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Surface tension, roughening, and lower critical dimension in the random-field Ising model

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A continuum interface model is constructed to study the low-temperature properties of domain walls in the random-field Ising model (RFIM). The width of the domain wall and its surface tension are computed by three methods: Simple energy accounting, dimensional arguments, and approximate renormalization-group calculations. All methods yield a surface tension which is positive at sufficiently low temperature for small random fields, h, provided that the dimensionality $d>2$. The lower critical dimension of the RFIM is thus argued to be 2. While effects due to discreteness of a lattice are argued to alter some of the continuum results quantitatively, they do not change these central conclusions. For $d<2$ the ferromagnetic correlation length of the RFIM behaves like $h^{-2(2-d)^{-1}}$ as $h\rightarrow 0$.

I. INTRODUCTION

In most problems of phase transitions, determination of the "lower critical dimension" d_c (the dimension below which the transition cannot occur) is accomplished by elementary arguments. In "frustrated" systems such as spin-glasses, however, competing interactions render detailed specification of the ground state extremely difficult; calculating d_c then becomes a formidable task. The ferromagnetic Ising model in a random magnetic field (RFIM) constitutes a particularly irritating example of this principle: For six years now theoreticians have disputed whether, for the transition of this model into a ferromagnetic state, d_c is 2 or 3.[1–9] Given the existence of several experimental realizations of the RFIM, notably random antiferromagnets in uniform magnetic fields[10,11] and commensurate charge-density-wave systems[12] with impurity pinning, determining whether the RFIM can exhibit ferromagnetism in three dimensions (3D) is not merely an academic exercise.[13]

The earliest analysis to predict $d_c=2$ was the domain argument of Imry and Ma.[1] We briefly review that reasoning now: Imagine creating a domain of linear size L of down spins in a d-dimensional RFIM with exchange constant J and random field strength h, assumed to be ferromagnetically ordered in the up direction. For large L the surface-energy cost of the domain varies as JL^{d-1}, while the field-energy gain grows like $hL^{d/2}$, there being, statistically, $(L^d)^{1/2}$ more down than up spins in the domain. For $d-1>d/2$ or $d>2$, the surface energy dominates, making it energetically unfavorable to form large domains of down spins and ensuring that the ground state of the RFIM is ferromagnetic. For $d<2$, on the other hand, the ferromagnetic ground state is unstable, even for arbitrarily small h/J, to the formation of down droplets of large size. One concludes that $d_c=2$. (Note that the domain argument is inconclusive when $d=2$, though it is typically assumed that the ferromagnetic state is marginally *unstable* to domain formation, a guess we will verify in Sec. III A.)

Though the domain argument is appealingly simple and physical, a formal correspondence, established order by order in perturbation theory, between the Ginzburg-Landau representations of the RFIM in d dimensions and the pure Ising model in $(d-2)$ dimensions,[14] suggests, since d_c for the pure Ising system is 1, that $d_c=3$ in the RFIM. One is forced to reevaluate the domain argument in light of this observation. How could it possibly be wrong? The most likely potential flaw involves[2] the neglect in the domain argument of the "roughness"[15] of the interface between the up and down spins; the argument implicitly assumes that the interface between domains is well defined and "smooth," i.e., that its width w remains finite even as its length L goes to infinity. This is almost surely false for $d\leq 3$ even at arbitrarily low T in the RFIM,[3–6] since the interface wanders (i.e., roughens) to gain random-field energy. But how rough must the interface be potentially to invalidate the domain argument? Presumably as long as $w/L\rightarrow 0$ as $L\rightarrow\infty$ the interface is *effectively* smooth even if $w\rightarrow\infty$ as $L\rightarrow\infty$. Thus the validity of the domain argument hinges[5] on the ratio w/L.

This suggests that one try to compute w/L by constructing and analyzing a model of a domain wall separating up Ising spins from down spins in the presence of random fields. Such a study has utility beyond merely testing the credibility of the domain argument: Having produced a plausible model of the interface one can in principle estimate from it the surface tension σ for the RFIM. Since σ is, by definition, the difference per unit surface area of the free energies of two RFIM's, one with periodic, the other with antiperiodic boundary conditions, its behavior in the thermodynamic limit distinguishes an ordered (in this case ferromagnetic) from a disordered (paramagnetic) phase. Free energies in the paramagnetic state are independent of boundary conditions, whereupon σ must be identically zero; the ferromagnetic state, by contrast is characterized by positive σ. To the extent that one can compute σ reliably, therefore, one can decide whether, in a given dimension, the RFIM can exhibit ferromagnetic order at low temperature for small h; that is, one can determine d_c.

In this paper we implement the program outlined in the preceding paragraph by considering a "solid-on-solid" model[16] for a domain wall in the RFIM. We analyze the model in the continuum limit where lattice effects are ignored and the interface regarded as an elastic membrane, with a combination of dimensional arguments, elementary power counting, and approximate renormalization-group (RG) calculations. Our results, most of which were first presented in a short note,[5] follow (many of them were derived independently by J. Villain[6]).

(1) Let $\Delta \equiv [h^2]_{\rm av}$, the average square of the random field, and let J represent the exchange constant of the interface model. For $\Delta \ll J^2$ and $T \ll J$, w diverges like $\Delta^{1/3} L^{(5-d)/3}$ as $L \to \infty$ for $d<5$. Thus the interface is indeed rough for all $d<5$, but

$$w/L \sim \Delta^{1/3} L^{(2-d)/3}\,, \tag{1.1}$$

so that $w/L \to 0$ as $L \to \infty$ for all $d>2$. The domain argument is therefore not invalidated by interface roughness for $d>2$; its prediction, $d_c=2$, survives the domain-wall wanderings. Two clarifying comments should accompany this result: First, the domain argument considers only energy accounting; entropy effects are ignored. As such it is valid at $T=0$ and implies only that for $d>2$ the *ground state* of the RFIM is ferromagnetic. Second, neglect of lattice effects in our continuum-interface model almost surely limits the possible applicability to the true lattice RFIM of result (1.1). A similar phenomenon has been elucidated for pure Ising systems, where continuum-interface models[17] predict a rough domain wall at all $T>0$ whenever $d \le 3$. Careful analysis[15,18] has shown, however, that the effect of the discrete lattice is to produce for $2<d\le 3$ a transition (the roughening transition) at low temperatures into a smooth phase characterized by w's which remain finite as $L\to\infty$. For $d \le 2$ domain walls in the pure Ising model are indeed rough for all $T>0$, in agreement with continuum theories. Presumably an analogous situation obtains in the RFIM: On the lattice there exists a dimension $d_R<5$ such that for $d_R<d\le 5$ the domain wall is smooth at sufficiently low T and Δ, whereas for $d\le d_R$ and $\Delta>0$ the wall is always rough. At present the value of d_R is unknown.[19] Although for $d_R<d\le 5$ our continuum result (1.1) is clearly inapplicable to the true lattice RFIM in the smooth phase (i.e., at low Δ and T) it provides an upper bound for the true w. It follows that w/L must vanish as $L\to\infty$ for $d>2$ in the lattice RFIM as well, whereupon the domain-argument prediction $d_c=2$ continues to hold.

(2) For T/J, $\Delta/J^2 \ll 1$, the surface tension σ for our model is positive in the thermodynamic limit for $d>2$. Since $\sigma>0$ is the distinguishing feature of the ferromagnetic state this implies that for $d>2$ the ferromagnetic state at $T=0$ predicted by the domain argument persists for some range of $T>0$. For $d\le 2$ we find $\sigma \to -\infty$ as $L\to\infty$ even for arbitrarily small Δ and T. That σ becomes negative and actually diverges in the thermodynamic limit is highly unphysical; it is presumably an artifact of an approximation—the neglect of droplets and overhangs (to be discussed in Sec. III)—in our interface model. One expects that in the true RFIM, $\sigma=0$ for $L=\infty$, $\Delta>0$, and all T whenever $d\le d_c$. We interpret the divergence of σ in our model for $d\le 2$ as a signal of the instability of the RFIM against domain formation and hence as further support for the conclusion that $d_c=2$. Our results for σ are summarized as follows (σ_0 being the surface tension of our interface model in the absence of random fields):

$$\sigma-\sigma_0 \sim -\Delta^{2/3},\quad -\Delta^{2/3}\ln L,\quad -\Delta^{2/3}L^{2(2-d)/3} \tag{1.2}$$

for d, respectively, greater than 2, equal to 2, and less than 2. These results are valid for $\Delta \ll J^2$, $T\ll J$, and $L\to\infty$.

(3) A simple RG scaling argument applied to the interface model permits calculation of the maximum linear size ξ_Δ which a ferromagnetically ordered domain can attain in the limit of small Δ. This length, which naturally serves as the limiting ferromagnetic correlation length at small T, is simply $\xi_\Delta \sim \Delta^{-(d_c-d)^{-1}}$. Since we assert $d_c=2$,

$$\xi_\Delta \sim \Delta^{-(2-d)^{-1}}\,. \tag{1.3}$$

In particular, $\xi_\Delta \sim \Delta^{-1}$ for $d=1$, in agreement with exact calculations on the one-dimensional (1D) RFIM.[20–22]

(4) It is obviously important to understand the extent to which these results are representative of interfaces in the discrete-lattice RFIM and not simply artifacts of the continuum approximation. To this end we consider a simple discrete-lattice version of the continuum-interface model. We argue that the effect of the discreteness is to change (1.1) and (1.2), respectively, to

$$w/L \sim \Delta L^{(2-d)}\,, \tag{1.4}$$

$$\sigma-\sigma_0 \sim -\Delta,\quad d>2 \tag{1.5a}$$

$$\sigma-\sigma_0 \sim -\Delta L^{2-d},\quad d<2\,. \tag{1.5b}$$

Note that while these results look very different from (1.1) and (1.2), the powers of L on the right-hand sides of (1.4) and (1.5b) still change sign at $d=2$, leaving unaltered the conclusion $d_c=2$. Equation (1.3) likewise continues to hold for the discrete model.

It is worth noting that the utility of considering interface models to extract information about Ising systems near their lower critical dimensionality has been recognized for some years. Wallace and Zia[17] used such an approach to derive critical exponents for the pure Ising model in $(1+\epsilon)$ dimensions. Indeed, at least two groups of authors[3,4] have already constructed and studied interface models in an attempt to describe the RFIM. The starting point for these models is the Ginzburg-Landau (ϕ^4) representation of the spin-$\frac{1}{2}$ RFIM, in contrast to our discrete-spin approach. Using replica methods and supersymmetry arguments to handle the random fields in their respective models, the two groups ultimately achieve unanimity in concluding $d_c=3$. We comment on these calculations and their relation to our work and very different results in Sec. V

The organization of this paper is as follows. In Sec. II we define the continuum model. Section III is devoted to the dimensional and RG calculations, a derivation of the results, a simple interpretation of them in terms of the domain argument, and a discussion of the discrete model. In Sec. IV we reconcile these results with perturbation-theory calculations. Section V contains comments on competing theories of the RFIM and their relation to our work.

II. INTERFACE MODEL

We are interested in the Ising model in d dimensions defined by the Hamiltonian

$$H=-J\sum_{\langle ij\rangle} S_iS_j-\sum_i S_ih_i \ , \tag{2.1}$$

where $\langle ij\rangle$ sums over nearest-neighbor pairs on a cubic lattice; h_i are fixed uncorrelated random fields. Let

$$\begin{aligned}&[h_i]_{\rm av}=0 \ ,\\ &[h_ih_j]_{\rm av}=\Delta\delta_{ij} \ .\end{aligned} \tag{2.2}$$

Our major concern is the effect of h_i at very low temperatures on the interface separating a region of up spins from one of down spins. Therefore, we define an interface model as follows. Let the system be divided into two regions as shown in Fig. 1; S_i is $+1$ if it is in the upper region and -1 if it is in the lower region. The boundary between the regions is the interface, whose position and shape are regarded as the dynamic variables. It is specified by

$$y=f(i) \ , \tag{2.3}$$

where i is the coordinate labeling lattice sites in the $(d-1)$-dimensional horizontal space and y is the vertical coordinate in Fig. 1(a). We set $f(i)=0$ along the boundary of the interface; $f(i)$ can be called the "interface profile."

Note that this description of an interface differs from what one would obtain by simply considering the RFIM with "antiperiodic" boundary conditions. Imposing such boundary conditions alone would give rise to configurations [such as shown in Fig. 1(b)] which contain "droplets" of the wrong phase within each domain and "overhangs" of the interface. The former have been eliminated from our model by construction, the latter by the requirement that $f(i)$ be a single-valued function of i. The hope (thus far unverified) is that neither of these omissions materially affects results for interfacial roughness or surface tension.[23]

The conventional definition of roughness[15] is that for any i, $f(i)$ diverges in the limit of large L, i.e.,

$$\langle f^2(i)\rangle^{1/2}\propto L^{\alpha}, \quad \alpha\geq 0 \ . \tag{2.4}$$

For the pure Ising model in the absence of the random field,[24,17]

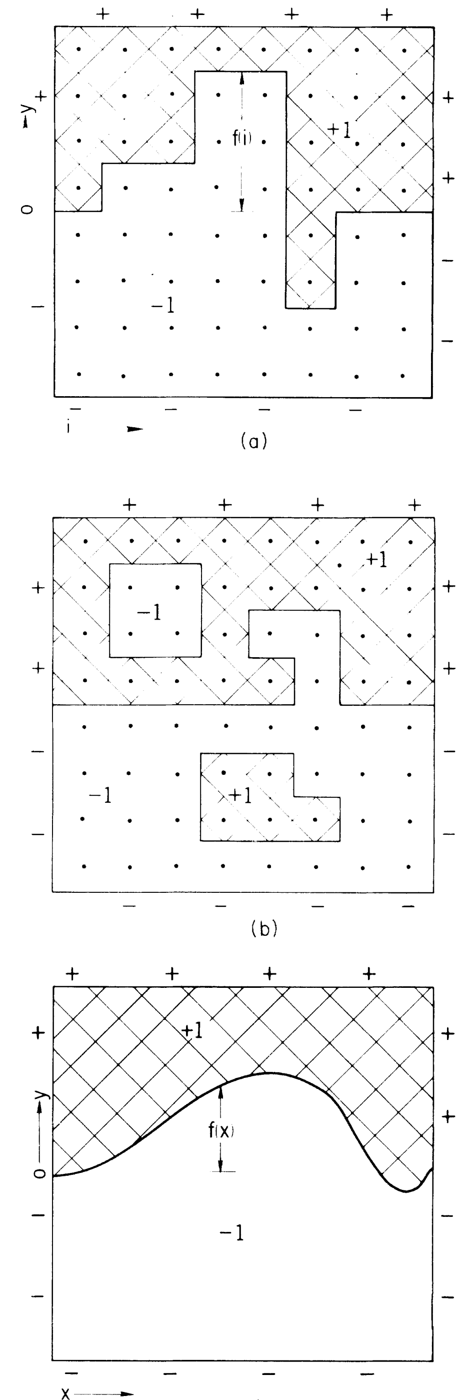

FIG. 1. Interface separating domains of $+1$ (hatched) and -1 Ising spins; (a) is on the discrete lattice with interface profile $y=f(i)$, where i labels columns ; (b) on the lattice, shows "overhang" of interface and "droplets" of wrong sign within each domain; (c) is in the continuum limit, with profile $y=f(x)$.

$$\alpha=\frac{3-d}{2} \tag{2.5}$$

in the rough phase. One must note that a rough interface does not necessarily imply the disappearance of the interface or the destruction of phase coexistence. In order for the phase coexistence to be destroyed the surface tension must vanish , or equivalently,

$$\alpha \geq 1 \ . \tag{2.6}$$

This means that the interface has to grow as thick as L itself in order to be destroyed.[24]

For the analysis in this paper, it is convenient to use a continuum model. We replace the discrete-lattice sites by a continuum and regard the interface profile $f(x)$ as a continuous function of the coordinate x characterizing the $(d-1)$-dimensional hyperplane transverse to y [Fig. 1(c)]. The interface is now regarded as an elastic membrane. We simulate the discrete model by the Hamiltonian

$$H = J\int d^{d-1}x[1+(\vec{\nabla}f)^2]^{1/2} - \int d^{d-1}x\, dy\, S(x,y)h(x,y) \ , \tag{2.7}$$

$$S(x,y)=\mathrm{sgn}[y-f(x)] \ , \tag{2.8}$$

$$\langle h(x,y)h(x',y')\rangle = \Delta\delta(x-x')\delta(y-y') \ . \tag{2.9}$$

The first term of (2.7) is simply the total area of the interface multiplied by J. The second is the random-field energy. Although this is a continuum model, we impose a short-distance cutoff, $a \equiv \Lambda^{-1}$, to prevent ultraviolet divergences in perturbation theory.

We can simplify (2.7) by adding a constant,

$$\int d^{d-1}x\, dy\, \mathrm{sgn}(y)h(x,y) \ ,$$

to H and obtain

$$H = J\int d^{d-1}x[1+(\vec{\nabla}f)^2]^{1/2} + 2\int d^{d-1}x \int_0^{f(x)} dy\, h(x,y) \ , \tag{2.10}$$

a more convenient form. We now proceed to study this model using a RG method similar to Wilson's first approximate recursion formula.[25]

III. ESTIMATING SURFACE TENSION AND ROUGHNESS

A. RG calculations

1. Surface tension

The RG procedure, a succession of coarse-graining and scale transformations,[25] is an efficient way to estimate how the interface roughens to gain energy from the random fields. Note that there is no length scale in the model (2.10) besides the size of the system L and the short-distance cutoff a, taken equal to unity in this subsection. For the moment, let us forget about the cutoff and see how the Hamiltonian behaves under the scale transformation

$$x \to bx', \quad y \to by', \quad f \to bf', \quad L \to bL' \ . \tag{3.1}$$

Substituting (3.1) in (2.10), one sees that

$$\begin{aligned} &H/T = H'/T' \\ &H' = J\int d^{d-1}x'[1+(\vec{\nabla}'f')^2]^{1/2} + \int d^{d-1}x' \int_0^{f'(x')} dy'\, h'(x',y') \ , \\ &T' = b^{-d+1}T \ , \\ &h'(x',y') = bh(bx,by) \ . \end{aligned} \tag{3.2}$$

From (2.9), one obtains a new Δ,

$$\Delta' = b^{2-d}\Delta \ . \tag{3.3}$$

The coupling strength J stays fixed. The free energy per unit area, i.e., the surface tension, $\sigma(T,\Delta)$, thus obeys the transformation law

$$\sigma(T,\Delta) = b^{-d+1}\sigma(b^{1-d}T, b^{2-d}\Delta) \ . \tag{3.4}$$

Now, to take into account the change of the short-distance cutoff a, under the transformation, we need first to increase it from unity to b by calculating the contribution to σ from variables describing short-scale variations of the interface, i.e., shorter than b. This contribution will be called σ_0. After the removal of these short-scale variables and the scale transformation (3.1), the remaining variables will again describe variations on scales longer than unity. The transformation formulas for H and σ are thus

$$\begin{aligned} &\frac{1}{T}H(T,\Delta) = \frac{1}{T'}H(T',\Delta') + \frac{L^{d-1}}{T}\sigma_0(T,\Delta) \ , \\ &\sigma(T,\Delta) = b^{-d+1}\sigma(b^{1-d}T, b^{2-d}\Delta) + \sigma_0(T,\Delta) \ . \end{aligned} \tag{3.5}$$

We have ignored any new terms in the Hamiltonian which may be generated by the elimination of the short-wavelength degrees of freedom and so just keep the two variables T and Δ at each stage. The task now is to calculate σ_0 and then repeat the transformation to obtain σ as a series of contributions from successively longer scales of interface variation. We shall do this approximately, following the lines of Wilson's original phase-space cell analysis.[25]

Let us expand $f(x)$ and $h(x,y)$ in terms of an orthonormal set of functions $\phi_\lambda(x)$,

$$\begin{aligned} &f(x) = \sum_\lambda \phi_\lambda(x) q_\lambda \ , \\ &h(x,y) = \sum_\lambda \phi_\lambda(x) u_\lambda(y) \ , \\ &\int \phi_\lambda(x)\phi'_\lambda(x) d^{d-1}x = \delta_{\lambda\lambda'} \ . \end{aligned} \tag{3.6}$$

Equation (2.9) can be written as

$$\langle u_\lambda(y)u_{\lambda'}(y')\rangle = \Delta\delta_{\lambda\lambda'}\delta(y-y') \ . \tag{3.7}$$

The $\phi_\lambda(x)$ are chosen to be "wave packets" with reasonably well-defined locations as well as magnitudes of $\nabla\phi_\lambda(x)$ or "momenta." The details will be discussed as the calculation proceeds.

Now we write the interface profile and the random field as

$$\begin{aligned} &f(x) = f_0(x) + f_1(x) \ , \\ &h(x,y) = h_0(x,y) + h_1(x,y) \ , \end{aligned} \tag{3.8}$$

where $f_1(x)$ and $h_1(x,y)$ are the "slowly varying" parts which are essentially constant over length scales shorter than b, while $f_0(x)$ and $h_0(x,y)$ describe the variations of f and h on scales between 1 (i.e., a) and b.

The quantities $f_0(x)$ and $h_0(x,y)$ can be written as a sum of wave-packet functions ϕ_λ, each of which covers a region of linear size b or less. The slowly varying quantities f_1 and h_1 are the sum of other ϕ_λ's which are more spread out. By orthogonality, we have

$$\int O_0^{(i)}(x)O_1^{(i)}(x)d^{d-1}x=0,\quad i,j=1,2 \tag{3.9}$$

where $O^{(1)}\equiv h$ and $O^{(2)}\equiv f$. The subscripts 0 and 1 in (3.9) denote quickly and slowly varying parts.

Now substitute $h=h_0+h_1$ and $f=f_0+f_1$ in the Hamiltonian (2.10). We obtain

$$H(f)=H(f_0+f_1)=H(f_1)+H_0(f_0,f_1)\ , \tag{3.10}$$

$$H_0(f_0,f_1)\equiv\int d^{d-1}x\left[\frac{J}{2}(\vec{\nabla}f_0)^2+2\int_0^{f_0+f_1}dy\,h_0(x,y)+2\int_{f_1}^{f_0+f_1}dy\,h_1(x,y)\right]. \tag{3.11}$$

We have approximated $[1+(\vec{\nabla}f)^2]^{1/2}$ by $1+\frac{1}{2}(\vec{\nabla}f)^2$ [a simplification which will be justified just below Eq. (3.30)] and dropped the term $J\int d^{d-1}x\,\vec{\nabla}f_0\cdot\vec{\nabla}f_1$. The task now is to calculate the contribution to σ from $H_0(f_0,f_1)$ for fixed f_1. We shall calculate the contribution for each packet independently. Over the region where a certain packets sits, the slowly varying quantities $f_1(x)$ and $h_1(x,y)$ can be considered as constants. The integration of $f_0(x)$ or $h_0(x,y)$ over this region is therefore zero as orthogonality [i.e., Eq. (3.9)] and the fact that the packets comprising h_0 are essentially localized over a linear distance b show,

$$\int_b d^{d-1}x\,h_0(x,y)=\int_b d^{d-1}x\,f_0(x)=0\ . \tag{3.12}$$

It follows that the last term of Eq. (3.11) vanishes, and that

$$\int_b d^{d-1}x\int_0^{f_1}dy\,h_0(x,y)=0\ . \tag{3.13}$$

Equation (3.11) now simplifies to a sum of packet Hamiltonians, each of which has the form

$$H_{0b}(f_0,f_1)=\int_b d^{d-1}x\left[\frac{J}{2}(\vec{\nabla}f_0)^2+2\int_{f_1}^{f_1+f_0}dy\,h_0(x,y)\right]. \tag{3.14}$$

The x integration is over an area of linear dimension b, i.e., over the spatial extent of a single small packet. For a given packet, (3.14) can be written in terms of the coordinate q_λ defined by (3.6). The first term of (3.14) is

$$H_1(q)\equiv JAb^{-2}q^2\ ,\qquad A\equiv\tfrac{1}{2}\int_b d^{d-1}x(\vec{\nabla}\phi)^2b^2\ . \tag{3.15}$$

A, the average of ∇^2 over the packet, is roughly independent of b, owing to the normalization (3.6) of ϕ. The index λ is suppressed. Note that $f_0(x)$ is just $q\phi(x)$ for this packet. The second term of (3.14) is more complicated. It is

$$H_2(q)\equiv2\int d^{d-1}x\int_{f_1}^{f_1+q\phi(x)}dy\,\phi(x)u(y)\ . \tag{3.16}$$

The only thing we know about $u(y)$ is its statistical property (3.7), from which we obtain

$$\langle[H_2(q)]^2\rangle=4\int d^{d-1}x\,d^{d-1}x'\,\phi(x)\phi(x')\int_0^{q\phi(x)}dy\int_0^{q\phi(x')}dy'\delta(y-y')\Delta=\Delta\,|\,q\,|\,b^{(d-1)/2}B^2\ , \tag{3.17}$$

$$B^2\equiv8\int d^{d-1}x\,|\,\phi(x)\,|^2\int d^{d-1}x'b^{-(d-1)/2}\,|\,\phi(x')\,|\,\theta(\,|\,\phi(x')\,|-|\,\phi(x)\,|\,)\ . \tag{3.18}$$

Note that since ϕ is normalized, its magnitude $|\,\phi\,|$ is proportional to $b^{-(d-1)/2}$. Hence B, like A, is roughly independent of b.

In view of (3.15) and (3.17), we write (3.14) as

$$H_{0b}=JAb^{-2}\left[q^2-C\frac{\sqrt{\Delta}}{J}\,|\,q\,|^{1/2}b^{(d+7)/4}\right]. \tag{3.19}$$

Here C is a function of q, random in both magnitude and sign, of order unity. It will be different for different packets. The ground-state energy for a given packet is obtained as the minimum of H_{0b}. Without knowing the details of C as a function of q, we can only estimate the magnitude of the ground-state energy E_{0b} by minimizing H_{0b} of (3.19), taking C as a constant. Setting $\partial H_{0b}/\partial\,|\,q\,|=0$, we obtain

$$|\,f_0\,|\sim|\,q\,|\,b^{(d-1)/2}\sim\Delta^{1/3}J^{-2/3}b^{(5-d)/3}\ ,\qquad b^{1-d}E_{0b}\sim-J^{-1/3}\Delta^{2/3}b^{(2-d)/3}\ , \tag{3.20}$$

for $C>0$, and $|\,f_0\,|=E_{0b}=0$ for $C<0$. The total ground-state energy per unit area can be obtained by repeating the transformation many times to account for interface fluctuations of progressively longer wavelength; one generates a series

$$\sigma_0\equiv L^{-d+1}E_0=J-J^{-1/3}\sum_{l=1}^{l_L}K_l[\Delta(b^{2-d})^l]^{2/3}\ , \tag{3.21}$$

where $l_L\equiv\ln L/\ln b$. Note that the value of Δ changes by b^{2-d} for every transformation [see Eq. (3.3)]. K_l is a non-negative random number of order unity.

It is clear from (3.21) that for $d>2$ the series will converge; the surface tension is well defined and, for sufficiently small Δ, positive, indicating the existence of a ferromagnetic state at $T=0$. For $d<2$, on the other hand, $\alpha_0 \to -\infty$ as $L \to \infty$, signaling the instability of the ordered ground state to domain formation. We conclude, consistent with the domain argument, that $d_c=2$, at least at $T=0$. Note that for $d=2$, $\sigma_0 \to -\infty$ as $\ln L$. This demonstrates the instability of the ferromagnetic ground state in $d=2$, the one dimension where the domain argument provides no information.

For nonzero temperatures, the surface tension contributed by H_{0b} [Eq. (3.19)] is

$$\sigma_{0b} = -b^{+1-d} T \ln Z_b \; , \qquad Z_b \equiv \int dq \, e^{-H_{0b}/T} \; . \tag{3.22}$$

From (3.19) one obtains

$$\sigma_{0b}(T,\Delta) \sim b^{1-d} \left[E_{0b} - T \ln \left[\frac{\pi b^2}{3J} \right] \right] , \tag{3.23}$$

for $T \ll T_\Delta \equiv J^{-1/3} \Delta^{2/3} \sim E_{0b}$, and

$$\sigma_{0b}(T,\Delta) \sim -b^{1-d} T \ln(Tb^2/J) - \tilde{C}^2 \frac{\Delta}{\sqrt{TJ}} b^{(3-d)/2} \; , \tag{3.24}$$

for $T \gg T_\Delta$; $\tilde{C}^2$ is a positive random number of order unity. In arriving at (3.24) we assumed that the average of C in (3.19) is zero and calculated the C^2 term in (3.22). The total surface tension $\sigma(T,\Delta)$ is obtained through repetition of this calculation for wave packets of progressively larger size,

$$\sigma(T,\Delta) = \sum_{l=1}^{l_L} \sigma_{0b}(T_l,\Delta_l) \; , \tag{3.25}$$

where [see (3.2) and (3.3)] $T_l \equiv b^{l(1-d)} T$, $\Delta_l \equiv b^{l(2-d)} \Delta$. Since $T_\Delta \equiv J^{-1/3} \Delta^{2/3}$, after l iterations T/T_Δ has become

$$T_l / T_{\Delta_l} = b^{-(1+d)l/3} (T/T_\Delta) \; . \tag{3.26}$$

Even for Δ's so small that T/T_Δ is very large, T_l/T_{Δ_l} decreases quickly with increasing l, becoming less than unity when l reaches

$$l_c \equiv 3 \ln(T/T_\Delta)/(d+1) \ln b \; .$$

Thus while it is appropriate to use (3.24) for σ_{0b} in (3.25) when $l \le l_c$, one must use (3.23) when $l > l_c$. One might have been tempted to conclude $d_c = 3$ from the fact that for $d \le 3$, (3.24) substituted in (3.25) would not give a convergent series for σ at large l. For sufficiently large l, however, (3.24) *always* gives way to the $T=0$ approximation (3.23) which, as we saw in (3.21), yields a convergent series for all $d>2$. We conclude that for low but nonzero T and small Δ, $\sigma(T,\Delta)$ is positive and finite as $L \to \infty$; ferromagnetism can therefore persist at finite T in the RFIM so long as $d>2$. This result is a consequence of the strong irrelevance of the variable T [Eq. (3.2)] under RG iteration.

Note that the dominant term in the series (3.25) occurs for $l=l_c$, where both (3.23) and (3.24) give a result proportional to

$$T^{2(2-d)/(d+1)} \Delta^{2(d-1)/(d+1)} \; . \tag{3.27}$$

Thus for T/T_Δ large the surface tension σ goes from $\Delta/T^{1/2}$ for d just below 3 to $\Delta^{2/3}$ for d just above 2.

2. *Interface roughness*

From the results (3.20), we can estimate the interface roughness by calculating

$$f = \sum_\lambda \phi_\lambda q_\lambda = \sum_{l=0}^{l_L} f_l b^l \; , \qquad f_l = a_l (b^{(2-d)l} \Delta)^{1/3} J^{-2/3} b^{(5-d)/3} \; , \tag{3.28}$$

where a_l is a random number of order unity. Assuming that the a_l's are independent one has

$$\langle f^2 \rangle \sim \Delta^{2/3} J^{-4/3} \sum_{l=1}^{l_L} b^{2(5-d)l/3} \sim \Delta^{2/3} J^{-4/3} \tilde{g}(L) \; , \tag{3.29}$$

where $\tilde{g}(L) \sim L^{2(5-d)/3}$, $\ln L$, and constant for $d<5$, $d=5$, and $d>5$, respectively.

According to the usual definition of roughness, the interface is therefore rough[26] for $d \le 5$. We note that

$$\frac{\langle f^2 \rangle^{1/2}}{L} \equiv \frac{w}{L} \sim L^{(2-d)/3} \; . \tag{3.30}$$

For $d>2$, this ratio decreases for large L, so that the interface is still well defined despite the roughness. This conclusion is consistent with the result of the surface-tension calculation, namely $d_c=2$.

Note that according to Eq. (3.29) the typical height $w(b)$ of an interface of linear dimension b (Fig. 2) is $w(b) \sim \Delta^{1/3} b^{(5-d)/3}$. That is, $\nabla f \sim w(b)/b \sim \Delta^{1/3} b^{(2-d)/3}$. For all $d>2$, therefore, $|\nabla f| \ll 1$ even for arbitrarily large length scales ($b \to \infty$), provided Δ is small. This justifies, for small random fields, our earlier approximation of $[1+(\vec{\nabla} f)^2]^{1/2}$ by $1+(\vec{\nabla} f)^2/2$.

It is worth emphasizing that [see Eq. (3.29)] in the presence of random fields $[\langle f^2 \rangle]_{av} \sim L^{2(5-d)/3}$ even at $T=0$ for $d<5$. In the pure Ising model, on the other hand, $\langle f^2 \rangle \sim L^{3-d}$ for $T>0$ due to thermal fluctuations. Since $2(5-d)/3 > 3-d$ for all d, the field-induced wandering of the interface always dominates the thermal wandering, re-

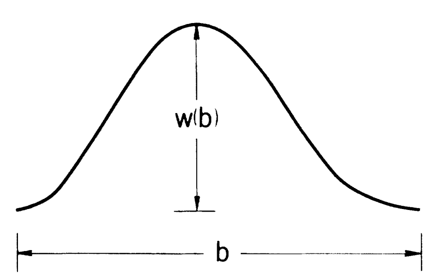

FIG. 2. Section of interface of length b and height $w(b)$.

flecting the strong irrelevance [see Eq. (3.2)] of the temperature in the RFIM.

3. Correlation length ξ_Δ

The lowest-order recursion relation (3.3) for Δ, $\Delta' = b^{2-d}\Delta$, can be used to compute the maximum length ξ_Δ over which spins can be correlated ferromagnetically when $d < d_c = 2$ in the RFIM for asymptotically small Δ. The calculation is trivial. Noting that J does not renormalize in our RG scheme we write for the dimensionless variable Δ/J^2

$$\Delta'/J'^2 = b^{2-d}(\Delta/J^2) . \tag{3.31}$$

Spins can be ferromagnetically aligned only so long as the effective random field Δ/J^2 is small. The limiting length $b = \xi_\Delta$ over which such correlations persist is obtained by setting $\Delta'/J'^2 \sim 1$ in (3.31). The result

$$\xi_\Delta \sim (\Delta/J^2)^{-(2-d)^{-1}} , \tag{3.32}$$

valid for $\Delta/J^2 \ll 1$, and in agreement with the domain argument,[1] follows immediately. Note that for $d=1$, $\xi \sim (\Delta/J^2)^{-1}$, which agrees with exact calculations on the 1D RFIM.[20]

For $d=2$, (3.32) provides no information about ξ_Δ. It seems reasonable to assume that the divergence of the power $(2-d)^{-1}$ in (3.32) as $d \to 2$ signals exponential behavior,[3] $\xi_\Delta \sim e^{\Delta_0/\Delta}$ for some Δ_0 in $d=2$. This result can be obtained by writing (3.31) in differential form, $\partial\Delta/\partial l = (2-d)\Delta$ where $b = e^l$, and assuming that for $d=2$ one has $\partial\Delta/\partial l = \Delta^2/\Delta_0$. Still higher terms [say of $O(\Delta^3)$] in the equation for $\partial\Delta/\partial l$ would give rise to power-law corrections, i.e.,

$$\xi_\Delta \sim \Delta^{-x} e^{\Delta_0/\Delta} \tag{3.33}$$

for some exponent x, to the exponential.[27]

B. Interpretation in terms of the domain argument

The crucial result (3.19), central to all our conclusions, has a simple interpretation in terms of the domain argument[28] reviewed in Sec. I. Imagine trying to estimate the height $w(b)$ of an interface of linear dimension b (Fig. 2), using this argument. Let us first approximate the exchange energy cost $J\int d^{d-1}x[1+(\vec{\nabla} f)^2]^{1/2}$ of the interface [Eq. (2.10)] by $J\int d^{d-1}x[1+(\vec{\nabla} f)^2/2]$. For the interface of Fig. 2, $|\vec{\nabla} f| \sim w/b$, whereupon the exchange-energy cost is just $Jb^{d-1}+Jb^{d-3}w^2$. The field-energy gain is proportional to the square root of the volume under the interface, viz., $(\Delta w b^{d-1})^{1/2}$. The total energy is then

$$E_w \sim Jb^{d-1} + Jb^{d-3}w^2 - \Delta^{1/2}|w|^{1/2} b^{(d-1)/2} . \tag{3.34}$$

[To compare this expression with (3.19), recall that there the height of the interface was simply (ϕq), and that ϕ was normalized so that $\phi \sim b^{-(d-1)/2}$. Substituting $w \sim b^{-(d-1)/2} q$ in (3.34) then reproduces the b, Δ, and q dependence of (3.19) identically.] Minimization of (3.34) with respect to w yields, not surprisingly, $w \sim \Delta^{1/3} J^{-2/3} b^{(5-d)/3}$, or

$$|\nabla f| \sim \frac{w}{b} \sim \Delta^{1/3} J^{-2/3} b^{(2-d)/3} , \tag{3.35}$$

in perfect agreement with (3.20). Again for Δ small and $d > 2$, $|\nabla f| \ll 1$ even on the longest length scales ($b = L \to \infty$); this justifies the approximation

$$[1+(\nabla f)^2]^{1/2} \sim 1 + (\nabla f)^2/2 .$$

Moreover, for small Δ and $d > 2$ the ratio w/b of the thickness of the interface to its length is small, vanishing as b becomes large. The domain argument therefore predicts that interface roughness is negligible for $d > 2$; this justifies using the argument in the first place, thereby establishing the self-consistency of its central result, $d_c = 2$. As a further test of consistency one can compute $\sigma \equiv E_w/b^{d-1}$ from (3.34),

$$\sigma \sim J - O(J^{-1/3}\Delta^{2/3} b^{2(2-d)/3}) . \tag{3.36}$$

For small Δ, σ remains positive even on the longest length scales, provided only $d > 2$; this supports the notion that, at least at $T=0$, ferromagnetism exists for all $d > 2$ at small Δ. The extension of this conclusion to finite T and the statement that there can be no ferromagnetism even at $T=0$ for $d=2$ requires the approximate RG analysis of the preceding section. It should be clear from the discussion above that that analysis is simply the application of the domain argument to an interface on *every* length scale. This allows the domain argument to predict its own self-consistency by accounting for the hitherto neglected roughness of domain walls and providing an estimate of surface tension. Its power and credibility are therefore considerably enhanced.

C. Derivation from dimensional arguments

Simple scaling arguments and dimensional analysis lead, for $d < 3$, to an alternate derivation of the results of the preceding subsections. To see this, consider the replicated version of the interface Hamiltonian (2.10),

$$\frac{H}{T} = \int d^{d-1}x \left[(J/T) \sum_{\alpha=1}^{n} [1+(\vec{\nabla} f_\alpha)^2]^{1/2} - (\Delta/T^2) \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \theta(f_\alpha f_\beta) \min(|f_\alpha|, |f_\beta|) \right] . \tag{3.37}$$

Here $\theta(x) \equiv 1, \frac{1}{2}, 0$ for $x > 0$, $x = 0$, and $x < 0$, respectively, the number of replicas n is to be set to zero at the end of any calculations, and the random fields at each site have been assumed distributed according to a Gaussian distribution of width $(\Delta)^{1/2}$. Defining a rescaled field $\tilde{f}_\alpha \equiv (J/T)^{1/2} f_\alpha$ one obtains

$$\frac{H}{T} \sim \int d^{d-1}x \left[\sum_{\alpha} [\tfrac{1}{2}(\vec{\nabla}\tilde{f}_\alpha)^2 - \tfrac{1}{8}(T/J)(\vec{\nabla}\tilde{f}_\alpha)^4 + O((T/J)^2(\vec{\nabla}\tilde{f}_\alpha)^6)] - (\Delta/T^{3/2}J^{1/2}) \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \theta(\tilde{f}_\alpha \tilde{f}_\beta)\min(|\tilde{f}_\alpha|, |\tilde{f}_\beta|) \right] . \tag{3.38}$$

In the low-temperature limit, the terms $(\vec{\nabla}\tilde{f})^4, (\vec{\nabla}\tilde{f})^6, \ldots$ in this Hamiltonian can be neglected, since their coefficients are at least of $O(T/J)$. Since H/T is dimensionless, $\tilde{f}_\alpha$, T/J, and $\Delta/T^{3/2}J^{1/2}$ have physical dimensions $L^{(3-d)/2}$, L^{d-1}, and $L^{-(d+1)/2}$, respectively. Now imagine computing $\langle \tilde{f}_\alpha^2(\vec{x}) \rangle$ perturbatively as a power series in $\Delta/T^{3/2}J^{1/2}$. The inverse lengths $a^{-1} \equiv \Lambda$ and L^{-1} serve, respectively, as ultraviolet and infrared cutoffs in each term of this expansion. On dimensional grounds alone one infers that $\langle \tilde{f}_\alpha^2(\vec{x}) \rangle$ can be written

$$\langle \tilde{f}_\alpha^2(\vec{x}) \rangle = L^{3-d} k(\Delta L^{(d+1)/2}/T^{3/2}J^{1/2}, a/L) \tag{3.39}$$

for some function k.

To proceed further we must argue about the perturbation expansion for $\langle \tilde{f}_\alpha^2(\vec{x}) \rangle$ in powers of $(\Delta/T^{3/2})$. It is not even immediately obvious how to generate this series from (3.38), which is nonanalytic in the field $\tilde{f}_\alpha$. Noting, however, that for any real number x, $|x| = x\theta(x) - x\theta(-x)$, where

$$\theta(x) = \int \frac{dq}{2\pi i} e^{iqx}/(q - i\epsilon)$$

(ϵ is a positive infinitesimal), we write (3.38) as

$$\frac{H}{T} \sim \int d^{d-1}x \left[\sum_{\alpha} \tfrac{1}{2}(\vec{\nabla}\tilde{f}_\alpha)^2 - \frac{4i\Delta}{T^{3/2}J^{1/2}} \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \tilde{f}_\beta \int_{-\infty}^{\infty} \frac{dq\, dq'}{(2\pi i)^2} \frac{\sin[q\tilde{f}_\alpha + (q'-q)\tilde{f}_\beta]}{(q-i\epsilon)(q'-i\epsilon)} \right] , \tag{3.40}$$

a form which readily facilitates the generation of graphical perturbation expansions. The graphs of model (3.40) are complicated since the sinusoidal interaction comprises arbitrarily high powers of the field $\tilde{f}$ and hence produces vertices of arbitrarily high order. This is not a complication of principle, however; the familiar sine-Gordon theory[29] [to which (3.40) bears an obvious similarity] possesses the same feature, for example. Moreover, (3.40) resembles any other theory of a single scalar field in that the most ultraviolet-divergent element of any graph is the single closed loop (Fig. 3) corresponding to the analytic expression $\int d^{d-1}p/p^2$ in d dimensions. For $d < 3$ this loop is ultraviolet convergent and so, therefore, is any diagram contributing to $\langle \tilde{f}_\alpha^2 \rangle$ (or any other correlation function). Thus $\langle \tilde{f}_\alpha^2 \rangle$ has a finite limit as $a \to 0$. Equation (3.39) then implies that for all $d < 3$,

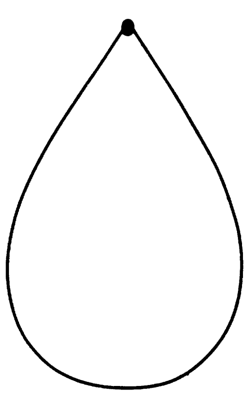

FIG. 3. Most ultraviolet-divergent element of any Feynman graph generated from (3.40). Solid line represents the propagator $1/p^2$.

$$k(\Delta L^{(d+1)/2}/T^{3/2}J^{1/2}, 0)$$

is finite, whereupon, for fixed a,

$$w^2 \equiv (T/J)\langle \tilde{f}_\alpha^2 \rangle \to TJ^{-1}L^{3-d}k(\Delta L^{(d+1)/2}/T^{3/2}J^{1/2}, 0) \tag{3.41}$$

as $L \to \infty$. From the fact that w^2 must have a finite limit as $T \to 0$ for L large but finite we infer that $k(x,0) \sim x^{2/3}$ as $x \to 0$, whence

$$w^2(T=0) \sim \Delta^{2/3}L^{2(5-d)/3} . \tag{3.42}$$

This is just result (3.29). Note that the foregoing demonstration involves the implicit assumption that

$$\lim_{x\to\infty}\lim_{y\to 0} k(x,y) = \lim_{y\to 0}\lim_{x\to\infty} k(x,y) .$$

Strictly speaking one wants to compute w^2 in the limit $T \to 0$ for fixed a and L with $a/L \ll 1$, that is, one should calculate $\lim_{y\to 0}\lim_{x\to\infty} k(x,y)$. In deriving (3.42) we have in fact calculated $\lim_{x\to\infty}\lim_{y\to 0} k(x,y)$. It seems plausible that the limits are indeed interchangeable though we cannot prove it.

The application of dimensional arguments similar to the above also provides a check of the consistency of results (3.21) and (3.27). Specifically, for $T/J \ll 1$ one has

$$\sigma = J - TL^{-(d-1)}\tilde{\sigma}(\Delta L^{(d+1)/2}/T^{3/2}J^{1/2}, a/L) , \tag{3.43}$$

for some function $\tilde{\sigma}$. Suppose we now assume that σ is finite in the limit $L \to \infty$; this is certainly true for sufficiently large d, and yields, from (3.43),

$$\sigma = J - Ta^{-(d-1)}\tilde{\tilde{\sigma}}(\Delta a^{(d+1)/2}/T^{3/2}J^{1/2}) , \tag{3.44}$$

for some $\tilde{\tilde{\sigma}}$. If σ is to have a finite, nonzero limit as $T \to 0$ then $\tilde{\tilde{\sigma}}(x) \sim x^{2/3}$ as $x \to \infty$, whereupon

$$\sigma - J \sim a^{(4-2d)/3} \Delta^{2/3} , \tag{3.45}$$

in agreement with (3.21). Note that, as expected, this formula exhibits no interesting behavior at $d=3$. Only at $d=2$, where the power of a [(3.45)] reverses sign and the small-a limit of $(\sigma - J)$ changes from ∞ to 0, does it show any qualitative change at all. This is consistent with the result $d_c = 2$: For $d < 2$ our assumption of the finiteness of σ in the $L \to \infty$ limit has broken down. Indeed, since for finite L and a, σ must be finite as $T \to 0$, one obtains, from (3.43),

$$\sigma - J \sim L^{(4-2d)/3} \Delta^{2/3} \Sigma(a/L) ,$$

for some function Σ. The assumption that, for $d < 2$ $\Sigma(x) \to$ const as $x \to 0$ then yields

$$\sigma - J \sim L^{(4-2d)/3} \Delta^{3/2} ,$$

consistent with Eq. (3.21) and the result $d_c = 2$.

At finite T we can invoke, for $d < 3$, the ultraviolet finiteness of the theory established earlier in this subsection to conlude that $\tilde{\tilde{\sigma}}(x) \sim x^{2(d-1)/(d+1)}$ as $x \to 0$. The result

$$\sigma - J \sim -T^{2(2-d)/(d+1)} \Delta^{2(d-1)/(d+1)} ,$$

in agreement with (3.27), follows immediately from (3.44).

D. Discreteness of the lattice

The analysis of the preceding subsections is useful only to the extent that the continuum model (2.10) accurately represents an interface in the true lattice RFIM. It is easy to convince oneself that the Hamiltonian appropriate to the lattice-interface model whose profile is described by (2.3) is simply

$$H_D = JL^{d-1} + (J/2) \sum_{\langle i,i' \rangle} |f(i) - f(i')| + \sum_i \mathrm{sgn} f(i) \sum_{y=\mathrm{sgn} f(i)}^{f(i)} h(i,y) . \tag{3.46}$$

Here i and i' label lattice sites in the $(d-1)$-dimensional horizontal space, y is the vertical coordinate in Fig. 1(a), and $\sum_{\langle ii' \rangle}$ denotes a sum over nearest-neighbor columns i and i'. The first two terms of (3.46) give the (standard) exchange-energy cost of the solid-on-solid model[15,16]: J times the total number of bonds cut by the interface. The final term is simply the discrete version of the random-field-energy term in (2.10). The exchange energy constitutes the most obvious difference between (3.46) and (2.10). The continuum generalization of the exchange terms of (3.46) would be

$$J \int d^{d-1}x (1 + |\vec{\nabla} f(x)|)$$

rather than the $J \int [1 + (\vec{\nabla} f)^2]^{1/2}$ of (2.10). We now argue that this difference has no consequences of significance.

The discussion is simplest in the "domain" picture of Sec. III B. Following the argument in that section we estimate the height $w(b)$ of an interface of linear dimension b. The exchange-energy cost from (3.46) should be well approximated by $J(b^{d-1} + b^{d-2} w)$. The field-energy gain remains proportional to $(\Delta w L^{d-1})^{1/2}$; the total energy is then [cf. (3.34)]

$$E_w \sim Jb^{d-1} + Jb^{d-2} w - \Delta^{1/2} w^{1/2} b^{(d-1)/2} . \tag{3.47}$$

Minimization with respect to w yields $w \sim \Delta b^{3-d}$ and a corresponding surface tension $\sigma \sim J - \Delta b^{2-d}$. These are just the results quoted in (1.4) and (1.5). While they look different in detail from the results (1.1) and (1.2) for (2.10), the crucial features indicating $d_c = 2$, viz., that $w(b) \sim b$ for $d = 2$ and that σ is finite as $b \to \infty$ for $d > 2$ continue to hold.

IV. PERTURBATION CALCULATIONS

The general analysis of the preceding section has shown that perturbation series in powers of the random field will not converge [see, e.g., (3.21)]. On the other hand, calculations to lowest order in Δ are so simple that they can reveal some important details inaccessible to the foregoing general analysis. In this section we calculate the surface tension of the interface to first order in Δ in the continuum-interface model (2.10).

A. Interfacial tension to $O(\Delta)$

From (2.1) we calculate the surface tension σ to lowest nonvanishing order in h_i, viz., $O(h_i^2)$. The total free energy F, prior to impurity averaging, is $F = F_0 + F_2$, where F_0 is the free energy of the pure system, and to $O(h_i^2)$,

$$F_2 = -\sum_{ij} h_i h_j \chi_{ij} / 2 .$$

Here

$$\chi_{ij} = \frac{1}{T} (\langle S_i S_j \rangle_0 - \langle S_i \rangle_0 \langle S_j \rangle_0) . \tag{4.1}$$

The subscript 0 denotes the pure system. Averaging over the random fields with (2.2) one obtains

$$[F_2]_{\mathrm{av}} = -\Delta \sum_i \chi_{ii} / 2 .$$

To obtain σ we subtract the $[F_2]_{\mathrm{av}}$ computed with periodic boundary conditions (i.e., with no interface) from that computed with antiperiodic boundary conditions (i.e., with the interface),

$$\sigma - \sigma_0 = (\Delta / 2L^{d-1}) \sum_i (\chi_{ii} - \chi'_{ii}) . \tag{4.2}$$

Here the prime indicates the presence of the interface. As expected, F_2 is directly related to the susceptibility χ_{ii} of the pure system. $\chi'_{ii} - \chi_{ii}$ should vanish if the location i is far from the interface. Therefore the summation over i is dominated by those spins located near the interface. At low temperatures $\langle S_i \rangle_0^2$ is nearly 1, and so is $(\langle S_i \rangle'_0)^2$ except for i near the interface. Previous calculations on pure systems[15–18] indicate that

$$\langle S_i \rangle_0' = m(y_i/w) , \tag{4.3}$$

where the function $m(\eta)$ approaches $\langle S_i \rangle_0$ for $\eta \gg 1$ and $-\langle S_i \rangle_0$ for $\eta \ll -1$. Here w is the "width" of the interface and y_i is the distance of the spin S_i from the center of the interface. Equation (4.3) applies for w and y_i large compared to the lattice spacing. F_2 of (4.1) is proportional to w. From (4.2) follows

$$\sigma - \sigma_0 \sim (\Delta/T) \int_0^\infty dy[m^2(y/w) - m^2(\infty)] \sim -C\Delta w/T ,$$
$$-C \equiv \int_0^\infty d\eta[m^2(\eta) - m^2(\infty)] . \tag{4.4}$$

For $d > 3$ it has been argued[15–18] that w is finite in the limit $L \to \infty$; for $d \le 3$ there is a roughening transition temperature T_R such that [30] $w \to \infty$ for $T \ge T_R$ and w is finite for $T < T_R$. More precisely, $w \sim L^{(3-d)/2}$ or $(\ln L)^{1/2}$ for $d < 3$ or $d = 3$, respectively, if $T \ge T_R$; $T_R = 0$ for $d \le 2$. Naively, then, it would seem from (4.4) that for $T < T_R$ where w is finite the random-field corrections to σ_0 are small so long as Δ is small. As we pointed out earlier, however, this conclusion is almost surely an artifact of the perturbation theory. That is, the w that appears in (4.4) is the w appropriate to the *pure* system and is therefore finite at sufficiently low T for all $d > 2$. Our strong belief[3,5] is that for any dimension of physical interest (i.e., $d \le 3$) the random fields roughen the domain walls so that w diverges even at $T = 0$. Since this roughening of the wall is invisible to the perturbation calculation, conclusions derived from (4.4) in the *smooth* phase (i.e., where w is finite) can be extremely misleading. It therefore makes sense to apply (4.4) only in the *rough* phase where $w \to \infty$ as $L \to \infty$, in which case

$$\sigma - \sigma_0 \sim -\Delta L^{(3-d)/2}, \quad -\Delta(\ln L)^{1/2} \tag{4.5}$$

for $d < 3$, $d = 3$, respectively. For $d > 3$ the pure system has no rough phase whereupon

$$\sigma - \sigma_0 \sim -\Delta , \tag{4.6}$$

i.e., σ remains finite to $O(\Delta)$ as $L \to \infty$. The perturbative results (4.5) and (4.6) are in perfect accord with the more detailed conclusion (3.27): For $d < 3$ the leading small $-\Delta$ correction to $\sigma - \sigma_0$ is proportional to $\Delta^{2(d-1)/(d+1)}$. The exponent $2(d-1)/(d+1)$ is less than unity for $d < 3$, whereupon any attempt to expand $\sigma - \sigma_0$ in integral powers of Δ will give rise to divergent coefficients as in (4.5). For $d > 3$ this exponent is greater than unity whereupon the coefficient of the term proportional to Δ in the perturbation expansion is finite, just as in (4.6). In the marginal case $d = 3$, the coefficient is very weakly (i.e., logarithmically) divergent.

To $O(\Delta)$ the response of each spin to the random field is independent of all other spins. As the discussion of Sec. III shows, however, the responses are not independent. The long-wavelength parts of the random field induce collective responses which are much weaker than (4.2) indicates. That is, (4.3) [or (4.4)] is an overestimate of the effect of the random field.

For $d = 3$, the result (4.4) is, nevertheless, not useless since $(\ln L)^{1/2}$ is not a large quantity for any system of interest in the laboratory. The collective effect is not too important.

In the remainder of this section we compute the form of $m(\eta)$ and the temperature dependence of w. It is extremely difficult to calculate these quantities directly for the Ising model (2.1); the utility of such a calculation is dubious anyhow, given our cautionary remarks about the reliability of perturbative calculations which predict that the system is in the smooth phase. A more realistic picture is obtained by evaluating (4.4) in the continuum-interface model (2.10) which correctly (we believe) predicts that the system is rough at least for $d \le 3$.

To do this calculation we simplify the first term of (2.10) to

$$H_0 = (J/2) \int d^{d-1}x (\vec{\nabla} f)^2 .$$

It follows from (4.1) and (4.2) that to evaluate $\sigma - \sigma_0$ to $O(\Delta)$ we need only compute

$$\langle S(\vec{x},y) \rangle_0 \equiv \langle \mathrm{sgn}[y - f(\vec{x})] \rangle_0$$

in the ensemble characterized by H_0. The identity

$$\mathrm{sgn}x = \int_{-\infty}^{\infty} \frac{dq}{\pi i} \frac{q}{q^2 + \epsilon^2} e^{iqx} \tag{4.7}$$

reduces this calculation to a trivial Gaussian integral. The results

$$\langle S(\vec{x},y) \rangle_0 = m(y/w) = (2/\pi)^{1/2} \int_0^{y/w} d\eta\, e^{-\eta^2/2} , \tag{4.8}$$

where

$$w^2 \equiv \langle f^2(\vec{x}) \rangle_0 = \frac{T}{J} \int \frac{d^{d-1}k}{(2\pi)^d} \frac{1}{k^2} \tag{4.9}$$

follow at once. As anticipated, w is finite only for $d > 3$ [the k integral has an upper (ultraviolet) cutoff of course]. Finally then, (4.4) becomes

$$\sigma - \sigma_0 \sim -\Delta h_d(L)/T^{1/2} , \tag{4.10}$$

where $h_d(L) \sim L^{(3-d)/2}$, $(\ln L)^{1/2}$, and const for $d < 3$, $d = 3$, and $d > 3$, respectively.

Note the divergence of (4.10) in the limit $T \to 0$. This divergence would not have appeared had we evaluated $\sigma - \sigma_0$ with either (2.1) or the *discrete* interface model defined near Eq. (2.3). It is easy to argue about the results of doing the discrete calculation; at low T roughly speaking,

$$\sigma - \sigma_0 \sim -\Delta \tilde{h}_d(L) e^{-J/T} , \tag{4.11}$$

where $\tilde{h}_d(L) \sim L^{1/2}$ and constant for $d = 2$ and $d > 2$, respectively. Thus the discrete models have $\sigma - \sigma_0$ finite as $L \to \infty$ for all $d > 2$ at sufficiently low T; the width w also vanishes exponentially as $T \to 0$. We have already argued several times that these results are artifacts of perturbation theory and do not give a reasonable picture of the physics of the RFIM.

V. CONCLUSIONS

A. RFIM for $d=2+\epsilon$

Our assertion that $d_c=2$ for the RFIM suggests that one might construct from the interface model (2.10) a systematic expansion for the critical exponents characterizing the paramagnetic-ferromagnetic transition in $(2+\epsilon)$ dimensions. Such an expansion would serve as the analog in our theory of the $(3+\epsilon)$ expansions of Refs. 3 and 4 and the $(1+\epsilon)$ expansion of Ref. 17 for the pure Ising model. We have been unable to produce such an expansion, the reason being that in contrast to the situation in Refs. 3 and 4, the RG equations for Δ and T are nonanalytic functions of T at $T=0$ beyond lowest order. This difficulty manifests itself in the lowest nontrivial order of perturbation theory. In Eq. (4.10) for σ, e.g., the term proportional to Δ diverges as $T\rightarrow 0$. Higher terms in the series exhibit similar singular behavior.

B. Relation to previous work

In this section we comment briefly on the several existing papers which argue that $d_c=3$. These calculations start, without exception, from the Ginzburg-Landau-Wilson (or ϕ^4) representation of the RFIM whereas we have used discrete Ising spins. While it is conceivable, given the complexity of the RFIM, that these two different representations belong to two universality classes so disparate as to have different critical dimensionalities, this possibility strikes us as extremely remote. We believe that our arguments for $d_c=2$ are quite compelling, but our analysis is far from exact. We cannot assert categorically that $d_c=2$. It is therefore crucial to consider the rigor of the conflicting theories: If they are correct our calculations must be in error. The point of view elaborated in this section is that none of the arguments for $d_c=3$ is completely convincing. We attempt to identify the approximations in each argument which are potentially dangerous, trying wherever possible to make contact with our own work. This discussion is suggestive rather than definitive; the various approaches are so varied formally as to obstruct direct comparison.

The original arguments suggesting $d_c=3$ were due to Aharony *et al.*,[14] who considered the Ginzburg-Landau-Wilson representation,

$$H_\phi=\int [\tfrac{1}{2}(\vec{\nabla}\phi)^2+(r/2)\phi^2+u\phi^4-h(x)\phi(x)]d^dx \ , \tag{5.1}$$

of the RFIM. They argued that a class of diagrams of (5.1)—the "tree-approximation" class containing no closed loops—is [once the averaging over the random field $h(x)$ is performed] the most-infrared singular class and therefore controls the critical behavior of the theory. In d dimensions, moreover, these diagrams were argued to be equivalent to those describing the critical properties of the corresponding *pure* system [$h(x)=0$ in (5.1)] in $(d-2)$ dimensions; the critical exponents of these two theories were thereby claimed equivalent. The inference that $d_c=3$ for the RFIM then follows from the fact that $d_c=1$ for the pure Ising model.

How could this argument fail? The only approximation is the neglect of the "nontree" diagrams of (5.1). The validity of this omission is conveniently considered in the framework of the RG. It is useful[14] to express the fully averaged RFIM in terms of the variables u and $w=u\Delta$, where $\Delta\equiv[h^2]_{\rm av}$. Elementary power counting shows that in a standard momentum-shell recursion-relation calculation,

$$w'=b^{6-d}(w+\cdots) \ , \tag{5.2a}$$

$$u'=b^{4-d}(u+\cdots) \ , \tag{5.2b}$$

where b is the usual RG scale factor. Six is, from (5.2a), the upper critical dimension for the RFIM. Since graphs of (5.1) with closed loops are proportional to at least one power of u when expressed in terms of u and w, they do not contribute at the fixed point (u^*,w^*) if $u^*=0$. Thus the tree approximation for the RFIM and the ensuing equivalence to the pure Ising model in two fewer dimensions seem entirely reasonable provided $u^*=0$. For d just slightly below 6, where w^* is small, the neglected higher-order terms in (5.2b) are small compared to the single term shown, the fixed point with $u^*=0$ is stable, and the $d\rightarrow(d-2)$ equivalence is on firm footing. As d decreases and w^* increases these higher-order terms might, in principle, become large enough to make the $u^*=0$ fixed point unstable and invalidate the tree approximation. It has, however, been shown[14] that (5.2b), linearized about the (nontrivial) fixed point with $u^*=0$, $w^*\neq 0$, is simply $u'=b^{-2}u$ to all orders in perturbation theory in $\epsilon\equiv 6-d$. Thus in perturbation theory the $u^*=0$ fixed point is stable for all d, suggesting the validity of the $d\rightarrow(d-2)$ equivalence. However, the possibility that nonperturbative effects destroy this result below some dimension, say $\tilde{d}$, less than 6, remains. The most natural candidate for a value of $\tilde{d}$ is 4, the dimension below which naive power counting predicts that the ϕ^4 operator in (5.1) is a relevant operator, but this is pure speculation. These observations do not, of course, prove the failure of the $d\rightarrow(d-2)$ analogy for any particular range of d, but rather indicate that the connection has only been established order by order in perturbation theory in ϵ.

An alternate derivation of the $d\rightarrow(d-2)$ connection, due to Parisi and Sourlas,[14] exploits the fact that the generating functional for the tree approximation for any Ginzburg-Landau-Wilson Hamiltonian can be expressed in terms of the solution of the classical (Ginzburg-Landau) equation for that Hamiltonian; in the case of (5.1) that equation is

$$-\nabla^2\phi(\vec{x})+r\phi(\vec{x})+4u\phi^3(\vec{x})-h(\vec{x})=0 \ . \tag{5.3}$$

Through this representation the generating functional can be neatly expressed in terms of a superfield, whence the $d\rightarrow(d-2)$ correspondence follows directly. This derivation, starting as it does from the tree approximation, involves the same uncertainties for values of d significantly below 6 as the original Aharony *et al.* argument.

Recently, Cardy[9] has used the supersymmetric formulation to argue about the three-dimensional (3D) RFIM at low temperatures. His idea is to treat the parameter r in

(5.1) as fixed and negative, independent of temperature, and introduce the temperature by considering the partition function $\mathrm{Tr}e^{-H_\phi/k_BT}$. In this formulation each closed loop in a diagram gives rise to a factor of T; in the $T\to 0$ limit in any dimension, therefore, one need only worry about the tree diagrams in each order of perturbation theory in u. As before, the trees lead, through (5.3), to the supersymmetric representation, from which follows $d\to(d-2)$; this correspondence is claimed to be exact for the ground state, $T=0$, of the RFIM, which is then argued to be disordered by analogy to the pure 1D Ising chain at finite T. However, for $r<0$, (5.3) has three solutions even for $h(\vec{x})=0$. To arrive at the supersymmetric representation it is necessary to average over all solutions of (5.3). It is therefore unclear that the result $d_c=3$ reflects a property of the true ground-state (i.e., lowest-energy) solution of (5.3) rather than of a weighted average of all solutions, some of which are presumably local maxima, others metastable minima. It seems to us reasonable that the inclusion of these extraneous extrema is responsible for the result $d_c=3$. The actual ground state may be considerably more ordered than any superposition of extrema.

It remains to consider the two field theoretic interface models which have been used to argue $d_c=3$. One, due to Kogon and Wallace,[4] starts from the tree approximation to the generating functional for (5.1) and so is accompanied by the uncertainties discussed above. The other, due to Pytte, Imry, and Mukamel,[3] utilizes the replica method to handle the random fields. The starting point is the replicated version,

$$H_n=\int d^dx \int \frac{1}{2}\sum_{\alpha=1}^{n}\left[(\vec{\nabla}\phi_\alpha)^2+r\phi_\alpha^2+2u_\alpha^4 - \Delta\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\phi_\alpha\phi_\beta\right], \tag{5.4}$$

of (5.1), the random fields having been assumed distributed according to a Gaussian distribution of width Δ. The number of replicas n is as always, to be taken to zero. For $r<0$ (where the system ought to order ferromagnetically if it does at all) one constructs an interface model by minimizing H_n with respect to the field $\phi_\alpha(\vec{x})$, subject to the boundary conditions that $\phi_\alpha(z)\to\pm|r|/4u$ as $z\to\pm\infty$, z being the direction perpendicular to the interface. Denoting by $\phi_\alpha^*(z)$ the function which minimizes $H_n\{\phi_\alpha\}$, one obtains the desired interface Hamiltonian, at least in principle, by substituting

$$\phi_\alpha(z)=\phi_\alpha^*\left[\frac{z-f_\alpha(\vec{x})}{[1+(\vec{\nabla}f_\alpha)^2]^{1/2}}\right] \tag{5.5}$$

in (5.4) and expanding the result in terms of the interface height variable f_α and its derivatives. If one assumes, as do Pytte *et al.*, that the replica symmetry remains unbroken, then the term $\Delta\phi_\alpha\phi_\beta$ of (5.4) does not contribute to the equation for ϕ^* in the $n\to 0$ limit, whereupon H_n is minimized by the field which minimizes the energy of the corresponding pure problem,

$$\phi_\alpha^*(z)=\phi_p(z)\equiv(|r|/4u)^{1/2}\tanh[(\tfrac{1}{2}|r|)^{1/2}z]\ .$$

This result is puzzling. Obviously the interface will wander in the presence of random fields to gain field energy; the function ϕ_p cannot possibly minimize the energy of the random problem. There are, however, no other replica-symmetric minima of H_n.[31] This is suggestive that for a proper description of the physics of the RFIM one should study the possibility[32] of broken replica symmetry, by analogy with spin-glasses,[33] but at the moment the usefulness of this idea is unclear.

Suppose one ignores this possibility and simply uses (5.5) to write H_n as a power series in $f_\alpha(\vec{x})$. Pytte *et al.* do not, strictly speaking, follow this laborious procedure but rather from ingenious symmetry considerations infer that the resulting Hamiltonian must take the form

$$H_n\{f_\alpha\}\sim\int d^{d-1}x\left[\sum_{\alpha=1}^{n}[1+(\vec{\nabla}f_\alpha)^2]^{1/2}+T^{-1}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}[1+(\vec{\nabla}f_\alpha)^2]^{1/2}g\left[\frac{f_\alpha-f_\beta}{[1+(\vec{\nabla}f_\alpha)^2]^{1/2}}\right]\right]. \tag{5.6}$$

Had one the strength to carry through the calculation we have outlined one could, via (5.5), compute the function $g(y)$ in (5.6) as an explicit power series in y^2. For purposes of analyzing (5.6) with the RG, however, it is necessary only to assume such a power series exists, not to know the coefficients. The crucial observation is that the critical dimension d_c of $H_n\{f_\alpha\}$ is determined by elementary power counting on the most relevant (lowest-order) term, $(f_\alpha-f_\beta)^2$, of this expansion. The coefficient of this term grows and shrinks, respectively, under RG iteration for $d\le 3$ and $d>3$; i.e., naive power counting suggests $d_c=3$. Thus in the calculation of Pytte *et al.*, $d_c=3$ results from the analyticity of $g(y)$ at $y=0$. This fact lies at the heart of the disparity between their results and ours.

To elaborate upon this point, we note that Mukamel[34] has argued that for large y, $g(y)$ ought to behave like $|y|$. If one puts $g(y)\sim|y|$ for all y in (5.6), thereby overturning the assumption that $g(y)$ is analytic near $y=0$, one obtains an $H_n\{f\}$ identical to our interface model (3.37) for which $d_c=2$. Setting $g(y)\sim y^2$ for all y, on the other hand, produces a Hamiltonian similar in structure and identical in its power counting properties and critical dimension (viz., $d_c=3$) to the interface representation of the "random-rod" model[5]—a RFIM in which the random fields are correlated along straight lines in one direction. Since the Imry-Ma domain argument[1] also predicts $d_c=3$ for the random-rod model, it seems possible that the choice $g(y)\sim y^2$ in (5.6) produces a model whose physics is essentially that of random rods rather than site-random fields. [Since the symmetry properties of the random-rod model, which is, unlike (5.6), clearly spatially anisotropic, are not identical to those of (5.6), however, this idea is highly speculative.] Suppose now one chooses a function $g(y)$ (such as $g \sim \ln \cosh y$) which behaves like y^2 and $|y|$ at small and large y, respectively. Should d_c be determined by the large-y or small-y behavior of g? We are biased in favor of the former; our prejudice is that d_c should be determined by the behavior of H_n near the true ground state of the system, i.e., at *large* values[35] of f_α. [There is no dispute over the fact that f_α is very large in the vicinity of the ground state. Indeed, Pytte *et al.* argue that even at $T=0$, $\langle f_\alpha^2(\vec{x}) \rangle$ diverges[3] in the thermodynamic limit for all $d \leq 5$. This is consistent with our assertion that for a typical collection of random fields the true lowest-energy interfacial profile of the random system lies far from the pure system profile $f_\alpha(\vec{x}=0.]$ On the other hand, it is not trivial to see how a RG calculation starting from (5.6) with $g(y) \sim \ln \cosh y$ would support our prejudice, which requires that the renormalized function $g(y)$ end up looking like $|y|$ rather than y^2 after many RG iterations. Hopefully further study will clarify these intriguing questions.

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[21] The result $\xi_\Delta \sim \Delta^{1/2}$ is derived for the interface model of Pytte *et al.* in Ref. 3, not surprisingly, since $d_c=3$ in that model. For $d=1$, then $\xi_\Delta \sim \Delta^{-1/2}$, in disagreement with the exact 1D result.

[22] Result (1.3) has been derived directly for the RFIM (rather than for an interface representation thereof) by Aharony and Pytte, Ref. 7.

[23] In the two-dimensional (2D) *pure* Ising model it is known that the effects of the omitted droplets and overhangs cancel exactly. See L. Onsager, Phys. Rev. 65, 117 (1944); H. N. V. Temperley, Proc. Cambridge Philos. Soc. 48, 683 (1952); M. E. Fisher, J. Phys. Soc. Jpn. Suppl. 26, 87 (1969); E. Müller-Hartmann and J. Zittartz, Z. Phys. B 27, 261 (1977). It is easy to show via perturbation theory that such exact cancellation does not occur in 3D in the pure system nor in any dimension for the RFIM. However, it is not unreasonable that such gross features as the dependence of w on L for large L are not affected by the omission.

[24] See, e.g., D. J. Wallace, in *Phase Transitions, Cargese 1980,* edited by M. Levy, J. C. LeGuillou, and J. Zinn-Justin (Plenum, New York, 1982), p. 423. Note from (2.5) that for the pure Ising model the condition $\alpha=1$ implies $d=1$, consistent with the lower critical dimension being unity.

[25] K. G. Wilson, Phys. Rev. B 4, 3174 (1971).

[26] We emphasize that as discussed in the Introduction these interface widths, computed for our continuum model, are upper bounds for the widths one would obtain from discrete-lattice models.

[27] Note that $\Delta \gg \Delta_0$ implies $\xi_\Delta \sim \Delta^{-x}$. If Δ_0 turns out to be very small compared to Δ's accessible in a typical 2D experiment, ξ_Δ may therefore be observed to vary algebraically with Δ.

[28] T. Nattermann (unpublished) has independently pointed out this interpretation.

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[31]We are grateful to H. Sompolinsky for helpful discussions of this point.

[32]G. Parisi (unpublished) has independently made this suggestion.

[33]See, e.g., G. Parisi, Phys. Rev. Lett. 43, 1754 (1980); J. Phys. A 13, 1101 (1980); A 13, 1887 (1980).

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[35]In other words, we believe that the statistics of *extremely large* blocks of random fields, i.e., the fact that the net random field in a volume V is proportional to $V^{1/2}$ for large V, ultimately determines d_c.