Comments

Comments are short papers which comment on papers of other authors previously published in the **Physical Review**. *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Scattering of electrons by impurities in a weak magnetic field: A comment

W. G. Chambers
Mathematics Department, Westfield College, University of London, London NW3 7ST, England
(Received 19 March 1982; revised manuscript received 8 December 1982)

Since the author's prediction that impurity scattering can contribute to an anomalous Hall effect has been queried, a modified derivation is given of the basic expression for the change to first order in the magnetic field of the differential scattering cross section of an impurity. The counterarguments are answered on the grounds that they are based on a method using wave packets which, because of the uncertainty principle, do not have a momentum sufficiently well defined to examine the possibility of such anomalous scattering.

INTRODUCTION

A suggestion by the author,[1] that resonant impurity scattering of noninteracting spinless electrons could lead to an anomalous Hall effect, has been queried by Huberman and Overhauser.[2] A new derivation of the basic formula is here given, which is based on ideas in the original presentation, but which, it is hoped, will make these ideas more plausible. (The original formulation has been criticized for its lack of rigor.[3]) The counterarguments are considered to show how a null result was obtained, and the uncertainty principle is invoked to show that a theory based on the use of time-dependent wave packets is not precise enough to discuss the existence of the effect proposed by the author.

EFFECT

Let us consider a scattering experiment where a free electron starts from a source at $\vec{R}$, scatters off a spherically symmetric impurity at the origin $\vec{O}$, and proceeds to a detector at $\vec{S}$ (Fig. 1). The electron may also be subject to a weak magnetic field $\vec{B}$ perpendicular to the plane of the diagram, this direction also being chosen for the z axis. We shall calculate to first order in B the change in the wave intensity at $\vec{S}$ when the field is applied and show that it is caused not only by the bending of the paths but also by a change in the differential scattering cross section. Naturally, the direct propagation must be removed by inserting a barrier as in Fig. 1.

The Hamiltonian H for a particle in a uniform field $(0,0,B)$ and subject to a spherically symmetric potential $V(r)$ may be written as

$$H=-\hbar^2\nabla^2/(2m)+V-\tfrac{1}{2}\omega L+\tfrac{1}{8}m\omega^2(x^2+y^2) \quad , \quad (1)$$

where m is the electron mass, $\omega=eB/m$ is the cyclotron frequency, and L is the z component of the orbital angular momentum $\vec{r}\times(-i\hbar\vec{\nabla})$. Here the symmetric gauge $\vec{A}=(-\tfrac{1}{2}By,\tfrac{1}{2}Bx,0)$ has been used. We shall also need the Hamiltonians h for the case

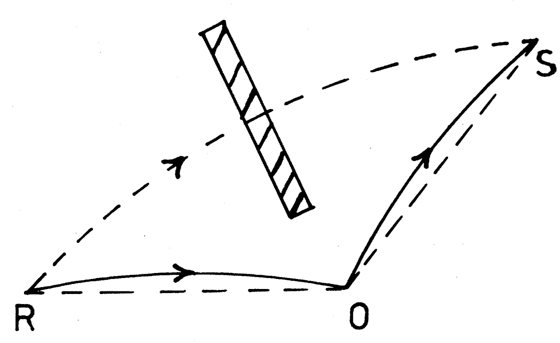

FIG. 1. Scattering geometry for an electron starting from a source at $\vec{R}$, being scattered by an impurity at $\vec{O}$, and then being detected at $\vec{S}$. The direct ray is removed by a block. The orbits of negative particles are curved as shown if the field is directed down through the plane of the diagram. The scattering angle is the angle between the continuation of $\overline{RO}$ through point O and $\overline{OS}$. (The arcs are implied if $B\neq 0$ and straight lines if $B=0$.)

 ©1983 The American Physical Society

with $B=0$, H_0 for the case with $V=0$, and h_0 with both B and V equal to zero. We define the propagators, or Green's functions,[4]

$$G=1/(E-H),\quad G_0=1/(E-H_0)\ ,$$

$$g=1/(E-h),\quad g_0=1/(E-h_0)\ .$$

To satisfy the boundary conditions at infinity and to avoid the singularities on the real axis, the energy E is given a small positive imaginary part $i\eta$. Thus, in particular, the propagator $g_0(E)$ in the configuration representation is given by[4]

$$g_0(\vec{r}',\vec{r},E)=-[m/(2\pi\hbar^2\rho)]\exp(ik\rho)\ ,$$

where $k=(2mE/\hbar^2)^{1/2}$ and $\rho=|\vec{r}-\vec{r}'|$. The effect of giving E an imaginary part $i\eta$ is to cause the propagator to die away exponentially with distance. The magnitudes of $\vec{S}$ and $\vec{R}$ will be chosen large enough to allow us to use the usual asymptotic formulas, but η will then be chosen small enough so that the attenuation in the propagation from $\vec{R}$ to $\vec{S}$ may be neglected. Finally, the field B will be chosen so small that for this value of η the propagation around a cyclotron orbit will be almost completely attenuated. Moreover, the radius of the orbit will far exceed R and S in magnitude. The effect of the direct path from $\vec{R}$ to $\vec{S}$ is removed by subtracting G_0 from G, so that the wave intensity at $\vec{S}$ is equal to $|G(\vec{S},\vec{R})-G_0(\vec{S},\vec{R})|^2$. In the case where $B=0$ the wave arriving at the scatterer from $\vec{R}$ will look like a plane wave of amplitude $-(m/2\pi\hbar^2R)\exp(ikR)$, and so the scattered wave at $\vec{S}$ will be[4]

$$g(\vec{S},\vec{R})-g_0(\vec{S},\vec{R})=C\exp[ik(S+R)]f(\vec{S},\vec{R})\ ,\quad (2)$$

where

$$C=-m/(2\pi\hbar^2SR)\ ,\quad (3)$$

and where $f(\vec{S},\vec{R})$ is the amplitude for scattering from the direction of $\vec{R}$ into the direction of $\vec{S}$. Naturally, because of the spherical symmetry, this amplitude depends on $\vec{S}$ and $\vec{R}$ only through the angle between them. The wave intensity at $\vec{S}$ is then the squared magnitude

$$|g-g_0|^2=|C|^2|f|^2\ .\quad (4)$$

We now consider changes of first order in B or ω. We have

$$G=g+\omega\left(\frac{\partial G}{\partial\omega}\right)_{\omega=0}$$

and similarly for G_0. By differentiating the equation

$$[E-h+\tfrac{1}{2}\omega L-\tfrac{1}{8}m\omega^2(x^2+y^2)]G=1$$

with respect to ω, we find that

$$\left[\tfrac{1}{2}L-\tfrac{1}{4}m\omega(x^2+y^2)\right]G+G^{-1}\frac{\partial G}{\partial\omega}=0\ ,$$

and so we obtain

$$\frac{\partial G}{\partial\omega}=-\tfrac{1}{2}GLG+\tfrac{1}{4}m\omega G(x^2+y^2)G\ .$$

We find that $GLG=LG^2$ since L generates rotations about the z axis and thus commutes with H [Eq. (1)] and hence with G. Moreover, by differentiating with respect to E we similarly obtain $G'=-G^2$, where the prime denotes $\partial/\partial E$, and so we have

$$\left(\frac{\partial G}{\partial\omega}\right)_{\omega=0}=\tfrac{1}{2}Lg'\ .$$

Similar results are found for G_0 by setting $V=0$, and by subtracting the two equations we find, to first order in ω,

$$(G-G_0)=(g-g_0)+\tfrac{1}{2}\omega L(g'-g_0')\ .\quad (5)$$

Since L generates infinitesimal rotations about the z axis we find that the configuration representation of Lg is given by $-i\hbar\partial g(\vec{S},\vec{R})/\partial\phi$, where ϕ is the azimuthal angle of $\vec{S}$. Finally, by using this result in (5) and by using (2) and (3), we find that to first order the wave intensity at $\vec{S}$ is altered by

$$|G-G_0|^2-|g-g_0|^2=\hbar\omega|C|^2\left[\tfrac{1}{2}k'(S+R)\frac{\partial}{\partial\phi}|f|^2+\mathrm{Im}\left(f^*\frac{\partial^2 f}{\partial\phi\,\partial E}\right)\right]\ .\quad (6)$$

[It should be noted that C in Eq. (3) is independent of ϕ and E.]

Since the differential scattering cross section $|f|^2$ is given by $|g-g_0|^2/|C|^2$ [Eq. (4)] we may interpret the right-hand side of (6) when divided by $|C|^2$ as a sum of two changes in the cross section. The first is $\Phi\partial|f|^2/\partial\phi$ with $\Phi=\tfrac{1}{2}\omega\hbar k'(S+R)$. The speed along a trajectory is $1/\hbar k'$ and the rate of change of direction is ω, and so Φ is just half the sum of the angles turned in the two legs of the flight. Thus this term is simply the effect of the change in the scattering angle which is apparent in Fig. 1 when the trajectories change from straight lines to arcs.

The other change in the differential cross section is

$$\hbar\omega\,\mathrm{Im}\left(f^*\frac{\partial^2 f}{\partial\phi\,\partial E}\right)\ .$$

This is the expression previously proposed by the author. The effect of this change on the Hall coefficient has already been discussed.[1]

DISCUSSION

An interpretation of the effect based on the Zeeman splitting of scattered partial waves has been given by Ballentine.[3,5] Another interpretation is as follows: If the scattering process is observed from a frame of reference rotating at the Larmor frequency $\frac{1}{2}\omega$, then the Coriolis force cancels out the Lorentz force, and so to first order there is no effect due to the field B. Now a resonant scatterer traps an incoming particle for a time with a mean value T of the order of $\hbar/\Delta$, where Δ is the width of the resonance. So the scattering pattern as observed from the fixed frame of reference is slewed round by an angle of order $\frac{1}{2}\omega T$, and hence loses its left-right symmetry. Interestingly enough, the argument in Ref. 2 at the end of Sec. III by which the null result is obtained is mathematically equivalent to this transform to a rotating frame.

The error in Ref. 2 is not introduced at this stage, but comes at a much earlier point through the introduction of wave packets. Time-dependent wave packets Ψ are employed for which the expectation values of the position operator $\vec{r}$ and of the velocity operator $\vec{P}/m$ follow the classical trajectories. With the convention used here $\vec{P}$ is equal to $(-i\hbar\vec{\nabla}-e\vec{A})$, Ψ satisfies the time-dependent Schrödinger equation, and $H_0\Psi=i\hbar\partial\Psi/\partial t$ with $V=0$. [In the conventions of Ref. 2, $\vec{P}$ is given by $-i\hbar\vec{\nabla}+(e/c)\vec{A}$.] Now the components P_x and P_y of $\vec{P}$ satisfy the commutation rule $[P_x,P_y]=-i\hbar eB$, so that there is an uncertainty relation for the components of the form $\Delta P_x \Delta P_y \gtrsim \hbar eB$, which must apply to the wave packets whatever their shape and size. (This uncertainty may be traced back to the zero-point motion of an electron in a magnetic field. It may also be discussed in terms of the probability current density which changes in direction across the wave packet.[5] In fact, this current swirls round in a motion which disappears when viewed from the rotating frame of reference mentioned above.) Since the shape of the wave packet is not discussed it must be assumed that ΔP_x and ΔP_y are roughly the same. Therefore they are of order $(\hbar eB)^{1/2}$. But any discussion of the anomalous Hall effect in terms of the kinetic momenta requires a precision of the order of $\hbar k_F(\frac{1}{2}\omega T)$, where $\hbar k_F$ is the Fermi momentum. This is a measure of the change of momentum transverse to the main part (of magnitude $\hbar k_F$) caused by the slewing of the scattering pattern. This term is of order B, and thus less than ΔP_x or ΔP_y if B is small, in fact, if $E_F\hbar\omega \lesssim \Delta^2$ where E_F is the Fermi energy $(\hbar k_F)^2/2m$.

It may be objected that this precludes any discussion of the normal Hall effect in terms of wave packets. This is not so. Let us consider an electron propagating in the x direction over a distance L, which is large in comparison with the de Broglie wavelength but small compared with the cyclotron radius. The electron acquires a transverse momentum of magnitude LeB to first order. We now let B tend to zero. If we may simultaneously let L tend to infinity keeping BL fixed, then the uncertainty $(\hbar eB)^{1/2}$ in the transverse momentum P_y becomes insignificant. But if L must remain fixed, then we use wave packets with $\Delta P_y << LeB$, which requires $\Delta P_x >> \hbar/L$ in compensation. The wave packets then would be very much elongated in the y direction, with a width of the order of $\hbar/\Delta P_x$ in the x direction. Thus this width must be very much less than L but, since L is very much greater than the de Broglie wavelength, this is not hard to arrange. An escape along these lines is not available when considering impurity scattering. Here the change in P_y to be discussed is of order $\hbar k_F(\omega T)$. So for ΔP_y to be much less than this we need $\Delta P_x >> m\Delta/\hbar k_F$ or, in other terms, $\Delta P_x >> \hbar k_F(\Delta/E_F)$. This would give rise to a large spread in energies unless Δ was very narrow.

[1] W. G. Chambers, J. Phys. C 6, 2441 (1973).
[2] M. Huberman and A. W. Overhauser, Phys. Rev. B 20, 4935 (1979).
[3] L. E. Ballentine, in *The Hall Effect and Its Applications: Proceedings of the Commemorative Symposium, Baltimore, Maryland, 1979* (Plenum, New York, 1980), pp. 201–213.
[4] L. I. Schiff, *Quantum Mechanics*, 2nd ed. (McGraw-Hill, New York, 1955), pp. 161–164.
[5] L. E. Ballentine (unpublished).