Brief Reports

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Two-fluid effects in the convective instability of ^{3}He-^{4}He superfluid mixtures

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The influence of two-fluid effects on the onset of convection in ^{3}He-^{4}He mixtures is discussed. The differences and similarities between the descriptions of Steinberg and Fetter are pointed out. The connections with recent experiments are also considered.

Recently there has been considerable interest in the theoretical[1-3] and experimental[4-6] study of the onset of convection in superfluid ^{3}He-^{4}He mixtures. ^{3}He-^{4}He mixtures are a unique system in the sense that it is possible to change important parameters as, e.g., the Prandtl number over many orders of magnitude [it can be changed from 0.01 (for alkali metals) up to values of the order 3 (for water)]. Therefore this system is very convenient to study nonlinear effects which are Prandtl number dependent as, e.g., the influence of vorticity. Since there is an additional variable in the system, the concentration of ^{3}He atoms (when compared to pure ^{4}He), additional types of behavior such as an oscillatory instability[2] and undamped second sound waves[1] have been predicted. The purpose of the present Brief Report is to evaluate the influence of two-fluid effects on the critical Rayleigh number for the stationary instability. Among other things, we find that the perturbations of the chemical potential reduce the stability of the system in agreement with experiment and contrary to the effects studied by Fetter. Furthermore, we clarify the relations between the descriptions given by Steinberg and Fetter.

In contrast to the heat-conducting state in normal-binary-fluid mixtures we have, in a superfluid mixture, nonzero values for both the normal-fluid and superfluid velocity, V_{n0} and V_{s0}, but with zero total mass flux. In detail we have, since $\partial V_{s0}/\partial t = 0$,

$$(\vec{\nabla}\mu_4)_0 = \vec{g}(\vec{\nabla}P)_0 = \rho_0\vec{g} \ , \tag{1}$$

$$(\vec{\nabla}C)_0 = -\frac{C\gamma}{T}(\vec{\nabla}T)_0 \ , \tag{2}$$

where $\gamma = -(\partial \ln C/\partial \ln T)|_{p,\mu_4}$, μ_4 is the chemical potential, C the concentration, and T the temperature. From mass conservation we have

$$\vec{i} + \rho C\vec{V}_{n0} = 0 \ , \tag{3}$$

with

$$\vec{i} = -\rho D\left(\vec{\nabla}C + \frac{k_T}{T}\vec{\nabla}T\right) \ , \tag{4}$$

where we use, as whenever possible in the following, the notation of Refs. 1 and 2. Combining Eqs. (3) and (4) we find, for the normal-fluid velocity in the heat-conduction state,

$$\vec{V}_{n0} = \frac{D}{C}\left(-\frac{C\gamma}{T} + \frac{k_T}{T}\right)\vec{\nabla}T_0 \ , \tag{5}$$

and, correspondingly,

$$\vec{V}_{s0} = -\frac{\rho_n}{\rho_s}\vec{V}_{n0} \ . \tag{6}$$

These expressions coincide with those given by Fetter. Introducing the deviations from the heat-conducting state,

$$\begin{aligned} &S = S_0 + S', \quad V_n = V_{n0} + V_n', \quad T = T_0 + T' \ , \\ &V_s = V_{s0} + V_s', \quad C = C_0 + C' \ . \end{aligned} \tag{7}$$

We arrive in dimensionless variables[2] at the following linearized equations for the deviations from the heat-conducting state:

$$\begin{aligned} &\operatorname{div}\vec{j} = 0, \quad \vec{j} = \vec{V}_n + \frac{\rho_s}{\rho_n}\vec{V}_s \ , \\ &\frac{\partial}{\partial t}(\nabla\operatorname{div} - \Delta)\vec{V}_n + \xi_0(\nabla\operatorname{div} - \Delta)\frac{\partial \vec{V}_n}{\partial z} = (\nabla\operatorname{div} - \Delta)\Delta\vec{V}_n - (\nabla\operatorname{div} - \Delta)\vec{\gamma}\left[R\sigma + \left(\frac{l}{l_0}\right)^3 L\mu_4\right] \ , \\ &\frac{\partial}{\partial t}\operatorname{div}\vec{V}_n - \frac{\rho_n}{\rho_s}\xi_0\frac{\partial}{\partial z}\operatorname{div}\vec{V}_n = \frac{\rho_s}{\rho_n}\left[\left(\frac{l}{l_0}\right)^3\Delta\mu_4 - m\Delta\operatorname{div}\vec{V}_n\right] \ , \\ &aP_TR\frac{d\sigma}{dt} + nP_T\left(\frac{l}{l_0}\right)^3 L\frac{\partial\mu_4}{\partial t} + \left(\frac{l}{l_0}\right)^3\operatorname{div}\vec{V}_n + aP_TR\xi_0\frac{\partial\sigma}{\partial z} + nP_T\left(\frac{l}{l_0}\right)^3 L\xi_0\frac{\partial\mu_4}{\partial z} = \frac{P_T}{P_C}a_1R\Delta\sigma + \frac{P_T}{P_C}n_1\left(\frac{l}{l_0}\right)^3 L\Delta\mu_4 \ , \\ &P_T\frac{\partial\sigma}{\partial t} + \vec{V}_n\cdot\vec{\gamma} + \xi_0P_T\frac{\partial\sigma}{\partial z} = \Delta\sigma + \left(\frac{l}{l_0}\right)^3\frac{Ld}{R}\Delta\mu_4 \ , \end{aligned} \tag{8}$$

where we have introduced the new abbreviation

$$\xi_0=\xi\frac{l\rho_n}{\eta}=\frac{D}{C}\frac{l\rho_n}{\eta}\left(\frac{k_T}{T}+K\right)\frac{dT_0}{dz},\quad m=\rho_n\frac{\zeta_4-\rho\zeta_3}{\eta}\ , \tag{9}$$

where $K=-\gamma C/T$. First, we consider for the stationary instability the case in which superflow is essential and weakly damped. This corresponds to $m\ll 1$; in this limit perturbations of the chemical potential can be neglected,[2] i.e., $\mu_4'=0$. We then see easily from the last equation (8) that

$$C'=-\frac{C\gamma}{T}T'\ . \tag{10}$$

Equation (10) is identical to the approximation made by Fetter,[3] however, in Ref. 3 terms proportional to $\mathrm{div}\vec{\nabla}_n$ have been neglected although they are of the same order of magnitude as the terms $\sim\vec{\nabla}_{n0}$. If we keep all two-fluid terms for the stationary instability in the limit $m\ll 1$ we obtain, from Eqs. (8),

$$\xi_0(\nabla\,\mathrm{div}-\Delta)\frac{\partial\vec{\nabla}_n}{\partial z}=(\nabla\,\mathrm{div}-\Delta)\Delta\vec{\nabla}_n-(\nabla\,\mathrm{div}-\Delta)\vec{\gamma}R\sigma\ , \tag{11}$$

$$\left(\frac{l}{l_0}\right)^3\mathrm{div}\vec{\nabla}_n+aP_TR\xi_0\frac{\partial\sigma}{\partial z}=\frac{P_T}{P_C}a_1R\,\Delta\sigma-aR\vec{\nabla}_n\cdot\vec{\gamma}\ , \tag{12}$$

$$\vec{\nabla}_n\cdot\vec{\gamma}+\xi_0P_T\frac{d\sigma}{dz}=\Delta\sigma\ . \tag{13}$$

Terms omitted in Ref. 3 include the second term on the left-hand side and the first term on the right-hand side of Eq. (12), as well as the second contribution on the left-hand side of Eq. (13). If we assume

$$(V_{nz},\sigma)=(V(z),\sigma(z))\exp(ik_\perp r)$$

we obtain, from Eqs. (11)–(13),

$$D^3\sigma=-k_\perp^2R\sigma+\zeta_0D^2\frac{d\sigma}{dz}-\delta_0D\frac{d^2\sigma}{dz^2}\ , \tag{14}$$

where

$$D\equiv\frac{d^2}{dz^2}-k_\perp^2,k_\perp^2=k_x^2+k_y^2\ ,$$

and where

$$\zeta_0=\xi_0(1+P_T)+\frac{P_Ta_1-P_Ca}{P_C}R\left(\frac{l_0}{l}\right)^3$$

$$=\left[\left(\frac{1+2P_T}{P_C}\right)\left(K+\frac{k_T}{T}\right)-K\right]\frac{1}{C}\frac{dT_0}{dz}l\ , \tag{15}$$

$$\delta_0=\xi_0\frac{1}{C}\left\{\frac{2P_T}{P_C}\left(K+\frac{k_T}{T}\right)-K\right\}\frac{dT_0}{dz}l\ . \tag{16}$$

Except for the term proportional to δ_0 in Eq. (14) the form is the same as Eq. (12) of Ref. 2(a). The first part of ζ_0 (due to $\vec{\nabla}_{n0}\neq 0$) has been given in Ref. 3 and the second part of ζ_0 (due to $\mathrm{div}\vec{\nabla}_n\neq 0$) has been discussed in Ref. 2. δ_0 is presented here for the first time and contains two-fluid effects coming from $\mathrm{div}\vec{\nabla}_n\neq 0$. Contrary to this case, Eq. (14) cannot be cast into self-adjoint form. Nevertheless, one can proceed as follows. We split up Eq. (14) into

$$D^2V=\psi\ , \tag{17}$$

$$\hat{L}\psi=-Rk_\perp^2V+\frac{d^2}{dz^2}DV\ . \tag{18}$$

Inserting $\psi=\exp(\zeta_0z/2)\phi$ into Eqs. (17) and (18) we get

$$\begin{aligned}&D^2V=e^{\zeta_0z/2}\phi\ ,\\ &e^{\zeta_0z/2}\left(D-\frac{\zeta_0^2}{4}\right)\phi=-Rk_\perp^2\sigma-\delta_0D\frac{d^2\sigma}{dz^2}\ .\end{aligned} \tag{19}$$

Contrary to the case studied in Ref. 2(a), Eqs. (19) are not self-adjoint; therefore it is impossible to establish a rigorous bound for R^C from Eqs. (19). Inserting the ansatz

$$\begin{Bmatrix}\phi\\ \sigma\end{Bmatrix}\sim\cos(\pi z)$$

into (19) in order to make a Galerkian-type analysis (we assume, for simplicity, free-free boundary conditions) we obtain

$$R=\frac{1}{k_\perp^2}\left(k^6+\frac{\zeta_0^2}{4}k^4+\delta_0\pi^2k^2\right)\ , \tag{20}$$

and where $k^2=\pi^2+k_\perp^2$. From Eq. (20) it follows that all contributions entering ζ_0 enhance the critical Rayleigh number which is obtained from Eq. (20) by minimization with respect to $k_\perp^2$. Inserting numbers appropriate for the experiments of Wheatley and co-workers[4,5] ($T=0.8$ K, $C=0.0024$, $P_T=0.66$), we obtain an enhancement of the Rayleigh number by 0.01%. This result can be expected to remain qualitatively correct for both rigid-rigid boundary conditions and a cylindrical geometry. For the latter case, Fetter,[3] taking into account the terms discussed above, also found an enhancement of the critical Rayleigh number due to two-fluid effects. The experiments,[4–6] however, show a decrease from what one would expect for a simple fluid in the same geometry. Thus the calculations sketched above and the results presented by Fetter do not support the hypothesis[3] that the superfluid ^{3}He-^{4}He mixture in the experimental conditions studied by Wheatley and co-workers behave essentially like a simple fluid. This is also obvious from the diagram in Fig. 4 of Ref. 7, which shows Nusselt number versus effective temperature. This diagram can be converted into a plot Nusselt number versus Prandtl number, from which it becomes clear that mixture effects are important. Therefore the assumption that T' is proportional to C' breaks down in this regime of the phase diagram.

From these considerations one is led immediately to two other possibilities to explain the experiments. The first concerns the assumption $m\ll 1$ entering the calculations made above and containing Fetter's approach as a special case. This assumption is only valid in a certain range of values for the concentration and the temperature; outside this regime the relation $C'=kT'$ breaks down and the perturbations of the chemical potential need to be taken into account leading to a destabilization of the system. This reduced stability seems to be a good reason to reduce the critical temperature gradient and might thus explain the smaller (compared to a simple fluid) critical Rayleigh number observed by Wheatley and colleagues. There is, however, a second problem for the comparison between the experiments and the theoretical estimates. All measurements

have been done in a system with a very small aspect ratio of order unity, and thus the system behaves essentially as if it would have only a few degrees of freedom contrary to the case of a container with a large aspect ratio. Therefore it would be very valuable to check experimentally for dilute solutions the stability in a temperature range much wider (e.g., 0.1 to 2 K) than that considered in Refs. 4 and 5. Then it will be much easier to establish quantitatively how two-fluid effects stabilize the system and how fluctuations of the chemical potential lead to a reduced stability.

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