Finite-size-scaling study of a two-dimensional lattice-gas model with a tricritical point

P. A. Rikvold,* W. Kinzel, J. D. Gunton,* and K. Kaski*
Institut für Festkörperforschung der Kernforschungsanlage Jülich GmbH, D-5170 Jülich, West Germany
(Received 16 December 1982)

We present a finite-size-scaling study of a two-dimensional square-lattice-gas model with nearest-neighbor repulsion and next-nearest-neighbor attraction. This system may be used as a simplified model of adsorbates on transition-metal (100) surfaces. An accurate determination of the tricritical temperature is obtained by the utilization of the degeneracy of the three largest eigenvalues of the transfer matrix at a tricritical point. We obtain the phase diagram and critical and tricritical exponents, all quite consistent with previously known results. We also find that a recent conjecture relating the critical exponent η to the correlation-length amplitude seems to work well even at the tricritical point.

I. INTRODUCTION

The phenomenological finite-size-scaling hypothesis for critical phenomena, which was introduced by Fisher and collaborators[1] and extended by Nightingale,[2] has already been successfully applied to study a variety of critical systems.[2–5] In the present work we apply finite-size scaling to obtain the phase diagram and critical properties of a two-dimensional square-lattice-gas model with nearest-neighbor repulsion and next-nearest-neighbor attraction. One of our main objectives has been to see to what extent this method can be used to determine the location and indices of a tricritical point. The system studied may be used as a model for certain adsorbates on (100) surfaces of transition metals, when surface reconstruction is either absent or simply results in a renormalization of the interaction energies.[6,7] We also briefly consider the question of whether this finite-size scaling yields meaningful results in the case of first-order transitions, where the correlation length as usually defined remains finite. Kinzel and Schick have previously presented a preliminary application to a square-lattice-gas system with nearest-neighbor exclusion and next-nearest-neighbor attraction. This system also has a tricritical point.[8,9]

The model studied in this work is defined in the grand canonical ensemble by the lattice-gas (LG) Hamiltonian

$$\mathscr{H}_{\mathrm{LG}}-\mu\Theta N=K\sum_{\mathrm{NN}}c_ic_j-\alpha K\sum_{\mathrm{NNN}}c_ic_k-(\mu+\epsilon)\sum_i c_i\ , \tag{1}$$

where $K>0$ and α is a real constant. The variable c_i is one or zero, depending on whether the site i is occupied or empty, and Θ is the fraction of occupied sites, or coverage. The chemical potential and binding energy per site are μ and ϵ, respectively. The sums $\sum_{\mathrm{NN}}$ and $\sum_{\mathrm{NNN}}$ extend over all nearest-neighbor pairs and over all next-nearest-neighbor pairs, respectively. The model is equivalent to the Ising metamagnet previously studied by Landau using Monte Carlo simulation[10] and by Landau and Swendsen using a Monte Carlo renormalization-group technique.[11] It is thus a two-dimensional model for which an excellent understanding of the tricritical behavior seems to exist.[11] Equation (1) is transformed to Ising-spin language by taking $c_i=(1-\sigma_i)/2$, where $\sigma_i=\pm1$, and takes the form

$$\mathscr{H}=J\sum_{\mathrm{NN}}\sigma_i\sigma_j-\alpha J\sum_{\mathrm{NNN}}\sigma_i\sigma_k-H\sum_i\sigma_i \tag{2}$$

with $J=K/4$ and $H=-(\mu+\epsilon)/2+(1-\alpha)K$. In this work we take $\alpha=\frac{1}{2}$. The magnetization M is related to the coverage Θ by $M=1-2\Theta$. The Ising model is invariant under the transformations $(\{\sigma_i\},H)\rightarrow(\{-\sigma_i\},-H)$. For weak fields $|H|/J<4$, the ground state is doubly degenerate antiferromagnetic (corresponding to an ordered $\sqrt{2}\times\sqrt{2}$ structure), while for strong fields $|H|/J>4$, it is singly degenerate paramagnetic (corresponding to a disordered lattice gas). Monte Carlo renormalization-group calculations[11] indicate that the tricritical point is located at $k_BT_t/J=1.208\pm0.009$, $H_t/J=3.965\pm0.017$, with tricritical indices $\nu_t=0.556\pm0.006$ and $\eta_t=0.14\pm0.02$. These values for the tricritical exponents are in agreement with the extension by Nienhuis *et al.* of den Nijs's conjecture for the thermal eigenvalue,[12] and with the conjecture by Pearson and by Nienhuis, Riedel, and Schick for the magnetic eigenvalue.[13] The coexistence curve is approximately known from Monte Carlo simulations.[10] The finite-size-scaling transfer-matrix calculations presented here reproduce these results with high accuracy, thus apparently confirming the applicability of finite-size scaling at tricritical points.

II. FINITE-SIZE SCALING AT CRITICAL POINTS

Detailed descriptions of the phenomenological finite-size-scaling method and transfer-matrix calculations on two-dimensional systems are given in Refs. 2 and 14, respectively. Therefore we give here only a brief description of the method.

Under a scaling transformation which transforms a system of linear size N into one of size N/L, the correlation length asymptotically scales as

$$\xi_N(K)\simeq L\xi_{N/L}(K_L)\ \text{ as } N\rightarrow\infty\ . \tag{3}$$

Here K is the set of coupling constants for the original system [in our case $(J/T,H/T)$] and K_L is the set for the rescaled system. Since a critical point K_c is a fixed point of the transformation, a finite-size estimate for K_c is provided by the solution of the equation

$$\xi_N(K_c)=L\xi_{N/L}(K_c) . \tag{4}$$

We now consider a strip of Ising spins of infinite length and width N with periodic boundary conditions. Since the ground state is antiferromagnetic, N must be even to avoid the introduction of interfaces. With $L=N/(N+2)$, Eq. (4) takes the form

$$\xi_N(K_c)/N=\xi_{N+2}(K_c)/(N+2) . \tag{5}$$

For the method to be of any use, these finite-size estimates for K_c must converge rapidly towards the infinite-system value. This appears to be the case both in this work and in previous applications of the method. As discussed in Sec. IV the convergence is rapid so that we have not found it necessary to use corrections to scaling or extrapolation methods to determine the line of transition points.

To find the correlation length ξ_N one needs the two largest eigenvalues of the transfer matrix $\underline{T}$. This $2^N\times 2^N$ matrix corresponds to the addition of one layer of N spins to the end of the strip, and is defined by its matrix elements

$$\langle S_i \mid \underline{T} \mid S_{i+1}\rangle=\exp[-\beta\mathscr{H}_i(S_i,S_{i+1})] . \tag{6}$$

Here $\mid S_i\rangle$ is the 2^N-dimensional column vector representing the state of the ith layer, $\beta=1/k_BT$, and the Hamiltonian (2) is written as a sum of single-layer contributions,

$$\mathscr{H}=\sum_i \mathscr{H}_i . \tag{7}$$

It is easily shown[2,14] that for a strip of infinite length and finite width N, the correlation length is given by the ratio of the two eigenvalues of $\underline{T}$ which are largest in absolute value,

$$\xi_N^{-1}=\ln(\lambda_1/\mid\lambda_2\mid) . \tag{8}$$

The largest eigenvalue λ_1 equals the partition function per layer and is thus always positive.

The thermal eigenvalue of the scaling transformation with scale change L is L^{y_T}, so that

$$\xi_N(K-K_c)\simeq L\xi_{N/L}[L^{y_T}(K-K_c)] \quad \text{as } N\to\infty . \tag{9}$$

Differentiating with respect to $(K-K_c)$ and again taking $L=N/(N+2)$, we obtain

$$y_T+1\simeq\left[\ln\frac{\left.\dfrac{\partial\xi_N}{\partial K}\right|_{K=K_c}}{\left.\dfrac{\partial\xi_{N+2}}{\partial K}\right|_{K=K_c}}\right]\left[\ln\frac{N}{N+2}\right]^{-1} \quad \text{as } N\to\infty . \tag{10}$$

According to the principle of smoothness, y_T is independent of the direction of the derivative in K space, as long as it is not taken parallel to a line of critical points.[15] The discontinuity in y_T as the direction of differentiation becomes parallel to the line of critical points becomes rounded out in finite systems. The estimates for y_T obtained from (10), therefore, in general depend on the direction of differentiation. For a wide range of directions around the normal to the line of critical points shown in Fig. 1, the estimates are, however, constant to within about 5×10^{-3}. To minimize these finite-size effects we therefore have chosen to perform the differentiations in this "orthogonal" direction. (It is worth noting that the orthogonality is not preserved under independent scale changes in the fields, as stressed by Griffiths and Wheeler.[16]) The question of how to obtain the best possible finite-size estimates for y_T has also been discussed by Kinzel and Schick.[17] The correlation length exponent ν is defined by $\xi_\infty(K)\sim\mid K-K_c\mid^{-\nu}$. From Eq. (9) it is easily seen to be given by $\nu=y_T^{-1}$.

The model studied here has a line of critical points $T_c(H)$, extending from the Néel temperature $T_N=T_c(0)$, to a tricritical point (H_t,T_t), as shown in Fig. 1. The tricritical point and the point $(H=4J,\ T=0)$ are connected by a line of first-order phase transitions.

Along the line of critical points we have determined the exponent η for the decay of the correlation function by using the interesting conjecture, due to Derrida and deSeze,[18] that η is related to the amplitude of the correlation length as

$$\xi_N/N=1/\pi\eta . \tag{11}$$

We also have applied this relation at the tricritical point to test its validity there. We find that it gives good agreement with the expected result, as discussed in Sec. IV.

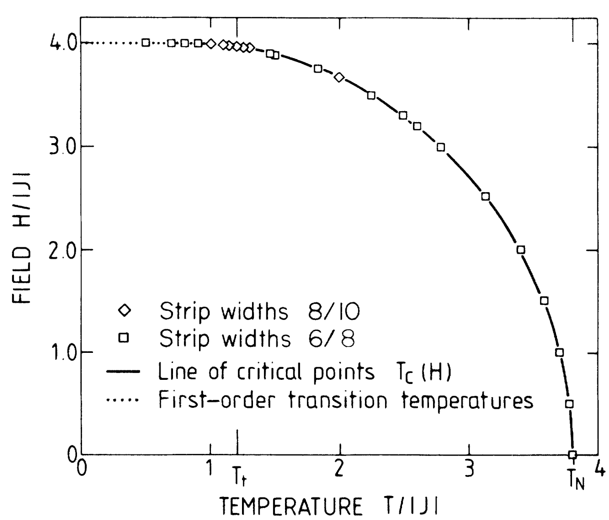

FIG. 1. Line of critical points $T_c(H)$ (solid line), and line of first-order transitions (dotted line). The data points are for $N/(N+2)=6/8$ (squares) and 8/10 (diamonds). The tricritical point is at $k_BT_t/J=1.205\pm0.003$, $H_t/J=3.965\pm0.001$ where T_t is determined from Fig. 2.

III. DETERMINATION OF THE TRICRITICAL POINT

The accurate determination of the tricritical temperature T_t requires some extension of the well-established ideas of finite-size scaling at critical points. In this section we discuss these extensions.

At the tricritical point the disordered paramagnetic and the two ordered antiferromagnetic phases become indistinguishable.[19,20] This requires the asymptotic degeneracy of the three largest eigenvalues of the transfer matrix $\underline{T}$. The degeneracy of the third eigenvalue with the two largest ones provides an additional constraint which fixes the position of the tricritical point along the line of critical points. In finite systems this asymptotic degeneracy shows up as a linear divergence with the strip width N of the quantity

$$\hat{\xi}_N = [\ln(\lambda_1/|\lambda_3|)]^{-1} ,$$

so for $T=T_t$,

$$\hat{\xi}_N \simeq A_t N \quad \text{as } N \to \infty . \tag{12}$$

One can interpret $\hat{\xi}_N$ as a length scale. It can be demonstrated that it characterizes the decay in the direction along the infinite strip of the correlation function for the magnetization of a single layer of N spins.

Along the line of critical points the magnetic susceptibility remains finite as $N \to \infty$, so for $T > T_t$ the characteristic length $\hat{\xi}_N$ must be asymptotically independent of N,

$$\hat{\xi}_N \simeq A_2 \quad \text{as } N \to \infty . \tag{13}$$

The behavior of $\hat{\xi}_N$ along the line of first-order transitions below T_t can be obtained from the following heuristic argument. At a first-order transition, phase coherence extends over distances which are much larger than N, and the three distinct phases coexist in a one-dimensional array of macroscopic single-phase domains. The average size of a domain of one of the two ordered phases is proportional to the dominant length ξ_N, while the average size of a domain of the disordered phase is proportional to $\hat{\xi}_N$. The ratio $\hat{\xi}_N/\xi_N$ is proportional to the fraction of the total volume taken up by the disordered phase so that $\hat{\xi}_N$ is proportional to ξ_N. The free energy required for the formation of an interface across the strip is $N\sigma$, where $k_B T\sigma(T,H)$ can be identified with the surface tension. The average domain size $\hat{\xi}_N$, therefore, should have the asymptotic behavior[21–23]

$$\hat{\xi}_N \simeq A_1 N e^{N\sigma} \quad \text{as } N \to \infty . \tag{14}$$

As σ vanishes at the tricritical point, this is consistent with (12). A result similar to (14), valid below the critical point of the two-dimensional Ising model, has previously been presented by Fisher.[21]

Based on the above reasoning we expect the asymptotic behavior as $N \to \infty$ of the length $\hat{\xi}_N$, evaluated on the transition line, to be as follows:

$$\hat{\xi}_N = \hat{\xi}_N^{(\text{reg})} + \begin{cases} A_1 N e^{N\sigma}(1+B_1 N^{-c_1}+\cdots), & T<T_t \\ A_t N(1+B_t N^{-c_t}+\cdots), & T=T_t \\ A_2(1+B_2 N^{-c_2}+\cdots), & T>T_t \end{cases} . \tag{15}$$

In (15) we have included a regular part

$$\hat{\xi}_N^{(\text{reg})} = A^{(\text{reg})} + B^{(\text{reg})} N^{-1} + O(N^{-2}) , \tag{16}$$

as well as corrections to scaling ($\sim N^{-c_i}$) due to irrelevant variables. The exponents $-c_i$ are the leading irrelevant scaling exponents and are expected to lie between -1 and 0. This has been pointed out by Privman and Fisher.[24] In general the coefficients A_i and the exponents c_i are expected to be different in the three cases.

For temperatures slightly different from T_t we expect $\hat{\xi}_N$ to show crossover from the tricritical to the appropriate critical or first-order behavior as N increases.[20] This can be utilized to obtain a numerical estimate for T_t. From Eq. (15) we thus expect a plot of $\hat{\xi}_N$ vs N for $T > T_t$ to exhibit a downward curvature as $N \to \infty$, while for $T < T_t$ an upward curvature is expected. Only for $T = T_t$ should the plot approach a straight line of nonzero slope. The tricritical temperature T_t can therefore be determined as the temperature for which the quantity (the "second derivative" with respect to N)

$$\hat{\xi}''_N = (\hat{\xi}_{N+2} - 2\hat{\xi}_N + \hat{\xi}_{N-2})/4 \tag{17}$$

vanishes asymptotically as $N \to \infty$. From (15) we obtain the asymptotic behavior of $\hat{\xi}''_N$ at T_t,

$$\hat{\xi}''_N(T_t) \simeq 2B^{(\text{reg})} N^{-3} + 2\begin{pmatrix} 1-c_t \\ 2 \end{pmatrix} A_t B_t N^{-(1-c_t)} \quad \text{as } N \to \infty . \tag{18}$$

The first term arises from the regular part of $\hat{\xi}_N$ and the second term from the corrections to scaling due to irrelevant variables. If the amplitude of the second term is small, the behavior for moderate N will be dominated by the rapidly convergent first term. If the second term vanishes, the best estimate for T_t is the temperature for which the exponent seen in the range of observed N is constant and approximately equals -3. It is well known, as pointed out in Ref. 24, that for the two-dimensional Ising model with $S=\frac{1}{2}$ and nearest-neighbor interactions the corrections due to irrelevant variables cancel exactly. This is, however, a coincidence and cannot be expected to hold in general. Similar rapid convergence has been observed in several previous works,[18,24] but since the asymptotic region may not have been reached for the system sizes studied, the accuracy of the resulting parameter estimates may be open to question.

If the term due to irrelevant variables does not vanish, the effective exponent $\zeta_N = \Delta \ln \hat{\xi}''_N / \Delta \ln N$ for finite N depends on the ratio of the amplitudes of the two correction terms and on the unknown exponent c_t as well as on N. In most cases, however, $c_t \in [0,1]$,[24] so that $\zeta_N \in [-3,-1]$. Using these bounds on the effective exponent, we can determine upper and lower bounds on T_t from the slopes between $N=8$ and $N=10$ in the doubly logarithmic plot of $\hat{\xi}''_N$ vs N for different T, shown in Fig. 2.

IV. RESULTS

The main numerical results of our investigation are shown in the figures. In Fig. 1 are shown the line of critical points $T_c(H)$, the tricritical point (H_t, T_t), and the line of first-order phase transitions. The whole line is the set of solutions of Eq. (5). The results converge rapidly with N, so that for $k_BT/J \simeq 1$, $N/(N+2)=8/10$ yields the transition field to a relative accuracy of 10^{-3}. The calculation with $N/(N+2)=6/8$ yields the value for the Néel temperature $k_BT_N/J=3.802$, as compared with the result from high-temperature series expansions,[25] $k_BT_N/J=3.809$. We therefore have not found it necessary to use larger strip widths or employ corrections to scaling in order to determine the transition line more accurately.

Figure 2 was used to determine the tricritical temperature T_t. It shows a doubly logarithmic plot of the quantity $\hat{\xi}''_N$, defined in (17), versus N. If we assume that $\hat{\xi}''_N$ converges with an exponent approximately equal to -3, we obtain the estimate $k_BT_t/J=1.207\pm0.001$ from the

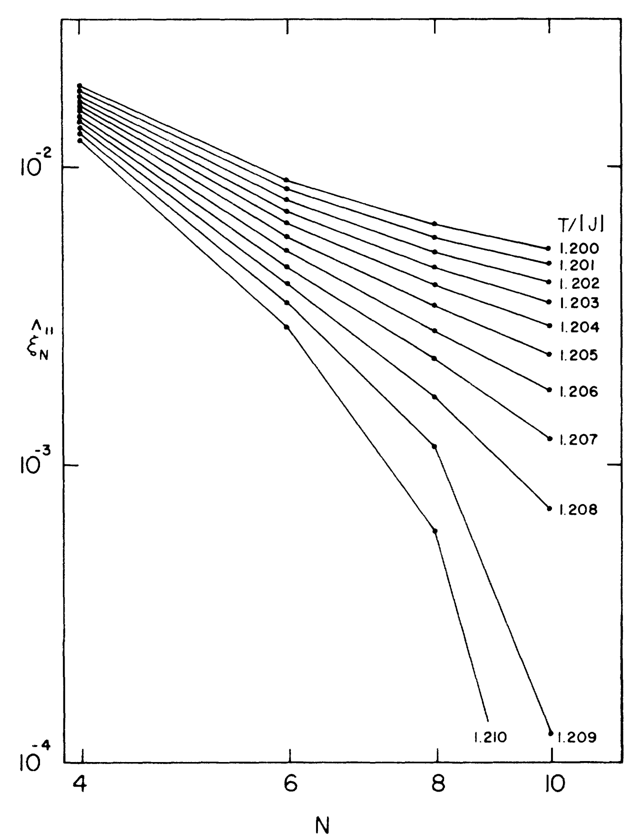

FIG. 2. Plot for determination of T_t, showing the quantity $\hat{\xi}''_N$, defined in (16) vs N on a log-log scale for different temperatures. The effective exponent ζ_{10} between $N=8$ and 10 for different temperatures is $\zeta_{10}(1.208)=-3.86$, $\zeta_{10}(1.207)=-2.79$, $\zeta_{10}(1.206)=-2.06$, $\zeta_{10}(1.205)=-1.69$, $\zeta_{10}(1.202)=-1.04$, and $\zeta_{10}(1.201)=-0.92$. If we assume a simple power-law convergence $\sim N^{-3}$, we thus obtain the estimate $k_BT_t/J=1.207\pm0.001$. If we assume convergence with an effective exponent $\zeta_N \in [-3,-1]$, we obtain the more conservative estimate $k_BT_t/J=1.205\pm0.003$.

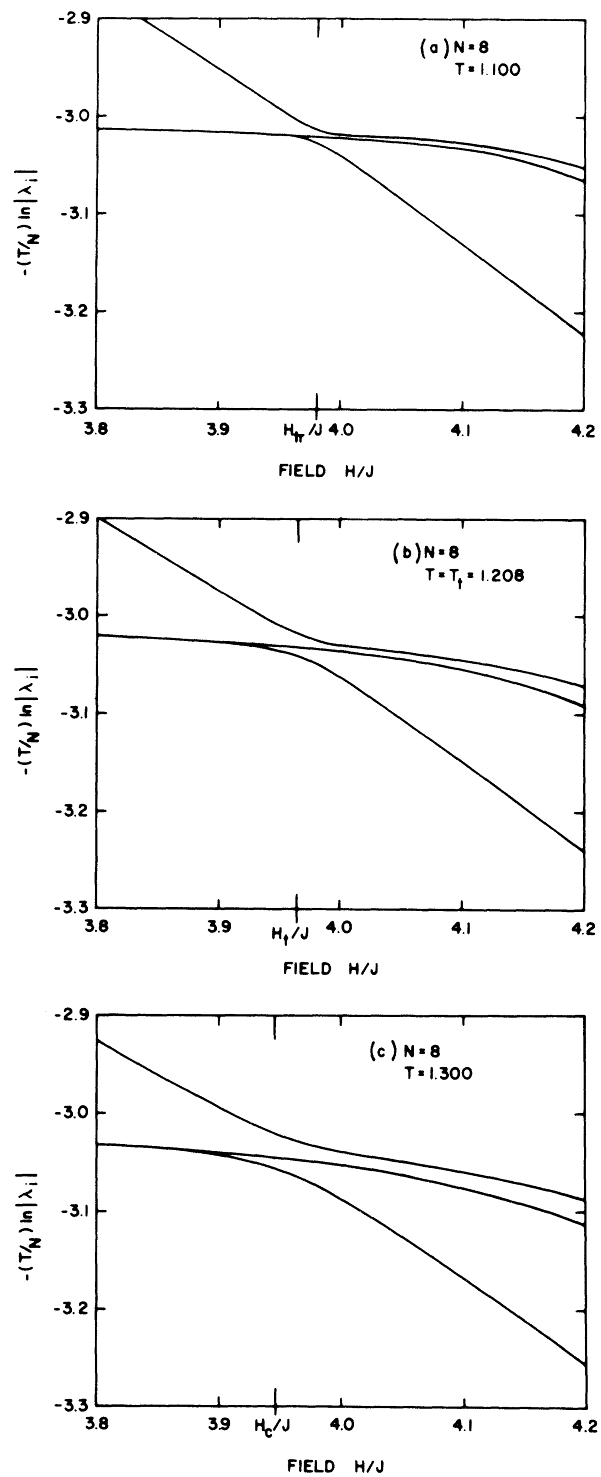

FIG. 3. $g_i(T,H)=-(T/N)\ln|\lambda_i|$ for the three largest eigenvalues λ_i for $N=8$ as functions of field across the first-order transition at (a) $k_BT/J=1.1$, near the tricritical temperature at (b) $k_BT/J=1.208$, and at the second-order transition at (c) $k_BT/J=1.3$. The branch for the largest eigenvalue λ_1 corresponds to Gibbs's free energy per spin.

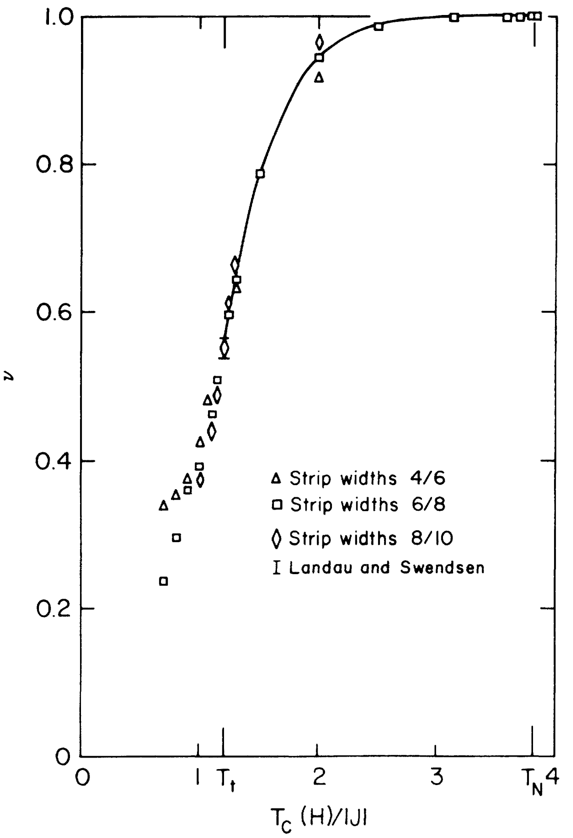

FIG. 4. Correlation-length exponent ν vs T. For finite systems ν changes continuously from the Ising critical value $\nu=1$ at the Néel temperature to a tricritical value $\nu_t=0.552\pm0.008$ at T_t. The change becomes more abrupt with increasing strip widths, indicating a discontinuous change for infinite systems. The line is a guide to the eye, drawn through the points for $N/(N+2)=6/8$. Our estimate for ν_t is consistent with the Landau and Swendsen result $\nu_t=0.556\pm0.006$, which is indicated by an error bar. The exponents obtained for $T<T_t$ have no physical significance.

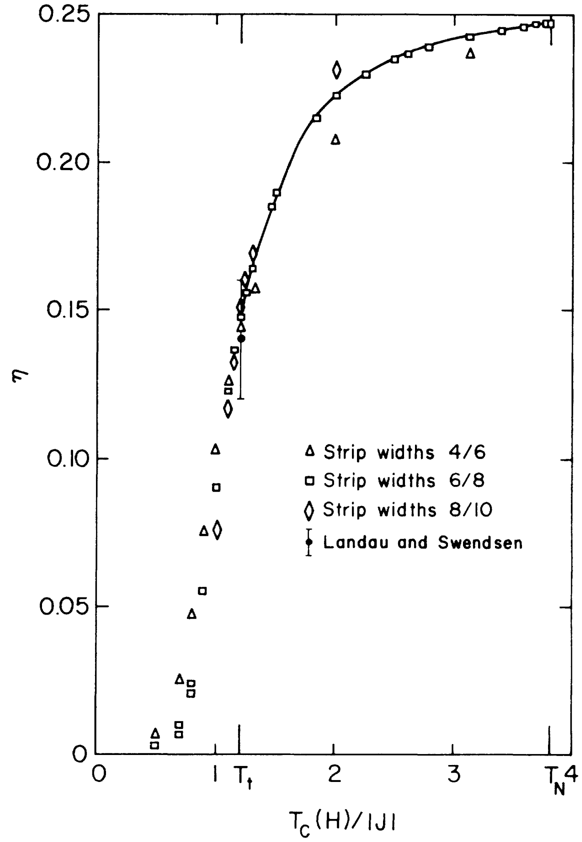

FIG. 5. Anomalous dimension exponent η vs T, as obtained from the conjectured relation $\eta=N/\pi\xi_N$ of Ref. 18. For finite systems η changes continuously from the Ising critical value $\eta=\frac{1}{4}$ to a different tricritical value η_t. The change becomes more abrupt with increasing strip widths, indicating a discontinuous change for infinite systems. The line is a guide to the eye, drawn through the points for $N/(N+2)=6/8$. We find a tricritical value $\eta_t=0.149\pm0.002$, within the uncertainty of the Landau and Swendsen result $\eta_t=0.14\pm0.02$ (error bar). Below T_t the exponents obtained have no physical significance.

line of smallest curvature in the figure. If we assume that $\hat{\xi}''_N$ converges with an effective exponent $\zeta_N\in[-3,-1]$, we obtain the more conservative estimate $k_BT_t/J=1.205\pm0.003$. This is within the error bounds of Landau and Swendsen's result $k_BT_t/J=1.208\pm0.009$. We will use the more conservative estimate in the following. The corresponding value of the field is $H_t/J=3.965\pm0.001$. This is the position of the tricritical point indicated in Fig. 1.

Figure 3 shows the change of $g_i(T,H)=-(T/N)\times\ln|\lambda_i|$ for the three eigenvalues of T which are largest in absolute value, as H is varied through the transition at (a) $k_BT/J=1.1$, at (b) $k_BT/J=1.208$, and at (c) $k_BT/J=1.3$. The data are for $N=8$. The Gibbs's free energy per spin is $g_1(T,H)$, corresponding to the largest eigenvalue λ_1. The transition field obtained from (5) is also indicated in the figure. The rapid change in the slope of $g_1(T,H)$ in the first-order case (a) corresponds to the discontinuity in the magnetization seen in Fig. 6.

In Fig. 4 is shown the correlation-length exponent $\nu=y_T^{-1}$, as obtained from Eq. (10), versus T. For an infinite system ν is expected to change discontinuously from its critical Ising value $\nu=1$, to a different value ν_t at T_t. For the finite systems studied here, ν changes continuously from unity to $\nu_t=0.552\pm0.008$. The uncertainty arises from the variation of y_T with the direction of differentiation in (10), and from the uncertainty in T_t. This result is consistent with that of Landau and Swendsen, $\nu_t=0.556\pm0.006$, and with the conjecture $\nu_t=\frac{5}{9}$.[12] The change in ν with T is seen to become more abrupt as the stripwidth N increases, indicating a discontinuous change as $N\to\infty$. The values of ν obtained for $T<T_t$ have no physical significance, as can be seen by applying (10) to (14).

In Fig. 5 is shown the exponent η, as obtained from the conjectured relation between η and the amplitude of the correlation length,[18] $\eta=N/\pi\xi_N$. For an infinite system one again expects a discontinuous change from the critical

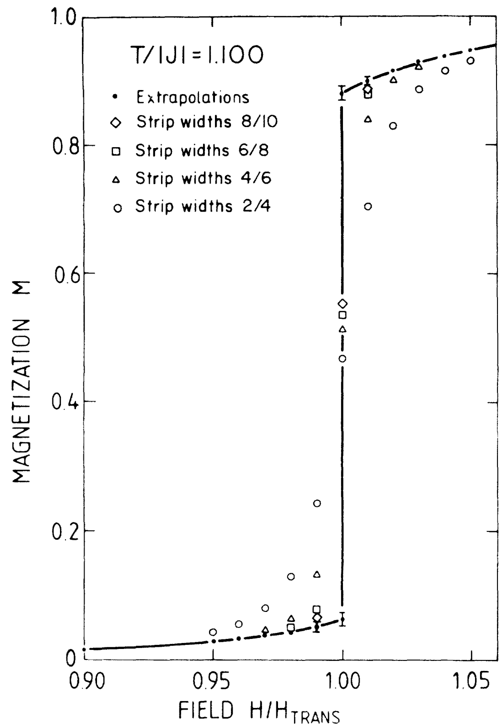

FIG. 6. Magnetization vs field across the first-order transition at $k_B T/J = 1.1$ The change across the transition is seen to become steeper with increasing strip width, approaching a discontinuity for infinite systems. Extrapolations to infinite strip widths are also shown. The magnetization is obtained by differentiation of Gibbs's free energy per spin.

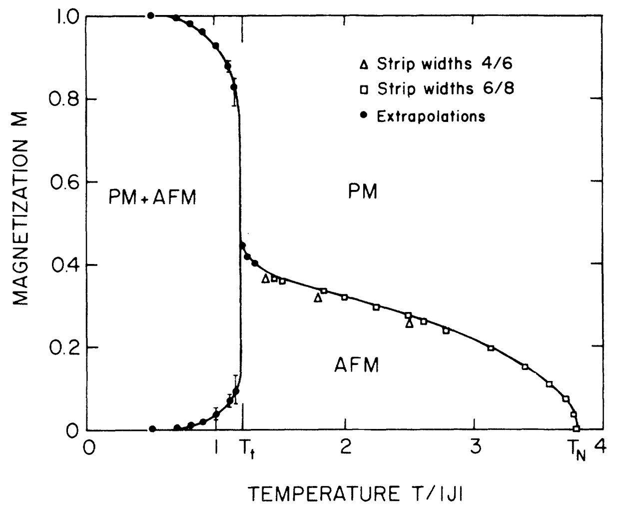

FIG. 7. Phase diagram, magnetization vs temperature at the transition field value. The coexistence gap is determined from plots like Fig. 6. Above T_t is shown $M(H_c)$ vs T. The accuracy close to T_t is poor, as indicated by the error bars, but well below T_t there is good agreement with Monte Carlo simulations. The disordered phase is paramagnetic (PM), the ordered phase is antiferromagnetic (AFM).

Ising value $\eta = \frac{1}{4}$, to a different value η_t at T_t. For the finite systems studied here we see a continuous change from $\eta = \frac{1}{4}$ to $\eta_t = 0.149 \pm 0.002$. This is within the uncertainty in the value found by Landau and Swendsen, $\eta_t = 0.14 \pm 0.02$, and in agreement with the conjectured result $\eta_t = 0.15$.[13] The small uncertainty in our estimate for η_t arises because the correlation length is very accurately determined. The main uncertainty in η_t comes from the uncertainty in the estimate for T_t. As in the case of ν, the change in η is continuous, but steepens as N increases. It is noteworthy that the conjecture appears to be valid at the tricritical point, as well as at critical points, for which it was originally proposed. The values of η obtained for $T < T_t$ have no physical significance.

The magnetization has been computed from the Gibbs's free energy per spin as $M = -\partial g_1(T,H)/\partial H$. In Fig. 6 is shown the magnetization versus field at $k_B T/J = 1.1$, i.e., below T_t. The change across the transition is gradual for small strip widths, but becomes steeper for larger systems, asymptotically approaching a discontinuity for infinite systems. It is, however, difficult to estimate the size of the discontinuity accurately. We have estimated the magnetization for an infinite system from plots of M vs N^{-1} for fixed H. We also have tried to fit our data to the scaling law

$$M_{\pm} \simeq Z_{\pm}(hN^d) , \tag{19}$$

where $h = 1 - H/H_{\text{trans}}$ and d is the spatial dimensionality, as has been suggested by Fisher and Berker.[21] However, we do not have enough data in the scaling region close to the transition point to test this suggestion.

In Fig. 7 is shown the phase diagram M vs T at the transition field. Below T_t the coexistence gap is determined from plots like Fig. 6. Above T_t is shown $M(H_c)$ vs T. Close to T_t the convergence with strip width is slow, so the phase diagram is not very accurately determined, as indicated by the error bars. The "hook" on the magnetization curve just above T_t may be a numerical artifact. Well below T_t we find good agreement with Landau's Monte Carlo simulation results.[10]

V. CONCLUSION

The purpose of this work has been to study the accuracy and effectiveness of the finite-size-scaling method when applied to a system with a tricritical point. For the system studied here the phase diagram is known from Monte Carlo studies,[10] and the critical and tricritical properties from exact results, renormalization-group calculations, and conjectures.[9–13] We have extended the method to determine the tricritical point, and tested a recently conjectured relation between the critical index η and the correlation-length amplitude. This conjecture seems to hold even at the tricritical point. We find that the finite-size-scaling method thus extended provides a single calculational scheme which yields results fully consistent with the previous calculations. The accuracy is comparable with that of the previous numerical studies.[10,11] As a tool for comprehensive studies of lattice-gas models with complicated phase diagrams the method seems to promise considerable computational savings compared to the use of a combination of other methods. The convergence with system size observed in finite-size-scaling studies is too rapid to be explained by corrections to scaling due to irrelevant variables, as has recently been pointed out by Privman and Fisher.[24] However, the apparent high accuracy and consistency of our results suggest that our extrapolations based on this rapid convergence are nevertheless valid.

ACKNOWLEDGMENTS

Three of us (P.A.R., J.D.G., and K.K.) are grateful for the generous hospitality shown to us at the Institut für Festkörperforschung der Kernforschungsanlage Jülich, where this study was carried out. Useful discussions with Dr. R. H. Swendsen and Professor D. P. Landau are acknowledged. The work was supported in part by National Science Foundation Grant No. DMR-80-13700. One of us (P.A.R) acknowledges support from the Norwegian Research Council for Science and the Humanities (NAVF).

---

*Permanent address: Physics Department, Temple University, Philadelphia, Pennsylvania 19122.

[1]M. E. Fisher, in *Proceedings of the International School of Physics, "Enrico Fermi" Course 51,* edited by M. S. Green (Academic, New York, 1971), p. 1; M. E. Fisher and M. N. Barber, Phys. Rev. Lett. 28, 1516 (1972); M. N. Barber, in Phase Transitions and Critical Phenomena, edited by C. Domb and J. L. Lebowitz (Academic, London, in press), Vol. 7.

[2]M. P. Nightingale, Physica 83A, 561 (1976); Phys. Lett. 59A, 486 (1977); Ph.D. thesis, University of Amsterdam, 1978 (unpublished); Proc. K. Ned. Akad. Wet. Ser. B 82, 235 (1979).

[3]L. Sneddon, J. Phys. C 12, 3051 (1979).

[4]J. M. Luck, J. Phys. Lett. (Paris) 42, L275 (1981); Thèse de Troisième Cycle, Université de Paris VI, 1981 (unpublished).

[5]C. J. Hamer and M. N. Barber, J. Phys. A 14, 259 (1981); B. Derrida, *ibid.* 14, L5 (1981); J. L. Pichard and G. Sarma, J. Phys. C 14, L127 (1981); 14, L617 (1981); W. Kinzel and J. M. Yeomans, J. Phys. A 14, L613 (1981).

[6]K. Binder and D. P. Landau, Phys. Rev. B 21, 1941 (1980); K. Binder and D. P. Landau, Surf. Sci. 108, 503 (1981).

[7]L. D. Roelofs, in *Springer Series in Chemical Physics,* edited by R. Vanselow and R. Howe (Springer, Berlin, 1982), Vol. 20.

[8]W. Kinzel and M. Schick, Phys. Rev. B 24, 324 (1981).

[9]After the present work was completed it was pointed out by Huse that the tricritical point, tricritical exponents, and the line of first-order transitions for this model have been found exactly by Baxter. The thermal and magnetic tricritical eigenvalue exponents are the same as those found in Refs. 12 and 13, namely $y_T=\frac{9}{5}$ and $y_H=\frac{77}{40}$, as expected from universality, and the tricritical temperature in units of the attractive next-nearest-neighbor coupling constant is $T_t=0.558\,02\ldots$. For comparison, Kinzel and Schick obtained $y_T\approx 1.7$ and $T_t\approx 0.617$. D. A. Huse, Phys. Rev. Lett. 49, 1121 (1982); R. J. Baxter, J. Phys. A 13, L61 (1980); R. J. Baxter, J. Stat. Phys. 26, 427 (1981); R. J. Baxter and P. A. Pearce, J. Phys. A 15, 897 (1982).

[10]D. P. Landau, Phys. Rev. Lett. 28, 449 (1972).

[11]D. P. Landau and R. H. Swendsen, Phys. Rev. Lett. 46, 1437 (1981).

[12]B. Nienhuis, A. N. Berker, E. K. Riedel, and M. Schick, Phys. Rev. Lett. 43, 737 (1979); M. P. M. den Nijs, J. Phys. A 12, 1857 (1979).

[13]R. B. Pearson, Phys. Rev. B 22, 2579 (1980); B. Nienhuis, E. K. Riedel, and M. Schick, J. Phys. A 13, L31 (1980); 13, L189 (1980).

[14]C. Domb, Adv. Phys. 9, 149 (1960).

[15]R. B. Griffiths, Phys. Rev. B 7, 545 (1973).

[16]R. B. Griffiths and J. C. Wheeler, Phys. Rev. A 2, 1047 (1970).

[17]W. Kinzel and M. Schick, Phys. Rev. B 23, 3435 (1981).

[18]B. Derrida and L. deSeze, J. Phys. (Paris) 43, 475 (1982).

[19]R. B. Griffiths and B. Widom, Phys. Rev. A 8, 2173 (1973).

[20]I. D. Lawrie and S. Sarbach, in Phase Transitions and Critical Phenomena, edited by C. Domb and J. L. Lebowitz (Academic, London, in press), Vol. 7.

[21]M. E. Fisher, J. Phys. Soc. Jpn. Suppl. 26, 87 (1969).

[22]M. E. Fisher and A. N. Berker, Phys. Rev. B 26, 2507 (1982); H. W. J. Blöte and M. P. Nightingale, Physica 112A, 405 (1982); J. L. Cardy and P. Nightingale, Phys. Rev. B 27, 4256 (1983); P. Kleban and C.-K. Hu (unpublished); V. Privman and L. S. Schulman, J. Stat. Phys. 29, 205 (1982).

[23]In fact there are two different kinds of interfaces, one separating two opposite ordered domains, and one separating an ordered domain from a disordered one. The energy associated with the former is approximately twice that of the latter. Order-disorder interfaces are therefore preferred, at least at low T and large N. This is the type of interface to which the surface tension $k_B T\sigma$ refers.

[24]V. Privman and M. E. Fisher (unpublished).

[25]N. W. Dalton and D. W. Wood, J. Math. Phys. 10, 1271 (1969).