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Fractional quantization of Hall conductance

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It is shown how the correlation energy of a system of two-dimensional electrons in a strong magnetic field may be enhanced if the electrons are in a regular array of the Landau orbitals. This gives an energy gap if the proportion ν of occupied orbitals is a simple fraction without giving rise to charge-density waves which may pin the system. The gap is determined self-consistently. Such a state can give rise to the plateaus in the Hall conductance observed at fractional multiples of e^2/h.

I. INTRODUCTION

Tsui and co-workers[1,2] have recently discovered that the Hall conductance of a two-dimensional system can be quantized at the fractional value $e^2/3h$ and the longitudinal resistance has a minimum at the corresponding value of the Landau-level filling factor ν equal to $\frac{1}{3}$. Anomalous behavior consistent with this observation has also been reported by Ebert *et al.*[3] Laughlin's arguments for integer quantization of the Hall conductance[4] seem to rule out any fractional values for noninteracting electrons, so it is generally supposed that this new effect is caused by the condensation of the system into a macroscopic collective ground state as a result of electron-electron interactions.

There are both experimental and theoretical results that restrict possible explanations. The observed linearity of current-voltage relations over a very wide range suggests the absence of pinning of the ground state,[2] while a charge-density wave (CDW) should be pinned and yields a threshold voltage for electrical conduction. Hartree-Fock calculations of a CDW[5-7] yield a condensation energy of the order of $e^2/\epsilon l$ per particle, where l is the magnetic length $(\hbar/eB)^{1/2}$, and this seems to be too high for the experimental results. Also, the Hartree-Fock energy is a smooth function of ν,[6] and so there is no commensurability energy favoring simple rational values of ν.

In this paper we present results for a different theory based on a many-body calculation of the correlation energy when the degeneracy of a Landau level is broken by Coulomb interactions between the electrons. This is in a sense a generalization of Laughlin's discussion of the problem of three electrons in a strong magnetic field[8] to the many-electron problem. If the electrons in a partially filled Landau level can be arranged in a regular manner in the space of Landau orbitals, then the correlation energy is enhanced, because fewer transitions to intermediate states are forbidden than would be if the electrons were randomly arranged. This regular arrangement leads to an energy splitting between occupied and empty orbitals which is calculated in Sec. II. In Sec. III it is shown how the theory leads to sharp V-shaped minima of the thermodynamic potential at rational values of ν. This state, with a superlattice in Landau-orbital space, gives no density variation in configuration space, so it is free to slide on a slightly disordered substrate. It should therefore lead to a Hall conductance $\nu e^2/h$ for stable values of ν. We have not yet determined which values of ν lead to states which are stable enough to give plateaus in the Hall conductance.

II. ENERGY GAP

We consider a standard two-dimensional electron gas model, with interacting electrons moving in a uniform positive background to form an electrically neutral system. In real inversion layer systems the positive background is provided by the depletion region on the gate and its density can vary; this is important for the observation of quantized Hall conductance, but we ignore the details. We also assume that the magnetic field is so strong that the Coulomb interaction does not significantly mix different Landau levels; this condition $e^2/\epsilon l << \hbar\omega_c$ is again unrealistic for real systems. For simplicity we assume that all electrons are in the lowest spin polarized Landau level, so $\nu < 1$, but there is no difficulty in generalizing the discussion to higher levels for $\nu > 1$.

We take the system to have area L^2 and to be periodic in the y direction, so that the Landau wave functions may be written in the form

$$\phi_s(\vec{r}) = \pi^{-1/4}(lL)^{-1/2} \times \exp[-2\pi i s y/L - \tfrac{1}{2}(x/l - 2\pi s l/L)^2] \ , \quad (1)$$

where s is an integer between 0 and $L^2/2\pi l^2$. The Coulomb interaction between the electrons in the lowest Landau level can be written in the form

$$H_c = \tfrac{1}{2} \sum V(s_1 - s_3, s_2 - s_3) a^\dagger_{s_1} a^\dagger_{s_2} a_{s_4} a_{s_3} \delta_{s_1+s_2, s_3+s_4} \ , \quad (2)$$

where $a^\dagger, a$, are creation and annihilation operators, and

$$V(s_1 - s_3, s_2 - s_3) = \frac{e^2}{\epsilon L} \int_{-\infty}^{\infty} \frac{dq}{[q^2 + 4\pi^2(s_1 - s_3)^2/L^2]^{1/2}} \exp\left[-\frac{l^2}{2}\left(q^2 + \frac{4\pi^2(s_1 - s_3)^2}{L^2}\right) + \frac{2\pi i q l^2}{L}(s_2 - s_3)\right] \ . \quad (3)$$

In addition to the Coulomb energy there is a constant energy per electron, and an interaction with the positive background which cancels the divergent first-order direct Coulomb interaction between the electrons. The system is equivalent to a one-dimensional quantum lattice gas with average occupancy ν. The interaction enables pairs of particles to make transitions over a distance of the order of L/l provided the center of mass is unchanged. Because of the particle-hole symmetry the behavior for $\nu > \frac{1}{2}$ can be deduced from the behavior for $\nu < \frac{1}{2}$.

We consider first the case $\nu = 1/p$, where p is an integer. We take as a broken-symmetry unperturbed ground state the configuration in which the occupied electron states (hole sites) are equally spaced with an interval p. This commensurate state has uniform density and is free to slide. We take the hole sites to be multiples of p. We assume that the energies of the hole sites ϵ_h and of the particle sites ϵ_p are different, and we use perturbation theory to make a self-consistent calculation of them.

First-order perturbation theory gives a Hartree-Fock energy

$$\epsilon_{\rm HF} = -p^{-1}(\pi/2)^{1/2}e^2/\epsilon l \ , \tag{4}$$

which is the same for particles and holes. In second order there is, formally, an energy difference, but the second-order term diverges for the Coulomb interaction, so we use one of the standard techniques of many-body theory and sum the infinite series of most divergent diagrams in each order.[9] A typical diagram of this sort is shown in Fig. 1. Here the unperturbed fermion propagator has the form

$$iG_0(s,\omega) = i(\omega - \epsilon_s + i\delta_s)^{-1} \ , \tag{5}$$

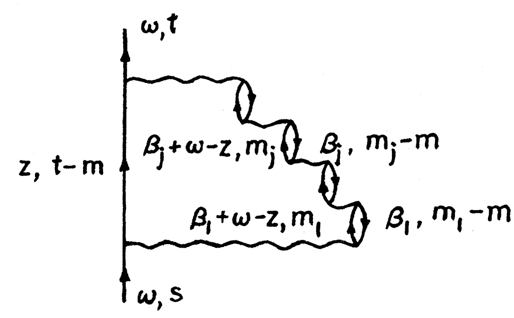

FIG. 1. Most divergent diagram.

where $\epsilon_s = \epsilon_h$ and $\delta_s = -\frac{1}{2}\delta$ for s a multiple of p, and $\epsilon_s = \epsilon_p$ and $\delta_s = \frac{1}{2}\delta$, otherwise. With substitution of Eqs. (3) and (5) into the expression for the self-energy contribution of this graph, and integration over the internal variables β_j and z, summation over the m_j and n gives the contribution of the diagram with K rings as

$$\sum_{n=1}^{p-1}\int\frac{dz}{2\pi}\frac{1}{z-\epsilon_{s-n}+i\delta_{s-n}}\left(\frac{e^2}{\epsilon lp}\right)^{K+1}\left(\frac{2(\epsilon_h-\epsilon_p+i\delta)}{(\epsilon_h-\epsilon_p+i\delta)^2-(\omega-z)^2}\right)^{K}\int_0^\infty\frac{d\rho\exp[-\frac{1}{2}(K+1)\rho^2]}{\rho^K} \ . \tag{6}$$

This contains a divergent integral, but the infinite sum over all values of K gives the finite result

$$\int\frac{dz}{2\pi}\int\frac{d\rho}{\rho}e^{-\rho^2}\left(\frac{e^2}{\epsilon lp}\right)^2\sum_{n=1}^{p-1}\frac{1}{z-\epsilon_{s-n}+i\delta_{s-n}}\,\frac{2(\epsilon_h-\epsilon_p)}{(\epsilon_h-\epsilon_p+i\delta)^2-(\omega-z)^2+2(\epsilon_p-\epsilon_h-i\delta)(e^2/\epsilon lp)\rho^{-1}e^{-\rho^2/2}} \ . \tag{7}$$

This gives the result

$$\begin{aligned}\epsilon_h &= E_0+\epsilon_{\rm HF}-(e^2/\epsilon_l)(1-1/p)f_1(\eta p) \ ,\\ \epsilon_p &= E_0+\epsilon_{\rm HF}+(e^2/\epsilon lp)f_1(\eta p)\\ &\quad -(e^2/\epsilon l)(1-2/p)f_2(\eta p) \ ,\end{aligned} \tag{8}$$

where

$$\eta = (\epsilon_p - \epsilon_h)\epsilon l/e^2 \tag{9}$$

defines the energy gap, and

$$\begin{aligned} f_1(x) &= \int_0^\infty \frac{e^{-\rho^2}d\rho}{(\rho^2x^2+2\rho xe^{-\rho^2/2})^{1/2}+\rho x+2e^{-\rho^2/2}} \ ,\\ f_2(x) &= \int_0^\infty\frac{e^{-\rho^2}d\rho}{\rho x+2e^{-\rho^2/2}} \ .\end{aligned} \tag{10}$$

Equations (8) and (9) show that the energy gap is determined by

$$\eta = f_1(\eta p) - (1-2/p)f_2(\eta p) \ . \tag{11}$$

Because of the symmetry between holes and particles, Eq. (11) is also true for $\nu = 1-1/p$ filling. Numerical results obtained from these equations, and from the corresponding equations for $\nu = 1-1/p$, are shown in Table I.

It must be emphasized that the energy gap results from the commensurate occupation of the states. If the states were randomly occupied, or even if there were discommensurations with a mean spacing small compared with L/l, Eqs. (8) and (11) would be replaced by

$$\begin{aligned}\epsilon_h &= E_0+\epsilon_{\rm HF}-(e^2/\epsilon l)(1-1/p)f_1+(e^2/\epsilon lp)f_2 \ ,\\ \epsilon_p &= E_0+\epsilon_{\rm HF}-(e^2/\epsilon lp)f_1+(e^2/\epsilon l)(1-1/p)f_2 \ ,\\ \eta &= f_1 - f_2 \ ,\end{aligned} \tag{12}$$

where the argument of f_1 and f_2 is $\eta p(1-1/p)$. Since $f_1 \leq f_2$, this equation has only the solution $\eta = 0$ and there is no gap. As a further example of the importance of the commensurate arrangement we also give in Table I the gap obtained by occupying the levels $s = 4n$ and $4n+1$ in a half-filled band; this is denoted by $\nu = \frac{2}{4}$. Equation (11) is replaced by

$$\eta = \tfrac{1}{2}f_1(2\eta) - \tfrac{1}{4}[f_2(4\eta) - f_1(4\eta)] \ , \tag{13}$$

so the gap is reduced by more than a factor of 2.

TABLE I. Energy gaps.

ν	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{4}$ [a]	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
Δ [b]	0.225	0.4557	0.1927	0.3174	0.229	0.1688

[a]Holes are $s = 4n$ and $4n+1$.
[b]$\Delta = \epsilon_p - \epsilon_h$. Energy unit is $e^2\sqrt{2\pi n}/\epsilon$.

III. THERMODYNAMIC POTENTIAL

For plateaus in the Hall conductance to be observed when the magnetic field is varied in a system with a uniform substrate potential, it is necessary that electrons should be able to flow into the inversion layer from some external reservoirs. This has been discussed for GaAs-$Ga_{1-x}Al_xAs$ heterjunctions by Baraff and Tsui.[10] We therefore need to find the minimum of a thermodynamic potential

$$\Omega(\mu,B) = \min_{N_e}[U_0(B,N_e) - \mu N_e] \quad , \tag{14}$$

where U_0 is the ground-state energy and μ is the chemical potential of the electrons determined by these reservoirs.

At fractional filling $\nu = 1/p$ and $1-1/p$, because

$$\epsilon_n < \mu = \epsilon_f < \epsilon_p \quad , \tag{15}$$

the thermodynamic potential Ω has a sharp V-shaped minimum, because

$$\Omega(N_e - \delta N_e) - \Omega(N_e) = \delta N_e(\mu - \epsilon_n) > 0 \tag{16}$$

and

$$\Omega(N_e + \delta N_e) - \Omega(N_e) = \delta N_e(\epsilon_p - \mu) > 0 \quad . \tag{17}$$

As the field is varied the number of electrons will adjust to minimize the thermodynamic potential, which includes the energy necessary to move an electron from the substrate to the inversion layer. Thus, ν will remain at fractional values for a range of magnetic field and give rise to characteristic plateaus of the quantized Hall conductance.[1] In such a state the whole electron system can slide freely in response to an applied electric field, so the Hall conductance has the fractional value $\nu e^2/h$.

IV. DISCUSSION

We have shown how the formation of a superlattice in the space of Landau oribtals can stabilize simple fractional values of the occupation ν of a Landau level without causing charge-density waves, and so lead to plateaus in the Hall conductance at fractional values $\nu e^2/h$. The method which we used to calculate the energy, although well established for the high-density electron gas, is without a good justification in this problem since there is no small parameter to describe the breaking of the degeneracy of a Landau level. Nevertheless, we believe that perturbation theory gives at least a plausible description of the preference for occupation of a simple rational fraction of the orbitals available in a Landau level. It will of course be interesting to see if accurate experiments confirm that the observed plateau is at exactly $e^2/3h$, and if other fractional values appear at even lower temperatures and electric field strengths.

In our theory, holes and particles are symmetric. Therefore, $\nu = 1-1/p$ filling can be achieved from $\nu = 1/p$ by replacing holes by particles and vice versa; Eq. (11) is also true for $\nu = 1-1/p$; but the magnetic length l in Eq. (9) at $\nu = 1-1/p$ is $[(p-1)/2\pi np]^{1/2}$ instead of $1/\sqrt{2\pi np}$ at $\nu = 1/p$. Thus, with the same density of electrons, we have

$$\eta_{1-1/p} = \eta_{1/p} \quad , \tag{18}$$

$$\Delta_{1-1/p} = (1/\sqrt{p-1})\Delta_{1/p} \quad . \tag{19}$$

Until a more detailed theory is developed we cannot make strong statements about the temperature dependence of our model, but the analogy with superconductivity theory might suggest that the energy gap η should determine the critical temperature at which the commensurate state becomes favorable. At a field of 15 T, with a dielectric constant of 13.1, the energy gap given by Eq. (11) for $\nu = \frac{1}{3}$ has a value of about $35k_B$, and so the analogy with superconductivity theory suggests a transition temperature of about 8 K, which is a factor of about 5 above the temperature at which the plateau can be observed.[1]

This theory, like any other theory with a broken symmetry, has modifications of the ground state that may change the energy by little or nothing. There may, for example, be Goldstone bosons.[11] One modification that is possible is to use, instead of the representation (1), a set of basis states such as

$$\chi_{s,t} = M^{-1/2} \sum_{s'=1}^{M} \exp(2\pi i s't/M)\phi_{s+s'} \quad , \tag{20}$$

and picking out $1/p$ of these states in a regular manner to form an unperturbed ground state. In this representation each Feynman diagram gives exactly the same contribution to the particle or hole energy that is given in the representation (1), since each diagram contributes a function of $1/p$, which is representation independent, multiplied by an integral that involves only the value of η and the unperturbed propagator for the full Landau level.

We have received a copy of Laughlin's unpublished work.[12] It is also possible in our theory to use symmetric gauge and angular momentum representation which Laughlin used. The advantage of our theory is a guarantee of the particle-hole symmetry.

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