Quantized Hall conductance in a relativistic two-dimensional electron gas

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The formula for the quantized Hall conductance in a two-dimensional electron gas is often derived by solving the Schrödinger equation for an electron in crossed electric and magnetic fields, and taking the expectation value of the current operator in its eigenstates. In this report we demonstrate explicitly, by using the Dirac equation, that there are no relativistic corrections to this expression, at least in the ideal case. This is true even if the drift velocity of the electrons approaches the speed of light or the Landau level splitting approaches the electron rest-mass energy and holds despite the appearance of a classical correction to the cyclotron frequency.

The transverse magnetoresistance of the two-dimensional metals formed at semiconductor-insulator or semiconductor-semiconductor interfaces has been observed[1] to be given by the expression

$$R_H = h/e^2 i \ , \tag{1}$$

where i is an integer equal to the number of Landau levels for which all extended states are occupied. Equation (1) holds an accuracy of better than 1 part in 10^6. It can readily be derived by considering a noninteracting two-dimensional (2D) electron gas in crossed electric and magnetic fields, and most theoretical activity has focused on establishing Eq. (1) in the presence of various classes of background potentials.[2-8] In this paper we report on an investigation of the possibility of relativistic corrections to Eq. (1), suggested by Girvin and Cage.[9] As discussed below, we find that the same result for an ideal 2D electron gas is obtained irrespective of whether Eq. (1) is derived from the Schrödinger equation or the Dirac equation. Correspondingly, although we do not focus on that aspect here, the general argument of Laughlin[10] can be generalized to the relativistic case and so we should expect no relativistic corrections to Eq. (1) in real quantum Hall systems.

We start by considering the Dirac equation for a 2D electron gas in the x-y plane in crossed electric $[\vec{E} = (-E, 0, 0), \phi = Eex]$ and magnetic $[\vec{H} = (0, 0, H), \vec{A} = (0, Hx, 0)]$ fields. Following Landau and Lifshitz[11] we first consider the auxiliary second-order equation

$$[(\mathscr{E} - eEx)^2 - (c\vec{p} - e\vec{A})^2 - m^2c^4 + e\hbar cH\Sigma_z + ie\hbar cE\alpha_x]\phi = 0 \ , \tag{2}$$

where $\mathscr{E}$ is the Dirac equation eigenvalue and other notations are standard. The first three terms on the left-hand side of Eq. (2) depend only on orbital coordinates while the last two depend only on spinor coordinates. This allows us to seek solutions to Eq. (2) which are the product of spinor and orbital functions. The normalized eigenspinors and eigenvalues for the last two terms in Eq. (2) are

$$\lambda_\pm = \pm e\hbar c\sqrt{H^2 - E^2} \ , \tag{3a}$$

$$\chi_+ = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ i\beta \end{bmatrix} \chi_- = \begin{bmatrix} 0 \\ \alpha \\ -i\beta \\ 0 \end{bmatrix} , \tag{3b}$$

$$\alpha = \left[1 + \left(\frac{H - \lambda_+/e\hbar c}{E}\right)^2\right]^{-1/2} , \tag{3c}$$

$$\beta = \left[1 + \left(\frac{E}{H - \lambda_+/e\hbar c}\right)^2\right]^{-1/2} . \tag{3d}$$

The orbital eigenfunctions for the first three terms in Eq. (2) may be taken as plane waves in the y and z direction,[12] with the x-dependent wave function obeying

$$\left[\mathscr{E}^2 - 2mc^2\left(\frac{p_x^2}{2m} + \frac{m\omega_c^2}{2}(x - x_0')^2\right) - e^2H^2x_0^2 + \frac{e^2(H^2x_0 - E\mathscr{E}/e)^2}{H^2 - E^2} - \hbar^2k_z^2c^2 - m^2c^4\right]\phi_n(x) \equiv \lambda_n\phi_n(x) \ , \tag{4}$$

where $\omega_c^2 = e^2(H^2 - E^2)/m^2c^2$, $x_0 = \hbar k_y c/eH$, and $x_0' = (H^2x_0 - E\mathscr{E}/e)/(H^2 - E^2)$. The only nonconstant term on the left-hand side of Eq. (4), the second term, is proportional to the one-dimensional harmonic-oscillator Hamiltonian, and so the eigenvalues and eigenfunctions are known. The equation $\lambda_n + \lambda_\pm = 0$ may be solved to determine $\mathscr{E}(\vec{k}_\perp, n, \pm)$ with the result

$$\mathscr{E}(\vec{k}_\perp, n, \pm) = eEx_0 + \left(\frac{H^2 - E^2}{H^2}\right)^{1/2} \times \{m^2c^4 + \hbar^2k_z^2c^2 + mc^2[\hbar\omega_c(2n + 1 \pm 1)]\}^{1/2} \ . \tag{5}$$

Apart from a normalization constant, the solution to the Dirac equation, ψ, is related to

$$\Phi(\vec{k}_\perp, n, \pm) = (L_yL_z)^{-1/2}\exp(ik_yy)\exp(ik_zz)\phi_n(x)\chi_\pm \tag{6}$$

by

$$\psi = \{\beta[\mathscr{E}(\vec{k}_\perp, n, \pm) - eEx] - \vec{\gamma}\cdot(c\vec{p} - e\vec{A}) + mc^2\}\Phi \ . \tag{7}$$

It is useful to compare these solutions with their nonrelativistic counterparts. In Eq. (4) we see that the cyclotron frequency ω_c has been reduced, compared to its nonrelativistic value, $\omega_c \equiv eH/mc$, by a factor of $[1 - (E/H)^2]^{1/2}$; this feature is shared with the corresponding classical problem.[13] The centers of the x-dependent orbitals are located at

$$x_0' = x_0 - \frac{E}{eH\sqrt{H^2-E^2}} \{m^2c^4 + \hbar^2 k_z^2 c^2 + mc^2[\hbar\omega_c(2n+1\pm 1)]\}^{1/2} \quad (8)$$

which can be compared with the nonrelativistic expression $x_0' = x_0 - Emc^2/eH^2$. The important point here is that even though the distance that the orbital centers are shifted by the electric field is changed in the relativistic treatment, the separation between orbitals with differing values of k_y is unchanged $[\delta_x = (\hbar c/eH)\delta k_y]$. With the use of Eq. (8) the energy can be reexpressed in terms of the electrostatic potential at x_0',

$$\mathcal{E}(\vec{k}_\perp, n, \pm) = eEx_0' + \frac{H}{\sqrt{H^2-E^2}} \times \{m^2c^4 + \hbar^2 k_z^2 c^2 + mc^2[\hbar\omega_c(2n+1\pm 1)]\}^{1/2} , \quad (9)$$

which can be compared with the nonrelativistic expression

$$\mathcal{E} = eEx_0' + mc^2 + \frac{h^2 k_z^2}{2m} + \hbar\omega_c(n + 1 \pm \tfrac{1}{2}) + \frac{m}{2}\left(\frac{cE}{H}\right)^2 + \cdots . \quad (10)$$

As we can see in going from Eq. (9) to Eq. (10), it is possible to distinguish two types of relativistic corrections, those which go as $(E/H)^2$ and those which go as $\hbar\omega_c/mc^2$. In typical experiments $\hbar\omega_c/mc^2 \sim 10^{-8}$ and, calculating on the basis of an ideal system, $E/H \sim 10^{-6}$. As we see more explicitly below E/H is the ratio of the "drift" velocity of the current-carrying electrons to the speed of light. If most of the current is carried by few electrons because of localization or because of electrostatic[14] effects, then this ratio could be several orders of magnitude large. Also, the drift velocity of electrons in edge states[15] can be several orders of magnitude larger. Thus, at first sight, it seems that there could be relativistic corrections to the quantum Hall effect at an accuracy level which is currently being approached experimentally.

To calculate the Hall current we assume that some integral number i of Landau levels is occupied. Then

$$I = L_y^{-1} \sum_{k_y, n, \lambda\pm}^{\text{occ}} \frac{\langle \psi(\vec{k}_\perp, n, \lambda\pm | j_y | \psi(\vec{k}_\perp, n, \lambda\pm)\rangle}{\langle \psi(\vec{k}_\perp, n, \lambda\pm) | \Psi(\vec{k}_\perp kn, \lambda\pm)\rangle} , \quad (11)$$

where the relativistic current operator $j_y = ec\alpha_y$, independent of $\vec{A}$. To evaluate its matrix element in Eq. (11) we use the operator relation[16]

$$\left[H, \vec{p} - \frac{e}{c}\vec{A}\right] = -\frac{ie}{h}\left(\vec{E} + \frac{\vec{j}\times\vec{H}}{ec}\right) ,$$

where $H = \vec{\alpha}\cdot(c\vec{p} - e\vec{A}) + \beta mc^2 + e\phi$ is the single-particle Dirac Hamiltonian. Since the expectation of the commutator is zero we have, for each eigenstate of H,

$$\langle \psi(\vec{k}_\perp, n, \lambda\pm) | j_y | \psi(\vec{k}_\perp, n, \lambda\pm)\rangle = (ecE/H)\langle \psi(\vec{k}_\perp, n, \lambda\pm) | \psi(\vec{k}_\perp, n, \lambda\pm)\rangle . \quad (12)$$

Equation (12) can also be obtained explicitly, if somewhat tediously, by using Eqs. (3), (6), and (7). Thus we have that

$$I = \sigma_0 i \left(\frac{ecV_H}{H}\right) , \quad (13)$$

where $V_H = EL_x$ and σ_0 is the number of states per unit area per Landau level. However, from Eq. (8) and taking periodic boundary conditions $(k_y = 2\pi n/L_y)$, it is clear that σ_0 is unaltered from the nonrelativistic case, and we recover Eq. (1).

There are essentially two elements influencing the expression for the Hall resistance of a 2D electron gas. The first element is the drift velocity of the electrons in the crossed fields. The argument leading to Eq. (12) shows that this must be cE/H, so that the Lorenz force is zero, independent of any relativistic treatments. The second element is the number of states per unit area in a Landau level. For an electron gas, whether relativistic or nonrelativistic, the electric and magnetic fields enter only in the two combinations $(\hbar k_y c - eHx)$ and $(E - eEx)$. Periodic boundary conditions require that k_y values be quantized in units of $2\pi/L_y$. The Dirac (or Schrödinger) equation for adjacent allowed values of k_y can be mapped into each other by changing the origin in the x direction to $x = x + \delta x$ $[\delta x = (hc/eHL_y)]$ and the zero of energy by $-eE\,\delta x$. Thus the number of states per unit area in a Landau level is also independent of the details of the solution and there are no relativistic corrections to the quantized value of the Hall resistance.

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