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Self-consistent calculation of the quasiparticle lifetime in two-dimensional disordered metals

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A self-consistent calculation of the quasiparticle inverse lifetime in two-dimensional disordered metals is presented in this paper. We confirm the $T\ln(T_2/T)$ temperature dependence found by Abrahams, Anderson, Lee, and Ramakrishnan but find a different prefactor and temperature T_2. In particular, the quasiparticle lifetime is not the same as the weak localization cutoff recently calculated by Fukuyama and Abrahams although it is quite simply related to it.

The scaling theory of localization[1,2] predicts that, for weak disorder in two dimensions, the usual Boltzmann metallic conductivity, $ne^2\tau/m$, should have a logarithmic correction given by (units with $\hbar=1$)

$$\Delta\sigma=-\frac{e^2}{2\pi^2}\ln\frac{\tau^{\text{loc}}}{\tau} \ ,$$

where τ is the lifetime of a Bloch state determined by static impurity scattering and τ^{loc} is the inelastic lifetime of the particle-particle diffusion propagator[3] which, according to an argument due to Thouless,[4] is the same as the quasiparticle lifetime. Abrahams, Anderson, Lee, and Ramakrishnan[5] (AALR) have recently calculated this quantity perturbatively to first order in the screened Coulomb interaction and found

$$\frac{1}{\tau^{\text{qp}}}=\frac{k_BT}{k_Fl}\ln\frac{T_1}{T} \ , \qquad (1)$$

with $k_BT_1=(k_Fl)^2\epsilon$, where k_F is the Fermi wave vector, l the impurity mean free path, and $\epsilon=D\kappa^2$, where D is the diffusion constant $k_Fl/2m$ and κ the inverse screening length $2me^2$.

Fukuyama and Abrahams[3] did a direct perturbative calculation of τ^{loc} in a similar approximation and found exactly the same result:

$$\frac{1}{\tau^{\text{loc}}}=\frac{k_BT}{k_Fl}\ln\frac{T_1}{T} \ . \qquad (2)$$

However, the calculation of AALR gives an imaginary part of the self-energy which diverges on the energy shell. This divergence is cut off by a procedure which leads to Eq. (1) but does not appear to be sufficiently justified.

In this paper it is shown that, with a self-consistent renormalization of the electron propagator in the self-energy, this divergence does not occur. One finds that the quasiparticle lifetime is not strictly identical to τ^{loc} and is given by

$$\frac{1}{\tau^{\text{qp}}}=\frac{k_BT}{2k_Fl}\ln\frac{T_2}{T} \ , \qquad (3)$$

with $T_2=4(k_Fl)^2\epsilon$.

The close relationship between τ^{qp} and τ^{loc} will become clear if we reformulate the calculation of AALR in momentum space. They define an average self-energy for exact impurity eigenstates of fixed energy by

$$\Sigma_E(\epsilon_{n'})\equiv\frac{1}{\Omega n_0}\left\langle\sum_\alpha\Sigma_\alpha(\epsilon_{n'})\delta(\omega_\alpha-E)\right\rangle_{\text{av}} \ , \qquad (4)$$

where α denotes an exact impurity eigenstate with unperturbed energy ω_α (all electronic energies are measured with respect to the chemical potential μ) and $\Sigma_\alpha(\epsilon_{n'})$ is the self-energy of such a state, due to interactions. The brackets denote average over impurity configurations, Ω is the volume of the sample, n_0 the one-spin density of states per unit volume, and $\epsilon_{n'}$ a Fermion-Matsubara frequency $\epsilon_{n'}=(2n'+1)\pi k_BT$. Noting that

$$\delta(\omega_\alpha-E)=\frac{1}{2\pi i}[G_{0\alpha}^A(E)-G_{0\alpha}^R(E)] \ , \qquad (5)$$

where $G_{0\alpha}^{A(R)}(E)$ is the advanced (retarded) unperturbed Green's function for the impurity eigenstates, it becomes clear that one can obtain the real and imaginary parts of the self-energy by suitable analytic continuations of the function

$$\sigma(\epsilon_{n'},\epsilon_n)\equiv\frac{1}{\Omega}\left\langle\sum_\alpha\Sigma_\alpha(\epsilon_{n'})G_{0\alpha}(\epsilon_n)\right\rangle_{\text{av}} \ . \qquad (6)$$

The imaginary part of the self-energy is

$$\Gamma_E(\omega)=-\frac{1}{4\pi n_0}\frac{1}{\Omega}\left\langle\sum_\alpha[\Sigma_\alpha^A(\omega)-\Sigma_\alpha^R(\omega)]\times[G_{0\alpha}^A(E)-G_{0\alpha}^R(E)]\right\rangle_{\text{av}}$$
$$=\frac{1}{2\pi n_0}\text{Re}[\sigma^{RA}(\omega,E)-\sigma^{AA}(\omega,E)] \ , \qquad (7)$$

with the usual definition of retarded and advanced functions. The quasiparticle inverse lifetime (neglecting quasiparticle renormalization factors) is

$$\frac{1}{\tau_E^{\text{qp}}}\equiv\Gamma_E(E)=\frac{1}{2\pi n_0}\text{Re}[\sigma^{RA}(E,E)-\sigma^{AA}(E,E)] \qquad (8)$$

[in the calculation presented in this paper $\Gamma_E(\omega)$ does not diverge at $\omega=E$].

The advantage of this formulation is that $\sigma(\epsilon_{n'},\epsilon_n)$ is the average of the trace over all states of the product of the matrices Σ and G_0 (it is assumed that the diagonal part of Σ in the α basis dominates the off-diagonal terms[5]) and thus is independent of representation. In momentum space

$$\sigma(\epsilon_{n'},\epsilon_n)=\frac{1}{\Omega}\left\langle\sum_{\vec{p},\vec{p}'}\Sigma(\vec{p},\vec{p}';\epsilon_{n'})G_0(\vec{p}',\vec{p};\epsilon_n)\right\rangle_{\text{av}} \ . \qquad (9)$$

We consider for $\Sigma(\vec{p},\vec{p}';\epsilon_{n'})$ an approximation given by the diagram of Fig. 1, which is similar to the one used by AALR except that the intermediate electron propagator is

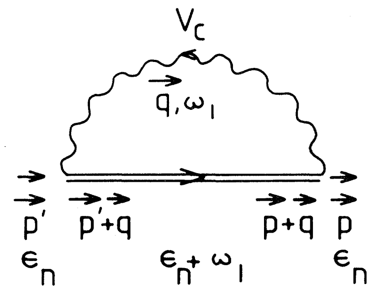

FIG. 1. Self-energy diagram explicitly to first order in the averaged screened Coulomb interaction and with a renormalized electron propagator.

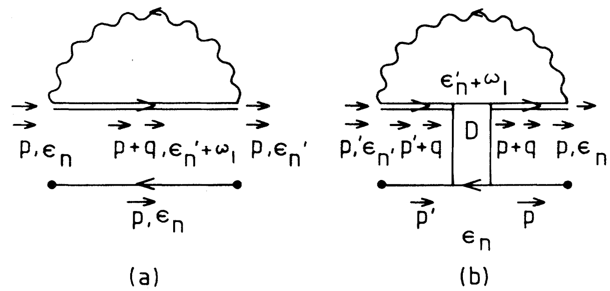

FIG. 2. Contributions to the function $\sigma(\epsilon_{n'},\epsilon_n)$ whose analytic continuations give the real and imaginary parts of the averaged self-energy: (a) $k_F l \rightarrow \infty$, pure case contribution; (b) leading $1/k_F l$ contribution. D is a renormalized particle-hole propagator.

renormalized by the interactions. The wavy line represents an averaged screened Coulomb interaction which in the low-$\vec{q}$, low-ω_l, diffusive regime has the form

$$V_c(\vec{q},\omega_l)=\frac{2\pi e^2}{q}\frac{|\omega_l|+Dq^2}{|\omega_l|+D\kappa q} \ , \tag{10}$$

where ω_l is a Matsubara frequency $\omega_l=2l\pi k_B T$. We shall only consider terms which are large for small $\vec{q}$ for these are seen to dominate $1/\tau^{qp}$. The function $\sigma(\epsilon_{n'},\epsilon_n)$ is then given by the diagrams of Fig. 2. The diagram 2(a) gives the clean limit contribution ($k_F l \rightarrow \infty$). To first order in $1/k_F l$ and in the approximation of AALR the block $D \equiv D(\vec{q},\Omega_\lambda+\omega_l)$ in Fig. 2(b) ($\Omega_\lambda \equiv \epsilon_{n'}-\epsilon_n$) is just the particle-hole diffusion propagator for the noninteracting system with the usual pole for $\epsilon_n(\epsilon_{n'}+\omega_l)<0$,

$$D_0(\vec{q},\Omega_\lambda+\omega_l)=\frac{1}{2\pi n_0\tau^2(|\Omega_\lambda+\omega_l|+Dq^2)} \ . \tag{11}$$

Here, however, the upper Green's function is renormalized with respect to interactions and D must be determined self-consistently. Note that D is *not* the particle-hole diffusion propagator of the interacting system. We will find that the pole of D is cut off by an inverse inelastic lifetime whereas the particle-hole diffusion progagator of the interacting system still has a pole. The calculation of D is very similar to the calculation of the particle-particle diffusion propagator of Fukuyama and Abrahams.[3] It obeys the equation represented diagramatically in Fig. 3. The block T contains the interactions, with the external Green's functions and the integration over their momenta included in its definition. As T only contains interaction lines in the upper Green's function there is no frequency transferred between the upper and lower Green's function, and Fig. 3 is represented by an algebraic equation with the solution ($\epsilon_{n'}\epsilon_n<0$),

$$D(\vec{q},\Omega_\lambda)=\frac{1}{2\pi n_0\tau^2(|\Omega_\lambda|+Dq^2+1/\tau_D)} \ , \tag{12}$$

where

$$\frac{1}{\tau_D(\epsilon_{n'},\epsilon_n)}=-\frac{1}{2\pi n_0\tau^2}T(\epsilon_{n'},\epsilon_n) \ . \tag{13}$$

In Eq. (13) we can take the limit $\vec{q}\rightarrow 0$, $\Omega_\lambda \rightarrow 0$ since we are interested in the leading term in $\vec{q}$ and Ω_λ in Eq. (12). We shall eventually do the analytic continuations $i\epsilon_{n'} \rightarrow E \pm i0^+$, $i\epsilon_n \rightarrow E \mp i0^+$, and it will turn out that the dominant term of $1/\tau_D$ is real. So we write

$$\frac{1}{\tau_D}=-\frac{1}{2\pi n_0\tau^2}T(-iE+0^+,-iE-0^+) \ .$$

Before actually calculating $1/\tau_D$ we will compute the diagram of Fig. 2(b), the leading $1/k_F l$ correction to the single-particle inverse lifetime. As usual, we keep only the terms for which the diffusive singularity of Eq. (12) occurs; we must have $(\epsilon_{n'}+\omega_l)\epsilon_n<0$. Hence

$$\sigma^{+-}(\epsilon_{n'},\epsilon_n)\equiv\sigma(\epsilon_{n'}>0,\epsilon_n<0)=-(2\pi n_0\tau)^2\int\frac{d^2q}{(2\pi)^2}\frac{1}{\beta}\sum_{\omega_l>-\epsilon_{n'}}V_c(\vec{q},\omega_l)D(\vec{q},\Omega_\lambda+\omega_l) \ , \tag{14a}$$

$$\sigma^{--}(\epsilon_{n'},\epsilon_n)\equiv\sigma(\epsilon_{n'}<0,\epsilon_n<0)=-(2\pi n_0\tau)^2\int\frac{d^2q}{(2\pi)^2}\frac{1}{\beta}\sum_{\omega_l>-\epsilon_{n'}}V_c(\vec{q},\omega_l)D(\vec{q},\Omega_\lambda+\omega_l) \ . \tag{14b}$$

The frequency sums can be performed by the usual method of contour integration. Doing the analytic continuations $i\epsilon_{n'}\rightarrow E+i0^+$, $i\epsilon_n \rightarrow E-i0^+$ in Eq. (14a) and $i\epsilon_{n'}\rightarrow E-i0^+$, $i\epsilon_n\rightarrow E-i0^+$ in Eq. (14b) one obtains

$$\sigma^{RA}(E,E)=4\pi n_0^2\tau^2\int\frac{d^2q}{(2\pi)^2}\int_{-\infty}^{+\infty}dx[N(x)+f(x+E)]\,\mathrm{Im}\,V_c^A(\vec{q},x)D^R(\vec{q},x)$$
$$-i2\pi n_0^2\tau^2\int\frac{d^2q}{(2\pi)^2}\int_{-\infty}^{+\infty}dx\,f(x+E)\,V_c^R(\vec{q},x)D^R(\vec{q},x) \ , \tag{15a}$$

$$\sigma^{AA}(E,E)=-i2\pi n_0^2\tau^2\int\frac{d^2q}{(2\pi)^2}\int_{-\infty}^{+\infty}dx\,f(x+E)\,V_c^R(\vec{q},x)D^R(\vec{q},x) \ , \tag{15b}$$

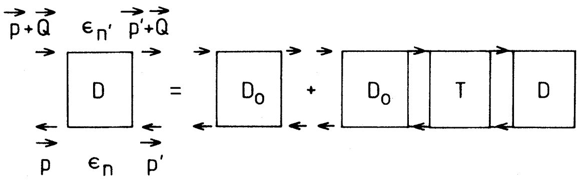

FIG. 3. Equation for the propagator D in terms of D_0, the particle-hole diffusion propagator in absence of interactions and T, the interaction vertex.

where $N(x)$ and $f(x)$ are the boson and Fermion thermal occupation factors.

Using Eq. (12) in Eqs. (15a) and (15b) and the corresponding expressions in Eq. (8) one gets

$$\frac{1}{\tau_E^{qp}}=2e^2\int\frac{d^2q}{(2\pi)^2}\int_{-\infty}^{+\infty}dx\,[N(x)+f(x+E)]\times\frac{xD\kappa}{x^2+\epsilon Dq^2}\,\frac{Dq^2+1/\tau_D}{x^2+(Dq^2+1/\tau_D)^2}\quad . \tag{16}$$

Apart from the appearance of $1/\tau_D$ in the last factor this equation is equivalent to Eq. (3.5) of AALR [in their paper Eq. (3.5) has a wrong sign]. With the change of variable $U\equiv Dq^2$ the q integral can be done exactly and for $E\ll k_BT$, i.e., on the Fermi level,

$$\frac{1}{\tau^{qp}}=\frac{e^2\kappa}{\pi\epsilon}\int_0^\infty dx\,\frac{x}{\sinh\beta x}\,\mathrm{Re}\left\{\frac{1}{(1/\tau_D)-ix-x^2/\epsilon}\times\ln\frac{\epsilon[(1/\tau_D)-ix]}{x^2}\right\}\quad . \tag{17}$$

This integral is divergent when $1/\tau_D\to 0$, and for $\tau_DT\gg 1$ it becomes

$$\frac{1}{\tau^{qp}}=\frac{e^2\kappa}{\epsilon}k_BT\ln(\tau_D^2\epsilon k_BT)=\frac{k_BT}{2k_Fl}\ln(\tau_D^2\epsilon k_BT)\quad . \tag{18}$$

It will soon be proven that $1/\tau_D$ is given by the right-hand side of Eq. (3) so that, to leading order in temperature,

$$\frac{1}{\tau^{qp}}=\frac{1}{\tau_D}=\frac{k_BT}{2k_Fl}\ln\frac{T_2}{T}\quad . \tag{19}$$

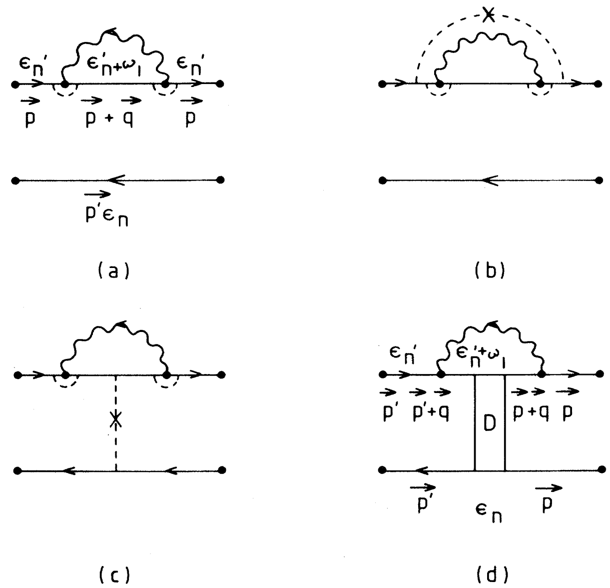

FIG. 4. (a)–(d) Leading $1/k_Fl$ corrections to the interaction vertex T. Dashed lines represent the particle-hole diffusion propagator D_0.

At low temperatures this term dominates the clean limit contribution, Fig. 2(a), which varies as T^2.

Let us turn now to the calculation of $1/\tau_D$. As was said, it is quite similar to the calculation of $1/\tau^{\mathrm{loc}}$ of Fukuyama and Abrahams.[3] The leading $1/k_Fl$ contributions to $1/\tau_D$ are given by the diagrams of Fig. 4. The dashed lines stand for the ordinary particle-hole diffusion propagator $D_0(\vec{q},\omega_l)$ [Eq. (11)], but in diagram 4(d) we used the renormalized propagator $D(\vec{q},\Omega_\lambda+\omega_l)$ [Eq. (12)] because in our approximation for the self-energy the intermediate electronic propagator is renormalized. Strictly speaking, the dashed lines should be replaced by D as well. However, we shall see that the contributions 4(a)–4(c), as opposed to 4(d), are not singular when $1/\tau_D\to 0$. Again keeping only the terms where the diffusive poles occur one has, for $\epsilon_{n'}>0$, $\epsilon_n<0$,

$$\frac{1}{\tau_D^{+-}}=2\pi n_0\tau^2\int\frac{d^2q}{(2\pi)^2}\left\{\frac{1}{\beta}\sum_{\omega_l>\epsilon_{n'}}V_c(\vec{q},\omega_l)D_0(\vec{q},\omega_l)-\frac{1}{\beta}\sum_{\omega_l>-\epsilon_{n'}}V_c(\vec{q},\omega_l)D(\vec{q},\Omega_\lambda+\omega_l)\right\}\quad . \tag{20}$$

The first term is the sum of 4(a)–4(c)—note that the leading terms cancel—and the second arises from 4(d). We performed the frequency sums and made the appropriate analytic continuations,

$$\begin{aligned}\frac{1}{\tau_D}&=in_0\tau^2\int\frac{d^2q}{(2\pi)^2}\int_{-\infty}^{+\infty}dx\,[f(x-E)D_0(\vec{q},x)-f(x+E)D(\vec{q},x)]\,V_c^R(\vec{q},x)\\&\quad+2n_0\tau^2\int\frac{d^2q}{(2\pi)^2}\int_{-\infty}^{+\infty}dx\,[N(x)+f(x+E)]\,\mathrm{Im}\,V_c^A(\vec{q},x)D^R(\vec{q},x)\\&\simeq 2n_0\tau^2\int\frac{d^2q}{(2\pi)^2}\int_{-\infty}^{+\infty}dx\,\frac{1}{\sinh\beta x}\,\mathrm{Im}\,V_c^A(\vec{q},x)D^R(\vec{q},x)\quad ,\end{aligned} \tag{21}$$

where, in the last line we kept only the most singular term and took the limit $E\to 0$. With the use of Eqs. (10) and (12) this can be transformed into

$$\frac{1}{\tau_D}=2e^2\int\frac{d^2q}{(2\pi)^2}\int_{-\infty}^{+\infty}dx\,\frac{x}{\sinh\beta x}\,\frac{D\kappa}{x^2+\epsilon Dq^2}\,\mathrm{Re}\frac{1}{-ix+Dq^2+1/\tau_D}\quad . \tag{22}$$

For the case $\epsilon_{n'} < 0$, $\epsilon_n > 0$ we would get the same result ($-ix$ changes into ix in the last factor).

The right-hand side of Eq. (22) is the same as in Eq. (16) for $E = 0$, which is not surprising since the dominant contribution to $1/\tau_D$ is entirely due to the diagram of Fig. 4(d) which is identical to that of Fig. 2(b). It follows, then, that

$$\frac{1}{\tau_D} = \frac{k_B T}{2k_F l} \ln(\tau_D^2 k_B T \epsilon) \quad , \tag{23}$$

and to leading order in T,

$$\frac{1}{\tau_D} = \frac{k_B T}{2k_F l} \ln\frac{T_2}{T} \quad . \tag{24}$$

In the calculation of $1/\tau^{\text{loc}}$ Fukuyama and Abrahams[3] have to sum an extra set of diagrams similar to those of Fig. 4 but with self-energy insertions in the lower Green's function. Their equation for $1/\tau^{\text{loc}}$ is identical to Eq. (22) but with an extra factor of 2 in the right-hand side, which accounts for the difference between $1/\tau^{\text{loc}}$ in Eq. (2) and $1/\tau^{\text{qp}}$ in Eq. (3).

In conclusion, the quasiparticle lifetime is not strictly identical to the localization cutoff although the relationship between the two for the case of Coulomb interactions is very simple. However, Fukuyama and Abrahams[3] point out that for interactions which are singular at $\omega_l = 0$, other diagrams, with interaction lines going between upper and lower Green's function, will contribute to $1/\tau^{\text{loc}}$. It is clear from this calculation that they will not contribute to the quasiparticle lifetime. In such cases this simple relationship between $1/\tau^{\text{loc}}$ and $1/\tau^{\text{qp}}$ may not apply.

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