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Localization, percolation, and the quantum Hall effect

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Noninteracting electrons in a smooth two-dimensional random potential are localized in the large magnetic field limit. In contrast to Anderson localization, eigenstates with large localization lengths occur with a probability proportional to a universal power of their size, with the power given in terms of percolation critical exponents. Adding a parallel electric field $\vec{\mathscr{E}}$ causes extended states to appear in numbers proportional to a power of $\mathscr{E}$. This implies a nonlinear broadening of steps in the quantized Hall conductivity. The results for a parallel electric field are obtained by considering a graded percolation problem, in which the probability that a site is occupied varies with position.

I. INTRODUCTION

It is believed that in two dimensions without a magnetic field any amount of disorder will localize all of the electronic wave functions.[1] Experiments have been performed in which an intense perpendicular magnetic field is applied to electrons that are confined to two dimensions in metal-oxide semiconductors and heterojunctions.[2,3] These experiments have demonstrated that the Hall conductivity is quantized in units of e^2/h.

The following question arises: Do the electronic states remain localized in the presence of a magnetic field? The simplest point of view[4] is that there must be extended (or arbitrarily large) states since localized states carry no current and nonzero Hall conductivity is observed. Purely theoretical arguments reaching this conclusion for sufficiently weak potential have been given by Aoki and Ando,[5] Laughlin,[6] and Halperin.[4]

It is of interest to know whether a finite fraction of the eigenstates is extended. If so, the steps in the quantum Hall conductivity will never become sharp, even in the limit where the magnetic field is large and the temperature and electric field go to zero. There have been speculations[7] that this may be the case (the assumption is also implied by Refs. 5 and 6). A further agrument in favor of a nonzero fraction of extended states is that is difficult to imagine a single delocalized electron carrying sufficient current to be observed in a macroscopic experiment.

Contrary to these expectations and in agreement with remarks in Refs. 3 and 8, we find that for a smooth random potential in the limit of an arbitrarily large magnetic field B, the fraction of extended states is zero. (Because the number of states per unit area in a given Landau level is proportional to B, it is possible that the number of extended states per unit area does not vanish as $B \rightarrow \infty$, even though the fraction of states that are extended goes to zero.) Our main result is to demonstrate a *quantitative* relation between the localization problem and two-dimensional percolation. [A qualitative correspondence has been previously noted; see Refs. 3 and 8–10(a).] The distribution of eigenstates of large spatial extent is given by a universal relation involving percolation critical exponents. A second relation is derived that describes the fraction of extended eigenstates as a function of the tangential electric field, which implies a nonlinear broadening of the Hall conductivity steps.

The contents of the remainder of this paper are as follows: Section II describes the electron eigenfunctions and their relation to continuum percolation. The distribution law for large eigenstates is derived. Section III treats the effect of adding a tangential electric field. A lower bound for the number of extended states in a general potential is derived, as well as a relation for a random potential. The model is shown to imply a linear relation between Hall voltage and current, but a nonlinear broadening of the steps in Hall conductivity versus electron density. Section IV includes a description of the electron Green's function and conclusions. The appendix is a rough calculation of the magnitude of the nonlinear step broadening, which appears to be in an experimentally accessible regime.

II. EIGENFUNCTIONS AND PERCOLATION

We consider the eigenfunctions of the two-dimensional Hamiltonian

$$H=\frac{1}{2m_{\rm eff}}\left[\vec{p}-\frac{e\vec{A}}{c}\right]^2+V(x,y)\ , \tag{1}$$

where $V(x,y)$ is a smooth position-dependent potential that arises from inhomogeneities. Electron-electron interactions are neglected. The magnetic field $\vec{B}=\vec{\nabla}\times\vec{A}$ is of constant magnitude B in the $\hat{z}$ direction. Let $l\equiv(\hbar c/eB)^{1/2}$ be the radius of the ground Landau orbit, and $\omega_c\equiv eB/m_{\rm eff}c$ be the cyclotron frequency.

In the limit in which the magnetic field becomes arbitrarily large (V slowly varying on the scale l), the eigenfunctions ψ become quite simple: $|\psi|$ is large only in the vicinity of a constant-energy surface of the potential V. Perpendicular to the line of constant V, the wave function extends a distance of order l, which vanishes in the limit of $B\to\infty$. Other authors, including Tsukada[9] and Prange and Joynt,[8] have noted that the eigenfunctions of energy $E+\hbar\omega_c(n+\frac{1}{2})$ are localized about the classical orbit $V(\vec{x})=E$ in the large-B limit. The latter work contains an illuminating derivation based on the path integral representation of the electron propagator. The motion of electrons along equipotential contours is analogous to that of superconducting or superfluid vortices in an inhomogeneous film. Percolation theory implies universal power-law behavior for vortices as well.[10(b)]

Although we will not need the explicit form of the electron eigenfunctions, they can be approximated by

$$\tilde{\psi}(u,v)=C(u)\chi_n(v)e^{i\phi(u,v)} \tag{2}$$

in the limit $B\to\infty$. The variable u parametrizes distance along the constant energy surface $V(\vec{x})=$const, and v parametrizes distance orthogonal to this line. The index $n=0,1,2,\ldots$ is the Landau level and the energy eigenvalue is $E=V+\hbar\omega_c(n+\frac{1}{2})$. The function χ_n is the nth harmonic oscillator function,

$$\chi_n=H_n\left[\frac{v}{l}\right]\exp\left[-\frac{v^2}{2l^2}\right]\ ,$$

$$C^2(u)\sim\frac{1}{|\vec{\nabla}V(u,v)|_{v=0}}\ ,$$

and $\phi(u,v)$ is a gauge-dependent phase.

One can show that the wave function $\tilde{\psi}(u,v)$ given by Eq. (2) satisfies the Schroedinger equation $H\tilde{\psi}=E\tilde{\psi}$ in the regions where $\tilde{\psi}$ is large, with fractional errors of order $r_c^{-1}(\hbar c/eB)^{1/2}$ and also of order

$$\hbar m_{\rm eff}\,|\,\vec{\nabla}V\,|(c/\hbar eB)^{3/2}$$

(r_c is the local radius of curvature of the constant-energy surface). These errors vanish on smooth constant-energy surfaces in the limit $B\to\infty$.

Because $\phi(u,v)$ must change by an integral multiple of 2π around a circuit, only a discrete set of constant-energy surfaces corresponds to electron eigenfunctions. In the high-B limit, each allowed surface encloses an additional area of hc/Be. The electron density is then everywhere equal to nBe/hc if n Landau levels are filled. (These points are discussed in more detail in Ref. 9.)

Consider two approximate eigenfunctions given by Eq. (2), ψ_1 and ψ_2, that are on opposite sides of a saddle and whose constant-energy surfaces are a distance d apart at their closest approach. The eigenfunctions are approximate because they were obtained by retaining terms in H only to second order in v. For a large but finite B, there is a nonvanishing overlap integral η between ψ_1 and ψ_2. A true eigenstate will then be an admixture of ψ_1 and ψ_2. Note, however, that

$$\eta\sim\exp(-d^2eB/4\hbar c)\ ,$$

whereas the minimum energy mismatch ΔE of ψ_1 and ψ_2 goes like $1/B$, barring an accidental degeneracy.[11] The ratio $\eta/\Delta E$ vanishes as $B\to\infty$. This implies that as $B\to\infty$, an electronic eigenstate has a large amplitude only in the neighborhood of a single (connected) constant-energy surface.

We now consider a random potential $V(\vec{r})$ which is assumed to have the following properties: $V(\vec{r})$ is a random variable whose distribution is given by $\text{Prob}[V(\vec{r})=\epsilon]=\rho(\epsilon)$. The distribution function $\rho(\epsilon)$ is continuous, bounded, and independent of $\vec{r}$ with $\int_{-\infty}^{\infty}\rho(\epsilon)d\epsilon=1$. We choose the zero of energy such that $\langle V(\vec{r})\rangle=0$, where enclosure by angular brackets denotes an average over $\vec{r}$. The correlation function $\langle V(\vec{r})V(\vec{r}+\vec{x})\rangle$ goes rapidly to zero for $|\vec{x}|\gg b$, where b is the correlation length of the potential.[12] Finally, we assume that V is slowly varying on a sufficiently small length scale. Many functions satisfy the above requirements, such as Gaussian white noise with a smooth high-frequency cutoff at $k\approx b^{-1}$.

As $B\to\infty$, the density of states is proportional to $\rho(E)$. If one seeks eigenfunctions of energy E,[13] a convenient construction is to color in all points $\vec{r}$ such that $V(\vec{r})\le E$. Then as $B\to\infty$, the eigenfunctions will be localized on the perimeters of the colored areas (see Fig. 1). For a random V, the

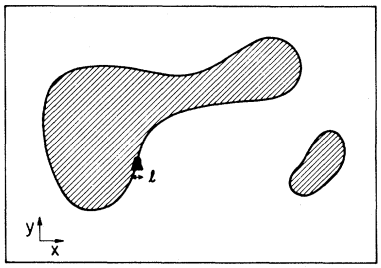

FIG. 1. Points $\vec{r}$ such that $V(\vec{r}) \leq E$ are shaded. Eigenfunctions of energy E are localized within a distance l of the perimeter of a shaded region.

properties of the connected colored regions are those of a continuum percolation problem.[14] In the usual site-percolation problem, a lattice point $\vec{r}_j$ is considered to be occupied (colored) with a probability p. One is interested in the properties of connected clusters as a function of p. In the continuum problem described above, $\vec{r}_j$ is replaced by the continuous variable $\vec{r}$ and p is replaced by E through

$$p(E) = \int_{-\infty}^{E} \rho(\epsilon) d\epsilon \ . \tag{3}$$

Percolation theory immediately implies a number of universal results for the case in which the potential V is random. There is a unique energy E_c (the percolation threshold) where E_c is the smallest energy for which an infinite connected colored region exists. For $E < E_c$ there is a length

$$\xi(E) \sim |E - E_c|^{-\nu} \text{ as } E \to E_c \tag{4}$$

such that connected clusters of spatial extent L greater than ξ are exponentially rare. (The spatial extent L could be loosely defined as the "diameter" of a connected region or quantitatively as the radius of gyration.) In two dimensions the percolation critical exponent ν is $\frac{4}{3}$.[15] Thus for $B \to \infty$ the electronic wave functions of energy E will typically have a spatial extent of no larger than the order of $\xi(E) = c_1 |E - E_c|^{-\nu}$. For $E > E_c$ an infinite connected colored region is present. One might suppose that this would imply eigenfunctions of infinite spatial extent. Such is not the case, however. For $E > E_c$ it is convenient to identify the wave functions with the perimeters of the *uncolored* regions.[16] Above E_c the connected uncolored regions are of finite extent, with maximum typical size again given by Eq. (4). Because of this symmetry, we will need to consider only the finite connected regions for $E < E_c$.

There are no states of infinite extent except perhaps at $E = E_c$, which is a set of measure zero. Thus we have that *for a random potential as* $B \to \infty$, *the fraction of eigenstates of infinite extent is zero.* Note that we have used the assumption that the distribution function $\rho(\epsilon)$ is continuous, so that in the vicinity of E_c Eq. (3) implies that $E - E_c$ is proportional to $p - p_c$. In contrast, Prange and Joynt[8] have considered a model potential in which $V = 0$ except in disconnected regions in which V is arbitrary. That potential has a δ function in $\rho(\epsilon)$ exactly at the percolation threshold, and in fact has a finite fraction of states of infinite extent for $B \to \infty$.

The energy dependence of the localization length ξ given in Eq. (4) disagrees with that given by Ono,[16(a)] who uses the self-consistent Born approximation to find $\xi(E) \sim l \exp(\gamma^2/E^2)$. One could attribute the difference to the fact that Ono considered a random potential that is a sum of δ functions, so that his $V(\vec{r})$ is never smooth on the scale of l, no matter how large the magnetic field. The localization of an eigenfunction on the perimeter of a percolation cluster may also be too subtle an effect to be seen in the self-consistent Born approximation.

In the absence of infinite eigenstates, we investigate those of large spatial extent. Let $f(R)$ be the fraction of eigenstates whose extent $L \geq R$. (The number of eigenstates per unit area is given by the fraction f multiplied by eB/hc.) We make the following assumptions, the first two of which are discussed by Stauffer[14]:

(1) The scaling hypothesis: The number density n_s of clusters of large area s is given by

$$n_s \sim s^{-\tau} g[(p_c - p)s^{\sigma}] \ ,$$

where $g(z)$ is analytic with $g(0) = 1$, and $g(z)$ goes rapidly to zero for large z.

(2) The ramification hypothesis: Large clusters of area s have a typical perimeter t with $s \sim t^{p'}$ and $p' = 1$.

(3) We further assume that as p is increased to $p + dp$, a cluster of area s and perimeter t will be coated with an additional area dA which is proportional to tdp. The fraction of the total area occupied by constant-energy surfaces of perimeter t or greater is then

$$f \sim \int_0^{1/2} d\epsilon \int_t^{\infty} ds\, s^{1-\tau} g(\epsilon s^{\sigma}) \to f \sim t^{2-\tau-\sigma} \ . \tag{5}$$

Using the relation between the perimeter and the diameter of a connected region $t \sim R^{1/\sigma\nu}$, we obtain the relation that for large R

$$f(R) \sim R^{-p_1} \ . \tag{6}$$

The exponent

$$p_1=(\tau-2+\sigma)/\sigma\nu=(1+\beta)/\nu ,$$

where the last equality follows from scaling relations among exponents. With the use of $\beta=\frac{5}{36}$ (Ref. 15), $p_1=\frac{41}{48}$. As expected, the fraction of states of extent larger than R vanishes as $R\to\infty$.

Equation (6) states that large eigenstates appear with a probability proportional to a power of their diameter. This is in sharp contrast to Anderson localization in the absence of a magnetic field, where $f(R)$ goes to zero exponentially.[1]

One subtlety should be mentioned with respect to the above derivation. Equation (5) is most naively obtained by assuming that each cluster of area s has a single (external) perimeter $t\sim s$. Clusters, however, have internal perimeter as well, and it is the total perimeter (internal plus external) that is proportional to s. To incorporate this fact, define the average perimeter $\bar{t}$ of a cluster as $\bar{t}\equiv\sum_j t_j^2/\sum_j t_j$, where the sum is over the external and all internal perimeters. If $\bar{t}\sim s$, one can show that $p_1=(1+\beta)/\nu$ still obtains. A sufficient condition for $\bar{t}\sim s$ is that the external perimeter remains a nonzero fraction of the total perimeter. Computer simulations by Leath and Reich[16(b)] show that as s increases, the external perimeter is a *very* slowly decreasing fraction of the total perimeter. The largest simulated clusters (2000 sites) still have over 75% external perimeter. This issue is not yet sufficiently well understood to know whether $p_1=(1+\beta)/\nu$ or whether this is a slight underestimate of the correct exponent.

III. TANGENTIAL ELECTRIC FIELDS

A. Lower bound for general $V(x,y)$

In experiments on quantized Hall conductivity, electrons move in the presence not only of a magnetic field normal to the surface, but also of an electric field tangent to the surface. When σ_{xx} vanishes, this electric field is simply the Hall voltage divided by the sample width. We will show that such an electric field has a drastic effect on the localization of electronic states. We assume a smooth potential $V(x,y)$ whose magnitude and slope are bounded: $|V(x,y)|\le M$, $|\vec{\nabla}V(x,y)|\le M'$. The potential need not be random (it could, for example, have long-range correlations). Then, in the presence of a tangential electric field $\vec{\mathscr{E}}$ in the limit $B\to\infty$, a nonzero fraction of the electronic eigenstates is delocalized. The fraction f_∞ is bounded below by f_b:

$$f_\infty(\mathscr{E})\ge e\mathscr{E}/(M'+e\mathscr{E})\equiv f_b(\mathscr{E}) . \tag{7}$$

Thus the fraction of states of infinite extent increases at least linearly with $\mathscr{E}$ for small $\mathscr{E}$. This, incidentally, resolves the paradox of how a vanishing fraction of delocalized states can carry a macroscopic Hall current. As the Hall voltage increases, additional extended states appear which carry the extra current.

The proof of Eq. (7) again uses the correspondence between area and number of eigenstates. One must show that for the potential $U(x,y)=V(x,y)-e\mathscr{E}y$ the constant-energy surfaces that are infinite in extent comprise a fraction of the total area no smaller than f_b. The sample is assumed infinite in the $\hat{y}$ direction, which is the direction of the electric field. It is convenient to define $\tilde{V}(x,y)$ as the function $V(x,y)$ multiplied by a factor $G(x)$, where $G(x)=1$ for $0<x<L_1$, $G(x)=0$ for $x\le-\Delta$ and $L_1+\Delta\le x$, and $G(x)$ interpolates smoothly between zero and one in between. Define

$$\begin{aligned}\tilde{V}(x,y)&\equiv V(x,y)G(x) ,\\ \tilde{U}(x,y)&\equiv\tilde{V}(x,y)-e\mathscr{E}y .\end{aligned} \tag{8}$$

Thus $\tilde{V}=V$ in the region $0\le x\le L_1$, and $\tilde{V}$ vanishes outside the region $-\Delta<x<L_1+\Delta$.

Consider a constant-energy surface of $\tilde{U}$ containing the point $(\bar{x},\bar{y})$ with $\bar{x}<-\Delta$. This constant-energy surface is simply the line $y=\bar{y}$ for $x\le-\Delta$ [see Fig. 2(a)]. As the line enters the region $-\Delta\le x\le L_1+\Delta$ it must remain in the band $|y-\bar{y}|\le M/e\mathscr{E}$ since it is a constant-energy surface. In particular it cannot escape to $y=\pm\infty$ for a fixed nonzero electric field. The line cannot reemerge at $x<-\Delta$ and $y\neq\bar{y}$ because then again it would not be a constant-energy surface. Thus the

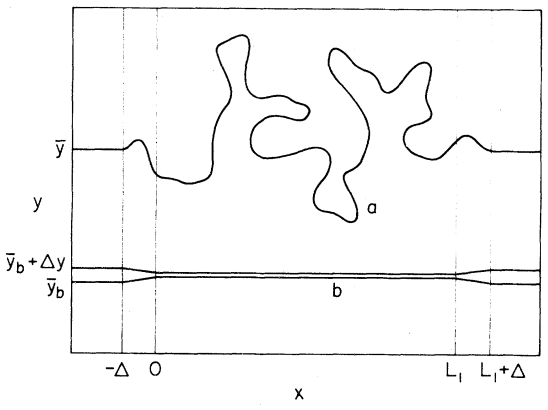

FIG. 2. Electric field is in the $\hat{y}$ direction. *a*, a constant-energy surface of $\tilde{U}$ for a random potential; *b*, a band of constant-energy surfaces that occupies the minimum possible area for $0\le x\le L$ (general potential).

line must reemerge at $x > L + \Delta$ where its equation is $y = \bar{y}$.[17] This is an extended constant-energy surface; it has an arc length of at least L_1 as the sample width $L_1 \to \infty$.

What fraction of the area is occupied by lines that extend across the sample? A band of states in the region $\bar{y}_b \le y \le \bar{y}_b + \Delta y$ for $x < -\Delta$ will occupy the smallest possible area in the region $0 \le x \le L_1$ if it moves straight across the sample in a region in which $|\vec{\nabla} U|$ has its maximum possible value of $M' + e\mathscr{E}$ [see Fig. 2(b)].[18] The minimum fraction of the area occupied by such states is then $f_b = e\mathscr{E}/(M' + e\mathscr{E})$, which proves Eq. (7).

Corollary. It is straightforward to show that if $e\mathscr{E} > M'$ all constant-energy surfaces are extended, so that $f_\infty = 1$.

B. Random $V(x,y)$

If V is a random potential then the constant-energy surfaces in the presence of a small electric field are tortuous and resemble the upper line in Fig. (2) more than the lower. The actual fraction $f_\infty(\mathscr{E})$ of extended states will then increase more rapidly with electric field than does the lower bound, Eq. (7). We propose that

$$f_\infty(\mathscr{E}) \sim \mathscr{E}^{p_2} \tag{9}$$

may be exact for $\mathscr{E} \to 0$, with

$$p_2 = (\tau - 2 + \sigma)/\sigma(1+\nu) = (1+\beta)/(1+\nu) = \tfrac{41}{84} .$$

Equation (9) is motivated as follows: One is interested in the connected regions of the plane such that

$$U(x,y) = V(x,y) - e\mathscr{E} y \le E .$$

This is a type of graded percolation problem in which the probability p that a point is occupied (colored) is not simply a constant, but rather a function of position. As a result, the locally defined correlation length $\xi(\vec{r})$ will vary from point to point in space. For fixed $\mathscr{E}$ and V, there will be a length which we denote by $L(\mathscr{E})$, such that finite clusters that extend a distance greater than L are exponentially rare. To estimate L, consider a cluster of extent l_y in the y direction. Such a cluster will be strongly suppressed if the local correlation length $\xi(\vec{r})$ is much less than l_y in large parts of the cluster. For nonzero $\mathscr{E}$, the local correlation length $\xi = c_1 |E - E_c|^{-\nu}$ can be no larger than the order of $c_1 |e\mathscr{E} l_y|^{-\nu}$ in large parts of the cluster. If the cluster is to occur with substantial probability, one must have

$$l_y \lesssim c_1 |e\mathscr{E} l_y|^{-\nu} ,$$

or

$$l_y \lesssim c' \mathscr{E}^{-\nu/(1+\nu)} = L(\mathscr{E}) . \tag{10}$$

The application of a small electric field $\mathscr{E}$ is not expected to substantially modify the statistics of clusters of extent $R \ll L(\mathscr{E})$, whereas it will strongly affect those with $R \gg L(\mathscr{E})$. It is reasonable to assume that the area that was occupied by constant-energy surfaces of extent $R \gtrsim L(\mathscr{E})$ in the absence of an electric field will be converted to f_∞ when the field is applied. Equations (6) and (10) then imply the desired result, Eq. (9).

C. Quantum Hall effect

The Hall current is carried by extended states. Since the number of such states is a singular function of the parallel electric field [Eq. (9)], one might expect a nonlinear relation between the current I_x and the Hall voltage V_H. Such is not, however, the case, and in fact this model predicts a linear relation $I_x = \sigma_{xy} V_H$. This relation has been previously obtained by Laughlin[6] as a consequence of gauge invariance. We include the following brief derivation, which obtains the same result from a different point of view.

Assume that the first Landau level is filled and for simplicity neglect spin and valley degeneracies. In the high-B limit, the electron number density $\tilde{n}$ is everywhere equal to eB/hc.[19] The expectation value of the local electron velocity operator is

$$\langle \vec{v}(\vec{r}) \rangle = c \frac{\vec{\mathscr{E}}(\vec{r}) \times \vec{B}}{B^2} . \tag{11}$$

$\vec{\mathscr{E}}(\vec{r})$ is the electric field in the x-y plane, which has contributions from the local random potential and from an external field,

$$\vec{\mathscr{E}}(r) = \vec{\mathscr{E}}_{\text{ext}} - (1/e)\vec{\nabla} V(\vec{r}) .$$

Equation (11) is correct both classically and quantum mechanically.[20(a)] The current I_x that passes through the line $x = \text{const}$ for a sample width L_2 in the $\hat{y}$ direction is

$$I_x = \int_0^{L_2} dy\, \tilde{n} e (\vec{v} \cdot \hat{x}) \to I_x = \frac{e^2}{h} \left[\left(\int_0^{L_2} \vec{\mathscr{E}}_{\text{ext}} \cdot \hat{y} \right) - \frac{1}{e} [V(x,L_2) - V(x,0)] \right] . \tag{12}$$

In an experiment, the measured Hall potential V_H is the sum of the electrostatic potential and the difference in Fermi levels[4] [respectively, the first and second terms inside the braces of Eq. (12)]. One recovers the result $I_x=\sigma_{xy}V_H$, with $\sigma_{xy}=e^2/h$. A simple generalization gives $\sigma_{xy}=ne^2/h$ for n occupied Landau levels.

Equation (9) implies a nonlinear broadening of the steps in σ_{xy} versus electron density. Consider a metal-oxide-semiconductor field-effect transistor (MOSFET) at $T=0$ in a nonzero parallel electric field $\vec{\mathscr{E}}$, with electrons added to the lowest Landau level by varying the gate voltage. The first electrons will occupy localized states with $E \ll E_c$. Then they will fill extended states at $E \approx E_c$, and finally the localized states with $E \gg E_c$. The current will increase only as the extended states are filled. By Eq. (9), the number of extended states is proportional to $\mathscr{E}^{(1+\beta)/(1+\nu)}$, with $\mathscr{E}$ proportional to the Hall voltage. One then has the following nonlinear effect: *The width of the steps in the Hall conductivity as a function of electron density is proportional to* $V_H^{(1+\beta)/(1+\nu)}$. The Appendix discusses the observability of this effect.

IV. DISCUSSION AND CONCLUSIONS

Percolation theory implies several additional properties of the electron eigenstates and Green's functions for very high magnetic fields. For $\mathscr{E}=0$ the eigenstates are closed curves of vanishing thickness $l \sim B^{-1/2}$. The statistics of the small closed curves are *not* universal, and depend in detail on the short-wavelength properties of the random potential V. In contrast, the large eigenstates have universal properties, including a density given by Eq. (6). They are rough objects (fractals), with a perimeter that grows faster than their diameter ($t \sim R^{1/\sigma\nu}$, with $1/\sigma\nu = \frac{91}{48}$).

We now consider some properties of the (retarded) electron propagator defined by

$$G(\vec{x}_2,\vec{x}_1,\omega)=\sum_m \frac{\psi_m^*(\vec{x}_2)\psi_m(\vec{x}_1)}{\omega-E_m/\hbar+i\eta} . \tag{13}$$

For fixed $\vec{x}_1$ there is a (connected) constant-energy surface S that contains $\vec{x}_1$. Let R_m be the distance to the point on S that is farthest from $\vec{x}_1$. $G(\vec{x}_2)$ will have a substantial amplitude for those points $\vec{x}_2$ that are near S. This amplitude tends on the average neither to grow nor diminish as $R_{12} \equiv |\vec{x}_2-\vec{x}_1|$ increases. Then suddenly as R_{12} increases past R_m, $\vec{G}(x_2)$ decreases very rapidly with increasing R_{12} for all $\vec{x}_2$. Even for the most favorable $\vec{x}_2$, G decreases like

$$\exp[-eB(R_{12}-R_m)^2/2\hbar c]$$

for R_{12} near R_m. This arises because increasing R_{12} past R_m forces $\vec{r}_2$ off of the constant-energy surface and into the Gaussian tail of the wave function. For random starting points $\vec{x}_1$, R_m will occur with a probability given by Eq. (6).

Conclusions. We have partially characterized the eigenstates of noninteracting electrons in a two-dimensional random potential and strong magnetic field. As $B \to \infty$, a fraction 1 of the eigenstates are localized. (As mentioned in the Introduction, this does not imply that no extended states exist, but rather that they comprise a vanishing fraction of the total.) Large eigenstates occur in numbers proportional to a power of their spatial extent, with the power given in terms of percolation critical exponents. The result is quite unlike Anderson localization that occurs for $B=0$, where large eigenstates are exponentially rare.

For $B \to \infty$, a tangential electric field $\mathscr{E}$ destroys localization, and creates extended states in numbers proportional to a power of $\mathscr{E}$. This implies a nonlinear broadening of the steps in quantum Hall conductivity, again given in terms of percolation critical exponents.

We have considered only the limiting distribution functions as $B \to \infty$. There is as yet no systematic way to perturb away from this limit, such as an expansion in powers of B^{-1}. In particular, the possibility remains that the fraction of extended states goes smoothly to zero as $B \to \infty$, but does not actually vanish at any finite magnetic field. One might investigate finite magnetic fields by considering a kind of Anderson localization problem in which the sites are constant-energy surfaces, and are connected by hopping terms. One feature of this model is that the sites themselves can become arbitrarily large near E_c, so that an eigenfunction near the band center could be localized when measured in terms of sites but extended in terms of distance.

Note added. After this work was completed, we learned of a study by Luryi and Kazarinov.[20(b)] These authors have adopted a similar point of view and have discussed (among other topics) finite-size effects, in contrast to the results for extended systems that are obtained here.

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APPENDIX: EXPERIMENTAL ASPECTS OF NONLINEAR BROADENING

The appendix contains a rough argument that the nonlinear broadening of the steps in the Hall conductivity may be experimentally observable. Let w be the width of the Hall step divided by the distance between Hall steps at successive Landau levels. (The width can be defined as the inverse of the maximum slope of $d\sigma_{xy}/d\tilde{n}$.) Equation (9) implies

$$w = c_2 \mathscr{E}^{P_2} . \tag{A1}$$

It is of interest to determine c_2 to know whether nonlinear broadening should be observable at reasonable Hall voltages.[21] An accurate estimate is difficult because c_2 is not universal and depends on details of the random potential V, including its correlation length b and average magnitude $\bar{V} \equiv \langle V^2 \rangle$. We will give what is at best an order-of-magnitude estimte.[22]

The discussion preceding Eq. (9) implies that $f(\mathscr{E})$ is 0(1) for an electric field the order of an average $|\vec{\nabla} V|$, where $\langle |\vec{\nabla} V| \rangle \approx \bar{V}/b$. Equation (A1) becomes

$$w = \alpha (e\mathscr{E} b/\bar{V})^{P_2} , \tag{A2}$$

with α a dimensionless constant of order unity. Both b and $\bar{V}$ are highly sample dependent. We use $\bar{V} \approx k_B T$ at the temperature T at which the steps experimentally get fairly broad due to thermal excitation of electrons. $T \approx 5$ K might be typical for some samples. At $B = 20$ T, the Landau length is 60 Å. The percolation arguments used are valid only for $b \gg l$; we assume a sample with $b \approx 500$ Å. For a sample of width 50 μm, Eq. (A2) implies that a Hall voltage of ≈ 1 mV will give a step of width $w = 0.05$. This is an experimentally accessible regime. In fact, the steps published by Klitzing *et al.*[2] may be nonlinearly broadened by the Hall voltage.

We are unaware of any published studies of $w(V_H)$. (Nonlinear studies of σ_{xx} do, however, exist.[23])

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[11]Such accidental degeneracies would occur systematically for a periodic potential, but can be neglected for the random potential that we will consider.

[12]More specifically, the assumption is that the correlations between $V(\vec{r}_1)$ and $V(\vec{r}_2)$ are sufficiently short ranged that the continuum percolation problem defined by V is in the universality class of the uncorrelated lattice percolation problem.

[13]The zero-point energy has been subtracted off, so that the eigenvalue of H is $E + \hbar\omega_c(n + \frac{1}{2})$.

[14]Review articles on percolation have been written by D. Stauffer, Phys. Rep. 54, 2 (1979) and by J. W. Essam, Rep. Prog. Phys. 43. 833 (1980).

[15]This presumably exact result follows from the equivalence of percolation and the $q = 1$ state Potts model; see the review by E. K. Riedel, Physica (Utrecht) 106A, 110 (1981), and references therein.

[16]Equivalently, the eigenfunctions lie on the boundaries of holes in the infinite colored cluster. (a) Y. Ono, J. Phys. Soc. Jpn. 51, 2055 (1982), and unpublished; (b) P. L. Leath and G. R. Reich, J. Phys. C 11, 4017 (1978).

[17]We have assumed that V is such that with probability 1, a constant-energy surface will not simply terminate or intersect itself at other than a simple saddle point.

[18]Note that trajectories that begin with different y for $x < -\Delta$ have different energies and cannot intersect, so that none of the area is double counted.

[19]See Ref. 9; corrections are of order $(m_{\text{eff}}\omega_c^2)^{-1}\nabla^2 V$.

[20](a) R. Kubo, S. J. Miyake, and N. Hashitsume, in *Solid State Physics,* edited by F. Seitz and D. Turnbull (Academic, London, 1965), Vol. 17, p. 269; (b) S. Luryi and R. F. Kazarinov, Phys. Rev. B 27, 1386 (1983), and unpublished. We also thank Dr. Luryi for an informative conversation.

[21]In an actual experiment, Eq. (A1) will not be obeyed down to $\mathscr{E} = 0$. For sufficiently small $\mathscr{E}$, the nonlinear

broadening will become smaller than the broadening due to nonzero temperature and an insufficiently large magnetic field; $w(\mathscr{E})$ will then cross over and approach a constant as $\mathscr{E} \rightarrow 0$.

[22]We implicitly assume that neither $\rho(\epsilon)$ nor $\langle V(\vec{r}_1)V(\vec{r}_2)\rangle$ have very long tails.

[23]M. A. Paalanen (private communication).