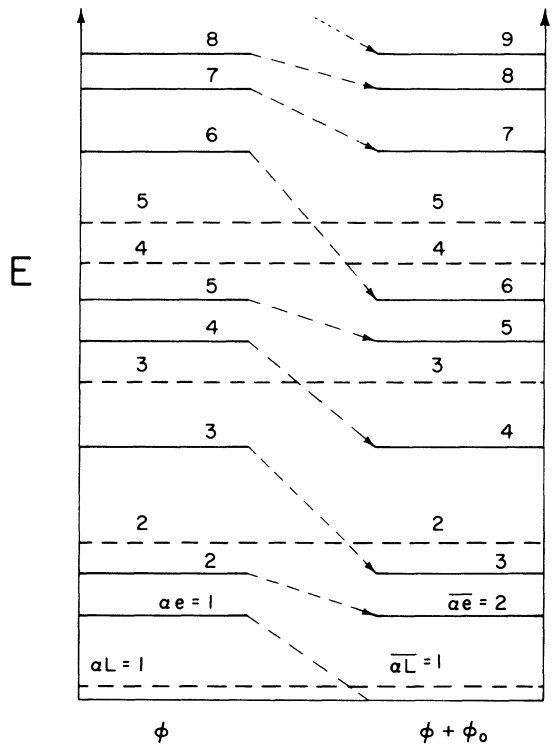

FIG. 5. Schematic illustration of how the spectrum of the system is modified by an adiabatic change of the flux threading the ribbon. αL and $\bar{\alpha L}$ label the localized states of H^{ϕ} and $H^{\phi+\phi_0}$; αe and $\bar{\alpha} e$ are the corresponding labels for the extended states. After a whole quantum of flux ϕ_0 has been added, the states αe are mapped into $\overline{\alpha e}=\alpha e-1$. The localized states are unaffected by the flux change.

carried out for a localized state $\psi_{\alpha L}$. However, the nature of a localized wave function is such that this electronic state will not be significantly modified by any change of trapped flux. This amounts to the fact that it is only for the localized states that a change in the trapped flux ϕ has the same effect of a gauge transformation. The situation for both extended and localized states is schematically pictured in Fig. 5.

If the external magnetic field is large enough the spectrum of the extended states of the system will still be characterized by a Landau-level structure. Each Landau level will, however, be broadened by the effect of the random potential. In such a case the label αe can be resolved into a Landau band index ν and a generalized orbit-position quantum number λ. With this notation Eq. (24) can be written as

$$E_{\nu\lambda}^{\phi+\phi_0}=E_{\nu,\lambda-1}^{\phi} , \tag{25}$$

where it has been recognized that for a macroscopic system ϕ_0 is a very small change of flux and cannot possibly induce a change of the Landau band index ν.

A word is in order here about the occupation of the extended states while the trapped flux ϕ is changed. If the step-by-step change in ϕ is adiabatic, we can make use of time-dependent perturbation theory as done in Sec. II. By a direct inspection of Eq. (14) (valid also in the presence of the potential V), it is easily realized that in this situation the occupation of the states does not change. If an electron occupies an extended state $\psi_{\nu\lambda}^{\phi}$ after a full quantum of flux ϕ_0 has been added to ϕ, the state $\psi_{\nu\lambda}^{\phi+\phi_0}$, i.e., $\psi_{\nu,\lambda-1}^{\phi}$ will be occupied. This is the transfer process originally suggested by Laughlin.[8] However, notice that for the extended states the flux change by no means can be thought of as a gauge transformation.

B. Hall current

Within our single-particle picture the Hall current I_H can be evaluated as follows:

$$I_H=\mathrm{Tr}(\rho j_\theta^{\mathrm{op}})=\sum_\alpha f(E_\alpha)\langle\alpha\,|\,j_\theta^{\mathrm{op}}\,|\,\alpha\rangle , \tag{26}$$

where the sum runs over the possible states of the system. $f(E_\alpha)$, ρ, and j_θ^{op} have been defined in Sec. II. Making use of the relation

$$j_\theta^{\mathrm{op}}=-c\frac{\partial H}{\partial A_0}=c\frac{\partial H}{\partial\phi} ,$$

(26) can be written as

$$I_H=c\sum_{\alpha e} f(E_{\alpha e})\frac{\partial E_{\alpha e}}{\partial\phi} , \tag{27}$$

where we have restricted the sum to the extended states only because as discussed in Sec. III, $\partial E_{\alpha L}/\partial\phi$ is zero. Notice that in absence of extended states the Hall current is zero.

In a macroscopic system we can expand Eq. (24) in powers of ϕ_0 and get

$$\frac{\partial E_{\alpha e}}{\partial\phi}\simeq\frac{1}{\phi_0}\frac{\partial E_{\alpha e}}{\partial\alpha e} , \tag{28}$$

where we have also made use of the fact that the extended states are closely spaced. Corrections to Eq. (28) are readily shown to vanish with the inverse of the ribbon radius. By using (28) in (27) we obtain

$$I_H=-\frac{c}{\phi_0}\sum_{\alpha e} f(E_{\alpha e})\frac{\partial E_{\alpha e}}{\partial\alpha e} , \tag{29}$$

after integrating by parts we find

$$I_H=\frac{c}{\phi_0}\sum_{\alpha e}\frac{\partial f(E_{\alpha e})}{\partial E_{\alpha e}}\frac{\partial E_{\alpha e}}{\partial\alpha e}E_{\alpha e} . \tag{30}$$

Here we have used

$$\frac{\partial f(E_{\alpha e})}{\partial\alpha e}=\left[\frac{\partial f(E_{\alpha e})}{\partial E_{\alpha e}}\right]\left[\frac{\partial E_{\alpha e}}{\partial\alpha e}\right] .$$

At low temperatures the derivative of the Fermi function is essentially the negative of a δ function which picks out the value of $E_{\alpha e}$ for which $E_{\alpha e}=\zeta$, the chemical potential. Because of the possible presence of an electric field along the z direction, ζ can, in general, vary with position and each extended state will experience some average value

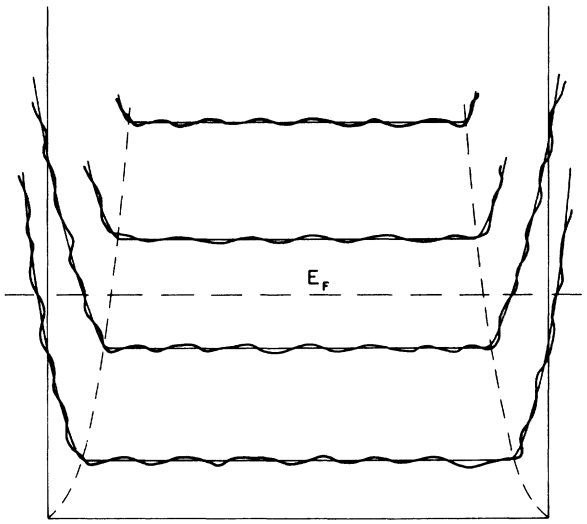

FIG. 6. Schematic energy spectrum for the extended states of a two-dimensional electron gas in the presence of an effective random potential $V(z,\theta)$ at a fixed value of θ. Here the strength of $V(z,\theta)$ is assumed to be much smaller than $\hbar\omega_c$ so that the Landau levels still provide an approximately good description scheme. Localized states levels (not shown) can be present everywhere in the energy range.

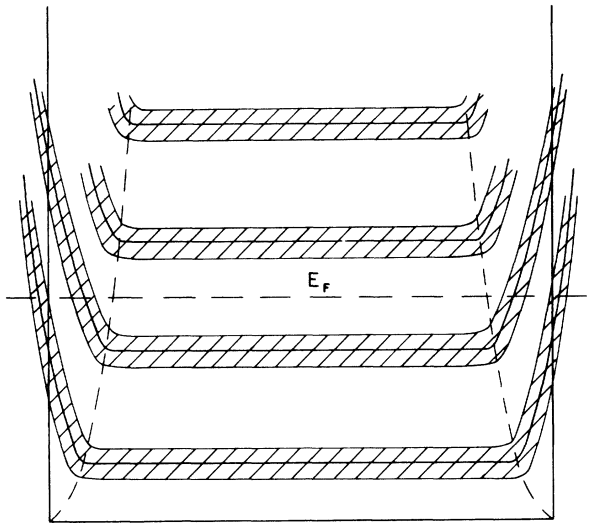

FIG. 7. When curves like those of Fig. 6 are drawn for all values of θ, the Landau levels are broadened into Landau bands labeled with the index ν, as shown schematically.

$\zeta_{\alpha e}$. We introduce the set of values αe_i for which $E_{\alpha e}=\zeta_{\alpha e}$, the local value of the chemical potential, and make use of the result

$$\delta(f(x))=\sum_i \delta(x-x_i)\left|\frac{\partial f}{\partial x}\right|^{-1}$$

to obtain

$$I_H=-\frac{e}{h}\sum_{\alpha e}\sum_i \delta(\alpha e-\alpha e_i)\,\mathrm{sgn}\left[\frac{\partial E_{\alpha e}}{\partial \alpha e}\right]\zeta_{\alpha e}\ . \tag{31}$$

When the Landau bands are well defined, Eq. (31) can be specialized to

$$I_H=-\frac{e}{h}\sum_{i,\nu}\mathrm{sgn}\left[\frac{\partial E_{\nu\lambda}}{\partial\lambda}\right]_{\lambda_i}\zeta_{\lambda_i}\ , \tag{32}$$

where we have made use of the indices ν and λ introduced in the preceding section. In (32) the sum over ν extends to all the Landau bands which cross the chemical potential at some value of λ.

First, let us apply Eq. (32) to the ideal case with boundaries for which the energy diagram of Fig. 2 applies. When the chemical potential lies between two Landau levels, say, n and $n+1$, then the only crossings occur close to the boundaries. In the absence of an applied electric field there is no current unless the local values of the Fermi energy at the two edges differ by ΔE_F. In this case Eq. (32) gives $I_H=-ne\Delta E_F/h$ which is the current carried by the edge states discussed by Halperin. If an electric field is applied along z with the use of (32) we obtain

$$I_H=-n\frac{e^2}{h}\Delta V\ , \tag{33}$$

where ΔV is the potential drop across the ribbon and n is the number of occupied bulk Landau bands. Equation (33) is a generalization of Eq. (19) to the case in which the electric field need not be homogeneous. In absence of impurities Eq. (33) is valid only for a very restricted range of values of N once B is fixed (or of B when N is fixed). The reason is that the chemical potential changes very rapidly between the Landau levels because the density of edge states is relatively small. In this case the width of the plateaus described by Eq. (33) would be a surface effect heavily dependent on the size of the sample.

The situation does not change in an essential way when the effect of the random potential is considered. In this case, the diagram of Fig. 2 no longer applies since l is no longer a good quantum number. However, for a particular value of θ we can still draw energy levels as a function of the orbit center (taken as the average value of z for that particular value of θ for the particular eigenstate in question). Then, instead of Fig. 2, something like the picture shown in Fig. 6 results. The wavy lines in Fig. 6 result from the particular distribution of impurities close to the value of θ for which the curves are drawn. If we repeatedly draw the equivalent of Fig. 6 for a large number of values of θ between 0 and 2π, we obtain a picture like that shown in Fig. 7. Again, if an electric field is applied and ζ lies between the broadened Landau bands in the bulk, the value (33) for the Hall current is obtained, in spite of the presence of not current carrying states.

Strictly speaking, I_H turns out to be slightly smaller than (33). In the evaluation of the chemical potential differences we have assumed the value $-e\Delta V$ for each Landau band. However, it takes the full energy $-e\Delta V$ to

transfer an electron in the $n=0$ Landau level across the ribbon, it takes somewhat less than $-e\Delta V$ to transfer an electron in a higher level across the ribbon.

We make one final argument which gives somewhat different insight into the mechanism of the Hall quantization. Let us once more think of slowly changing, in time, the trapped flux ϕ by an amount $\Delta\phi \propto \exp(i\omega t)$.

According to Faraday's law the rate of change of ϕ gives rise to the induced emf, $\vec{F}=-c^{-1}(\partial\phi/\partial t)\hat{\theta}$. If we assume that $\sigma_{xx}=0$, then the only response the system can make to this emf is a current flow I_z perpendicular to $\vec{F}$,

$$I_z=\sigma_{xy}\left[-\frac{1}{c}\frac{\partial\phi}{\partial t}\right] . \tag{34}$$

If we integrate this to obtain the charge transfer associated with a flux change $\Delta\phi$ we find

$$Q=\int I_z dt=-\frac{1}{c}\sigma_{xy}\Delta\phi . \tag{35}$$

But due to the fact shown above that for $\Delta\phi=\phi_0$ each orbit center moves exactly one step into the position of its neighboring orbit center, we know that $Q=en$, where n is the number of filled Landau levels. Thus we obtain

$$en=\sigma_{xy}(-c^{-1})\frac{hc}{e} , \tag{36}$$

and we again find $\sigma_{xy}=-n(e^2/h)$. The point to be emphasized is that, for the topology of the metallic ribbon, the change in flux *is not* a gauge transformation. There is a real emf associated with the rate of change of flux. The electrons sense this emf and their response leads to a flow of charge across the sample.[26]

IV. DISCUSSION

In this paper we have provided a microscopic theory of the Hall conductance for a two-dimensional electron gas in the presence of an impurity random potential. Our main result is Eq. (32) where the value of the Hall current I_H is related to the electronic spectrum at the Fermi level.

In previous works on quantized Hall conductance, the results are first derived for free electrons and then qualitative arguments are given why the impurities cannot destroy the quantized nature of the Hall conductivity. No explicit calculations were done for the disordered system. In addition, the question on the accuracy of the quantization was not satisfactorily addressed.

We have analyzed the electronic level structure in the presence of impurities. We show explicitly that for the extended states, when the trapped flux is adiabatically changed, the spectrum shifts in a way analogous to the behavior of free electrons and maps into itself when the change in flux is precisely one quantum. For the localized states, the change of the flux does not shift the level and amounts to just a gauge transformation. These are exact results that enable us to *derive* the quantized Hall conductance for electrons in the presence of impurities and the result can be considered as a nontrivial generalization to the many-impurity case of Prange's simple result concerning the single-impurity problem.[17] It is also worth mentioning that the analysis of Sec. III provides a firm theoretical ground for justifying and assessing the limits of the picture put forward by Laughlin.[8] Our general expression (32) for the Hall conductance can also be used in the case in which a weak periodic substrate potential acts on the electrons.[11] Making use of it one easily finds that if the Fermi level lies between two of the magnetic subbands in which each Landau band is split by the periodic substrate potential, the Hall conductance is still an integer multiple of e^2/h in agreement with the conclusions of Thouless and co-workers.[11]

The other major point in our theory is the careful inclusion of the edge effects. We find that whenever the energy of an extended state crosses the local Fermi level ζ, a term $\pm e^2\zeta/h$ is contributed to I_H, the sign being the same as that of $\partial E_{\alpha e}/\partial\alpha e$. When the Fermi level lies between two Landau bands, for instance, n and $n+1$, the only crossings of this type occur at the edges and I_H is found to assume the quantized value $-ne^2\Delta V/h$, ΔV being the voltage drop across the sample. As discussed at the end of Sec. III, however, the local values of ζ at the two crossing points at the edges of a given Landau band differs in general by an amount smaller than ΔV. The correction is found to be dependent on the ratio of the corresponding skipping-orbit radius to the size of the sample. If our picture is correct the exactness of the quantization of I_H must depend upon the specific shape of the potential confining the electron gas in the plane. In the case of a shallow confining potential the Hall conductance will not be quantized as in Eq. (33). In the Appendix we present a model calculation in which this phenomenon is explicitly demonstrated.

It must be stressed here that the Hall current is not carried by the edge states only but is typically a bulk phenomenon. The restriction of I_H to the nature of the confining boundary potential is the result of a great number of cancellations of bulk contributions. This situation is reminiscent of the Landau diamagnetism in which a similar phenomenon occurs. Finally, as clear from Eq. (27), temperature effects will also cause I_H to deviate from the quantized values.

The Hall conductance will maintain its quantized value as long as the Fermi level remains in the localized region of the density of states between two Landau bands. The width of such plateaus depends, therefore, on the capacity of the impurity potential to localize the electronic states. The presence of extended states in the bulk of the sample is necessary for our model to give a finite value for I_H.[21] When the Fermi level is within one of the Landau bands, i.e., within one of the shaded regions of Fig. 7, Eq. (33) need not be valid because several crossings of the local chemical potential by extended states can occur, leading via (32) to a completely different value of I_H. This situation is schematically exemplified in Fig. 8 where the Hall conductivity is plotted against the fractional occupation number a_B^2N/r_0d.

Our approach is based on a single-particle picture and no explicit reference is made to the electron-electron interaction. It must be understood, however, that the present theory implicitly contains some of the effects asso-

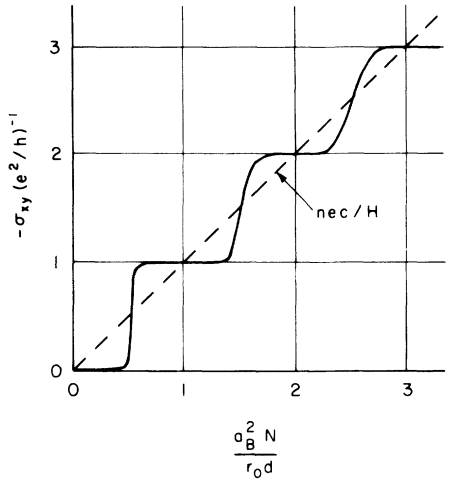

FIG. 8. Behavior of the Hall conductivity σ_{xy} in the quantized region. The plateau regions correspond to a situation in which the chemical potential lies between two Landau bands and localized states are being filled. Corrections associated with edge effects discussed in the text are neglected here.

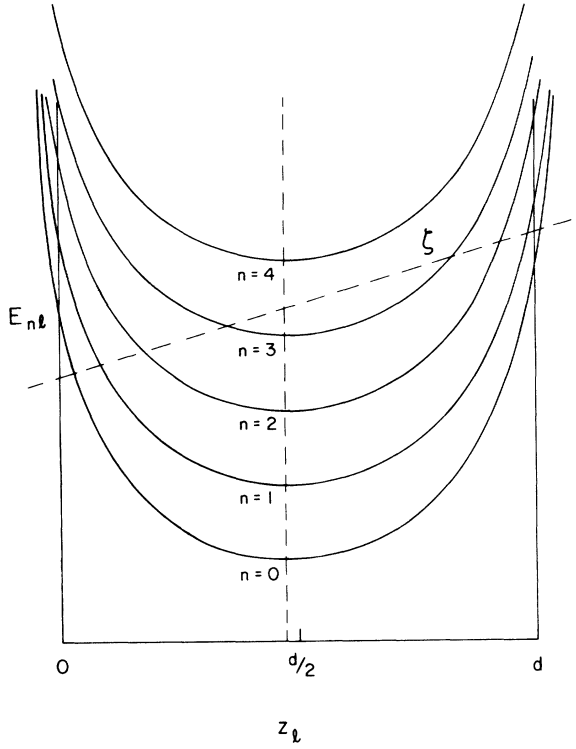

FIG. 9. Schematic energy spectrum of a two-dimensional electron gas confined on a ribbon by a harmonic potential. The Landau levels are parabolas. Energy is plotted vs $z_l = a_B^2 l / r_0$. Various Landau levels are separated by an energy $\hbar\tilde{\omega}_c$ [Eq. (A5)]. Crossing points between the parabolas and the local chemical potential ζ (dashed line) determines the Hall current I_H via Eq. (32). "Edge corrections" are very large in this case and σ_{xy} is not quantized.

ciated with this interaction. In particular, the random potential-energy term $V(z,\theta)$ of Eq. (21) can be thought of as the total effective potential seen by a single quasiparticle including the Hartree-Fock contributions within the normal state. Many-body effects[15–17] possibly responsible for an instability of the normal state[14] have been neglected and are currently under investigation.

ACKNOWLEDGMENT

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APPENDIX: HALL CONDUCTANCE OF AN IDEAL TWO-DIMENSIONAL ELECTRON GAS CONFINED BY A HARMONIC POTENTIAL

In Sec. III we have shown that within the single-particle picture the occurrence of the quantized Hall effect crucially depends on the shape of the confining potential. If the latter is too shallow the Hall conductance will not be expressed by Eq. (33). In particular, for a realistic system, an edge correction will arise which is of the order of the ratio of a_B to the sample size. For the sake of illustration, we explicitly demonstrate this phenomenon here, making use of the exactly solvable model of an ideal system confined by a harmonic potential. Within the geometrical setup discussed in the text, we write the following Hamiltonian:

$$H_\Omega = \frac{p_z^2}{2m} + \frac{1}{2m}\left[\frac{p_\theta}{r_0} - m\omega_c z\right]^2 + eE_z z + \tfrac{1}{2}m\Omega^2\left[z - \frac{d}{2}\right]^2 , \tag{A1}$$

where we have chosen $A_1 = 0$. The last term represents a confining potential. H_Ω can be readily solved exactly in the same way as done for H_0 in Sec. II, as p_θ is still a conserved quantity. The problem reduces to the following Schrödinger equation for the z-dependent part of the wave function:

$$\left[\frac{p_z^2}{2m} + \frac{m\tilde{\omega}_c^2}{2}(z - \tilde{z}_l)^2 + \Delta E_l - E\right] u(z) = 0 , \tag{A2}$$

where the integer l has the same meaning as in Eq. (5) of the text. In (A2) we have defined

$$\tilde{\omega}_c = (\omega_c^2 + \Omega^2)^{1/2} , \tag{A3}$$

$$\tilde{z}_l = \frac{1}{1 + (\Omega/\omega)^2}\left[z_l + \left[\frac{\Omega}{\omega_c}\right]^2 \frac{d}{2} - \frac{v_D}{\omega_c}\right] , \tag{A4}$$

$$\Delta E_l = \frac{m\omega_c^2}{2[1 + (\omega_c/\Omega)^2]}\left[z_l - \frac{d}{2} + \left[\frac{\omega_c}{\Omega}\right]^2 \frac{v_D}{\omega_c}\right]^2 + \text{const} , \tag{A5}$$

with $z_l = a_B^2 l / r_0$. The eigenfunctions and eigenvalues of H_Ω can still be classified by making use of the quantum

numbers n and l as in the free case. We have

$$\psi_{nl}^{\Omega}(z,\theta)=\frac{1}{\sqrt{2\pi}}e^{il\theta}u_n(z-\tilde{z}_l)\ ,$$
$$E_{nl}^{\Omega}=\hbar\tilde{\omega}_c(n+\tfrac{1}{2})+\Delta E_l\ . \tag{A6}$$

Notice that as $\Omega\to 0$ these equations reduce to Eqs. (8).

The spectrum of the system is composed by a series of parabolic Landau levels separated by an energy $\hbar\tilde{\omega}_c$ as described in Eqs. (A6). This spectrum is represented in Fig. 9.

The Hall current carried by each state ψ_{nl}^{Ω} is easily evaluated and is given by

$$I_H^{nl}=-\frac{e\omega_c}{1+(\omega_c/\Omega)^2}\left[z_l-\frac{d}{2}+\left[\frac{\omega_c}{\Omega}\right]^2\frac{v_D}{\omega_c}\right]\ . \tag{A7}$$

Notice that in this case even in the absence of an electric field every state carries a finite current proportional to $z_l-d/2$. This is in contrast with the picture discussed by Halperin in which only the eigenstates carry a finite current in this field as the localizing potential is flat in the bulk region.[21]

Making use of the Eq. (A7), it is straightforward to evaluate the total Hall current. The result for $T\simeq 0$ K is given by the general expression Eq. (32) obtained in the text. Looking at Fig. 9 it is now obvious that I_H is not an integer multiple of e^2/h since the value of the local chemical potential cannot be approximated by either zero or $e\Delta V$ as in the case of a sharp confining potential such as, for instance, the one represented in Fig. 2.

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[20]If we take $\Delta\phi=\phi_0$, then this is actually the Laughlin argument; for that case $z_l+\Delta z_l=z_{l+1}$, and each state moves to its neighboring orbit. The net effect is to transfer one electron per occupied Landau level across the ribbon giving $\Delta E_T=\nu e\,\Delta V$, where ν is the number of occupied Landau level and $\Delta V=dE$ in the potential drop associated with $\vec{E}$. Using Eq. (12), $I_H=-c\eta e\,\Delta V/\phi_0$ and we are led once more to results (19) and (20).

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[24]As according to our definition extended and localized states are topologically distinct it is reasonable to assume that an adiabatic change of the trapped flux does not modify the nature of the given state.

[25]We have assumed here that the applied electric field and the random potential lift any possible degeneracy of the Hamiltonian operator. The accidental degeneracy with an extended state close to one of the edges does not have any bearing on the argument because of the spatial separation.

[26]This argument has been first put forward in Ref. 18. A similar reasoning due to Takemori can be found in Ref. 6.