Reply to "Scattering of electrons by impurities in a weak magnetic field: A comment"

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(Received 17 May 1982; revised manuscript received 16 February 1983)

An alternative proof is given which shows that scattering of electrons by spherically symmetric impurities cannot contribute to the Hall coefficient.

The Hall effect is caused by the change in the average drift velocity $\langle \vec{v} \rangle$, resulting from the Lorentz force $(-e/c)\vec{v}\times\vec{B}$. Chambers argues[1] that scattering cross sections acquire a right-left asymmetry, proportional to B, and that *this* contributes an additional term to the Hall constant.

In our paper[2] we presented a mathematical proof that impurity scattering (from a spherically symmetric, spin-independent potential) does not modify the Hall constant. We agree with Chambers that for impurities having a finite size, and therefore a finite collision duration T (not to be confused with the time between collisions τ), the scattering cross section becomes skewed. However, the rotation of $\langle \vec{v} \rangle$ attributable to the skewing is $\omega_c T$, the same angle through which the Lorentz force term of the transport equation rotates $\langle \vec{v} \rangle$ during a time T, i.e., from

$$\left(\frac{\partial f(\vec{v},t)}{\partial t}\right)_{\text{Lor.F.}} = \frac{e}{mc}(\vec{v}\times\vec{B})\cdot\vec{\nabla}_v f(\vec{v},t) \ . \quad (1)$$

The presence of this term in the transport equation, which describes the time evolution of the velocity distribution, guarantees that $\langle \vec{v} \rangle$ will be rotated at the cyclotron frequency ω_c at *all* times. If one wishes to include this rotation *during the collision duration T* by means of a skewed cross section, then one must delete the action of Eq. (1) during that same interval. This could be done with a collision operator

$$\left(\frac{\partial f(\vec{v},t)}{\partial t}\right)_{\text{coll}} = F_{\text{op}}[f(\vec{v},t-T)] \ , \quad (2)$$

where F_{op} relates the velocity distribution emerging from the scattering center (at time t) to the velocity distribution reaching the center *at an earlier time*, $t-T$. Chambers has failed to incorporate into transport theory the time lag T of the collision operator (2), and has therefore double counted the rotation $\omega_c T$ *during* the collision.

The foregoing discussion explains the subtlety of the issue in classical terms. It is possible to reach our conclusion quantum mechanically.[2] An alternative approach, less encumbered by formalism, is the following: In the presence of a magnetic field $\vec{B}=\text{curl}\vec{A}$, the velocity operator is

$$\vec{v}_{\text{op}} = \vec{p}/m + e\vec{A}/mc \ . \quad (3)$$

The acceleration operator $\vec{a}_{\text{op}}$ is obtained from the commutator of (3) with the Hamiltonian

$$H = H_0 + V_{\text{scatt}}(\vec{r}) \ , \quad (4)$$

with $H_0 \equiv m v_{\text{op}}^2/2$. Accordingly,

$$\vec{a}_{\text{op}} = (i\hbar)^{-1}(\vec{v}_{\text{op}}H - H\vec{v}_{\text{op}}) \ . \quad (5)$$

This can be divided into two parts:

$$\vec{a}_{\text{op}} = (i\hbar m)^{-1}[(\vec{p}+e\vec{A}/c)H_0 - H_0(\vec{p}+e\vec{A}/c)] + (i\hbar m)^{-1}[(\vec{p}+e\vec{A}/c)V_{\text{scatt}}(\vec{r}) - V_{\text{scatt}}(\vec{r})(\vec{p}+e\vec{A}/c)] \ . \quad (6)$$

The first term is easily recognized as the acceleration of a free electron in a magnetic field. The second term (the only one involving V_{scatt}) simplifies to

$$(\vec{a}_{\text{op}})_2 = (i\hbar m)^{-1}(\vec{p}V_{\text{scatt}} - V_{\text{scatt}}\vec{p}) \ , \quad (7)$$

since $\vec{A}(\vec{r})$ commutes with $V_{\text{scatt}}(\vec{r})$. Note that the magnetic field does *not* appear in (7). It follows that any acceleration caused by the scattering potential is *independent of* $\vec{B}$. Consequently, the scattering centers cannot contribute to the Hall coefficient. There is no acceleration (linear in B) other than the Lorentz force term, and that is taken into account completely by Eq. (1) (in the transport equation), which rotates $\langle \vec{v} \rangle$ *continuously*— even during those short intervals (of duration T) during which the acceleration of Eq. (7) occurs.

Although the acceleration $(\vec{a}_{\text{op}})_2$, Eq. (7), does not involve B, we must still verify that its expectation value has no term linear in B (which might conceivably arise from the influence of the magnetic field on the scattering wave function).

At this point we make use of the (agreed upon) assumption that we are dealing with a spherically symmetric scattering potential. The exact solution of such a problem, for an electron of energy $\hbar\omega$, can be written in terms of partial waves:

$$\psi_0 = e^{-i\omega t} \sum_{l,m} a_{lm}(r) Y_{lm}(\theta,\phi) \quad , \tag{8}$$

where $\{Y_{lm}(\theta,\phi)\}$ are spherical harmonics. The scattering center is located at $\vec{r}=0$. If a magnetic field $B\hat{z}$ is turned on, the perturbation (to order B) is $\hbar\omega_c L_z/2$, where L_z is the $\hat{z}$ component of the angular momentum operator. But the $\{Y_{lm}\}$ are eigenfunctions of L_z. So the new solution of the scattering problem can be written

$$\psi' = e^{-i\omega t} \sum_{l,m} a_{lm}(r) Y_{lm}(\theta,\phi) \exp(-im\omega_c t/2) \quad . \tag{9}$$

This wave function solves the time-dependent Schrödinger equation. It defines a scattering event, with a given incident-velocity vector, only during a small time interval, which we may take to be at $t \approx 0$. Suppose we select an incident velocity in the x-z plane. Then (at $t=0$) the ϕ dependence for each $|m|$ in (9) is $\cos m\phi$,[3] and only the component of $(\vec{a}_{op})_2$ in the $\hat{y}$ direction is relevant to a left-right asymmetry. From Eq. (7),

$$[(\vec{a}_{op})_2]_y = \frac{-1}{m}\frac{dV_{scatt}}{dr}\left(\frac{y}{r}\right) \sim \sin\theta \sin\phi \quad . \tag{10}$$

It is now obvious that

$$\langle \psi' [(\vec{a}_{op})_2]_y | \psi' \rangle = 0 \quad , \tag{11}$$

since each term in (11) involves an integral,

$$\int_0^{2\pi} \cos(m'\phi)\cos(m\phi)\sin\phi \, d\phi = 0 \quad . \tag{12}$$

This completes our alternative proof. The simplicity of the derivation leading to Eq. (11) derives from the fact that the perturbation $\hbar\omega_c L_z/2$ does not modify the $\{a_{lm}(r)\}$, but merely introduces time-dependent phase factors, as shown in Eq. (9).

We find Chambers's analysis of the uncertainty relation, $\Delta v_x \Delta v_y \gtrsim \hbar\omega_c/m$, to be selective and faulty. This relation imposes limitations to physical meaning for *all* wave functions, not just for (our) wave packets. One must recognize that the electrical current contributed by a wave packet is proportional to the *expectation value* of the velocity operator for that packet. We insist that there is no injunction against following the evolution of this expectation value as the packet travels through the field $\vec{B}$ and past scattering centers. Chambers's implicit assertion that changes in expectation value are meaningless unless they exceed the velocity uncertainty of the packet contradicts elementary quantum mechanics.

We emphasize that the transport equation commonly used to calculate the (weak-field) Hall coefficient employs Jones and Zener wave packets.[4] Consequently, our choice of these (same) basis functions to treat the scattering term of that transport equation is singularly appropriate.

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[1]W. G. Chambers, J. Phys. C 6, 2441 (1973).

[2]M. Huberman and A. W. Overhauser, Phys. Rev. B 20, 4935 (1979).

[3]This follows from the fact that at $t=0$, $\psi'=\psi_0$, the scattering state of a spherically symmetric object in zero field, which can have no left-right asymmetry.

[4]H. Jones and C. Zener, Proc. R. Soc. London Ser. A 144, 101 (1934).