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Transverse elastic waves in periodically layered infinite and semi-infinite media

R. E. Camley*
Department of Physics, University of Colorado, Colorado Springs, Colorado 80907

B. Djafari-Rouhani and L. Dobrzynski
Laboratoire des Surfaces et Interfaces, Institut Supérieur d'Electronique du Nord, 3 rue Francois Baes, F-59046 Lille Cedex, France

A. A. Maradudin
Department of Physics, University of California, Irvine, California 92717
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The propagation of transverse elastic waves both perpendicular and parallel to the laminations of infinite and semi-infinite periodically layered media is studied. The displacement fields and the dispersion relations for such waves are obtained analytically, and the latter are solved numerically to yield the corresponding dispersion curves. It is shown that a semi-infinite medium layered periodically parallel to its stress-free surface can support shear horizontal surface acoustic waves that have no counterpart in a homogeneous medium. Finally, the dynamical elastic Green's function for a semi-infinite medium layered periodically parallel to its stress-free surface is obtained, and its possible application to Brillouin scattering studies of the surface acoustic waves on such a medium is discussed.

I. INTRODUCTION

The propagation of acoustic waves through layered media has been the object of a great deal of theoretical study over the past 70 years. The motivation for much of the earlier work was the desire for an accurate description of the propagation of seismic shocks through Earth's crust, whose density and elastic properties vary with distance from the surface of Earth. This more or less continuous variation of the material parameters of Earth's crust was modeled by first neglecting the curvature of Earth's surface as large compared with the wavelengths of the disturbances of interest, and then regarding the crust as a stack of homogeneous elastic plates, each with its own elastic moduli and density.

The study of elastic waves propagating in such layered media, parallel to the surface, was thus reduced to the solution of the equations of motion of elasticity theory for the displacement field in each layer, and the satisfaction of the boundary condition at each interface. In general, the latter step yields the dispersion relation for such waves.

In the kinds of calculations described in the preceding paragraph there was usually no particular regularity in the thickness of the successive layers nor in the variation of the material properties from layer to layer. The number of layers treated was limited by the computational resources available. Good summaries of these kinds of calculations are contained in books by Ewing *et al.*[1] and Brekhovskikh.[2]

With the development of techniques for fabricating artificial semiconductor superlattices consisting of a periodic sequence of very thin layers of, for instance, two different materials such as InAs-GaSb (compositional superlattices)[3,4] interest is beginning to turn to the study of elastic waves in such periodically modulated structures.

Even before the advent of such artificial superlattices elastic wave propagation in a medium that consists of alternating layers of two different materials had been studied theoretically by several authors. The focus of much of this work was on the determination of the effective elastic moduli of such laminated media which, in general, are now anisotropic, even if the lamina out of which they are constructed are themselves anisotropic.[5–11]

While such an approach to wave propagation in periodically layered systems, usually called an effective modulus theory, simplifies all subsequent calculations based on its results, it has the drawback that it predicts bulk and surface waves that are nondispersive, while calculations that take the laminations into account explicitly yield dispersive behavior. Recent work dealing with such systems has avoided the use of effective modulus theories.[12–16]

In this paper we seek to add to the literature dealing with elastic wave propagation in layered media

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by considering several structures and propagation conditions that do not appear to have been investigated up to now. The emphasis in this work is on obtaining results analytically as much as possible, although computer calculations are ultimately required to translate these analytical results into more easily understood plots. On the basis of continuum theory, we study here the propagation of transverse elastic waves both perpendicular and parallel to the laminations of infinite and semi-infinite periodically layered media.

After an introductory discussion of the propagation of transversely polarized elastic waves in an infinitely extended layered structure, in Sec. II, we study the effects on these waves, of introducing various types of boundaries into the structure. Specifically, we consider the cases of a semi-infinite layered medium, in which the laminations are parallel to the stress-free surface of the medium; a semi-infinite layered medium in contact with a semi-infinite homogeneous medium; and a semi-infinite layered medium in contact with an elastic film. Of particular interest are the surface and interface acoustic waves that can propagate in these structures. The theoretical results obtained in Sec. II are illustrated by numerical calculations in Sec. III for specific examples of the various structures considered. In Sec. IV we present a determination of the dynamic elastic Green's function for a semi-infinite medium laminated periodically parallel to its stress-free surface. This function describes the dynamic response of the medium to a time-dependent, externally applied point force in it, and can be used in calculations of various physical properties of the medium, including Brillouin scattering of light from thermally excited elastic waves. The conclusions reached on the basis of the work described in this paper are presented in Sec. V.

II. THEORY

We initially consider propagation of transverse elastic waves in an infinitely extended layered structure. We will then consider the effects on these waves of introducing various types of boundaries into the structure.

The geometry of our problem is illustrated in Fig. 1. The elastic properties of the media of thickness d_1 are described by two parameters—the density ρ and the transverse sound velocity c_t. In the media of thickness d_2 the density is ρ' and the transverse sound velocity is c_t'. We consider transverse elastic waves where the displacement is in the x_2 direction, parallel to the plane of the layers. The propagation direction is in the x_1x_3 plane.

With this assumption, the elastic equation of

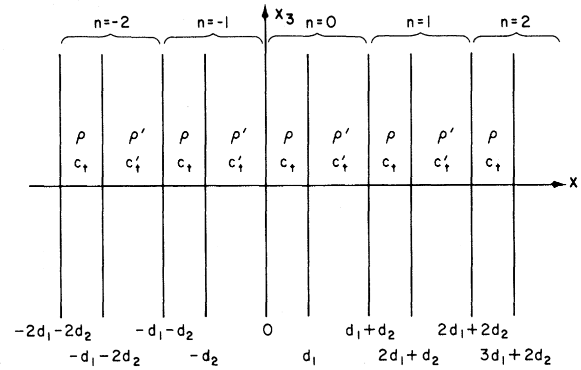

FIG. 1. Infinitely extended periodic layered structure. We consider waves propagating in the x_1x_2 plane with displacement in the x_2 direction.

motion in the umprimed media for the displacement $u_2(x_1,x_3,t)$ may be written

$$\ddot{u}_2(x_1,x_3,t)=c_t^2\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_3}\right]u_2(x_1,x_3,t)\ . \tag{2.1}$$

We initially look for a solution of a plane wave form with propagation along x_3. Thus, let

$$u_2(x_1,x_3,t)=e^{i(k_3x_3-\omega t)}u_2(x_1)\ . \tag{2.2}$$

Substitution of Eq. (2.2) into Eq. (2.1) gives

$$(c_t^2k_3^2-\omega^2)u_2(x_1)=c_t^2\frac{\partial^2}{\partial x_1^2}u_2(x_1)\ . \tag{2.3}$$

The general solution of this is clearly

$$u_2(x_1)=A_+e^{\alpha_1x_1}+A_-e^{-\alpha_1x_1}\ , \tag{2.4}$$

where

$$\alpha_1=[k_3^2-(\omega^2/c_t^2)]^{1/2}\ . \tag{2.5}$$

However, due to the periodicity of the layered structure in the x_1 direction, we want our solution for $u_2(x_1)$ to be in the form of a Bloch wave. Thus $u_2(x_1)$ must have the form

$$u_2(x_1)=e^{iqx_1}u_2(q,x_1)\ , \tag{2.6}$$

where

$$u_2(q,x_1)=u_2(q,x_1+L)\ . \tag{2.7}$$

Here $L=d_1+d_2$ is the period of the structure in the x_1 direction.

From Eqs. (2.4) and (2.6) we may find $u_2(q,x_1)$:

$$u_2(q,x_1)=e^{-iqx_1}(A_+e^{\alpha_1x_1}+A_-e^{-\alpha_1x_1})\ . \tag{2.8}$$

We must make $u_2(q,x_1)$ a periodic function with period L. We can do this by replacing x_1 by x_1-nL when x_1 is in the nth layer, where n indexes the layers as shown in Fig. 1. This redefines the arbitrary constants A_+ and A_-. Thus

$$u_2(q,x_1)=e^{-iq(x_1-nL)}\times(A_+e^{\alpha_1(x_1-nL)}+A_-e^{-\alpha_1(x_1-nL)}) , \tag{2.9}$$

where

$$nL<x_1<nL+d_1 . \tag{2.10}$$

It is easily seen that $u_2(q,x_1)$ in Eq. (2.9) satisfies the periodicity condition of Eq. (2.7). Thus, using Eqs. (2.6) and (2.9) we find

$$u_2(x_1)=e^{iqnL}(A_+e^{\alpha_1(x_1-nL)}+A_-e^{-\alpha_1(x_1-nL)}) \tag{2.11}$$

when

$$nL<x_1<nL+d_1 . \tag{2.12}$$

Similarly, one may obtain an expression for the displacement in the primed regions

$$u_2(x_1)=e^{iqnL}(B_+e^{\alpha_2(x_1-nL-d_1)}+B_-e^{-\alpha_2(x_1-nL-d_1)}) , \tag{2.13}$$

where

$$nL+d_1<x_1<(n+1)L \tag{2.14}$$

and

$$\alpha_2=[k_3^2-(\omega^2/c_t'^2)]^{1/2} . \tag{2.15}$$

There are now four arbitrary constants A_+, A_-, B_+, and B_- which determine the displacement field. To find these constants and to determine the dispersion relation, we employ the boundary conditions at the interfaces between the different elastic materials. The boundary conditions are (1) the continuity of displacement and (2) the continuity of the normal components of the stress across the interface. Condition (2) is that T_{11},T_{12}, and T_{13} be continuous. Of these only T_{12} is nonzero for the wave considered here. T_{12} is given by $\rho c_t^2(\partial u_2/\partial x_1)$ in the unprimed media and by $\rho' c_t'^2(\partial u_2/\partial x_1)$ in the primed media.

Continuity of displacement at $x_1=nL+d_1$ gives

$$A_+e^{\alpha_1 d_1}+A_-e^{-\alpha_1 d_1}=B_++B_- . \tag{2.16}$$

Continuity of stress at $x_1=nL+d_1$ gives

$$\alpha_1\rho c_t^2(A_+e^{\alpha_1 d_1}-A_-d^{-\alpha_1 d_1})=\alpha_2\rho' c_t'^2(B_+-B_-) . \tag{2.17}$$

Continuity of displacement at $x_1=nL$ gives

$$A_++A_-=e^{-iqL}(B_+e^{\alpha_2 d_2}+B_-e^{-\alpha_2 d_2}) . \tag{2.18}$$

Continuity of stress at $x_1=nL$ gives

$$\alpha_1\rho c_t^2(A_+-A_-)=e^{-iqL}\alpha_2\rho' c_t'^2(B_+e^{\alpha_2 d_2}-B_-e^{-\alpha_2 d_2}) . \tag{2.19}$$

Equations (2.16)–(2.19) are four equations in the four unknowns A_+,A_-,B_+,B_-. The solvability condition yields an implicit dispersion relation for ω as a function of q and k_3. We find

$$\cos(qL)=\cosh(\alpha_1 d_1)\cosh(\alpha 2d_2)+\tfrac{1}{2}(F+1/F)\sinh(\alpha_1 d_1)\sinh(a_2 d_2) , \tag{2.20}$$

where

$$F=\frac{\alpha_1\rho c_t^2}{\alpha_2\rho' c_t'^2} . \tag{2.21}$$

In general the dispersion relation giving ω as a function of k_3 and q must be found numerically from Eq. (2.20). This relation has been obtained earlier in Ref. 17.

We may now consider the effects of introducing additional boundaries into the structure of Fig. 1.

A. Boundaries at $x_3=\pm d/2$

This geometry is shown in Fig. 2. We now must consider the boundary conditions at $x_3=\pm d/2$. The stress-free conditions at these two surfaces require that T_{31},T_{32}, and T_{33} be zero. For the wave considered here only T_{32} is not identically zero everywhere. Thus we must have

$$T_{32}=\rho c_t^2\frac{\partial u_2(x_1,x_3,t)}{\partial x_3}\bigg|_{x_3=\pm d/2}=0 , \tag{2.22}$$

if x_1 lies in the unprimed media, and

$$T_{32}=\rho' c_t'^2\frac{\partial u_2(x_1,x_3,t)}{\partial x_3}\bigg|_{x_3=\pm d/2}=0 , \tag{2.23}$$

if x_1 lies in the primed media. It is easy to see that one way to satisfy these boundary conditions is to let

$$u_2(x_1,x_2,t)=\cos(k_3x_3)e^{-i\omega t}u_2(x_1) , \tag{2.24}$$

where k_3 is quantized by

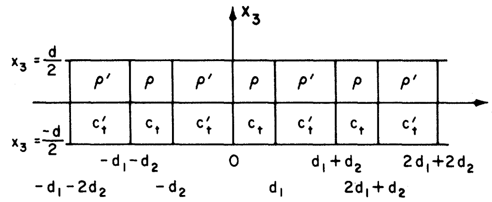

FIG. 2. Slab composed of a periodic layered structure. We find solutions with a standing wave character in the x_3 direction and propagating in the x_1 direction.

$$k_3=\frac{n\pi}{d}\ ,\quad n=0,2,4,6\ . \tag{2.25}$$

Similarly there is also a solution

$$u_2(x_{1,}x_3t)=\sin(k_3x_3)e^{-i\omega t}u_2(x_1)\ , \tag{2.26}$$

where k_3 is quantized by

$$k_3=\frac{n\pi}{d}\ ,\quad n=1,3,5.\ .\ .\ . \tag{2.27}$$

It is now easy to show that the dispersion relation for the modes in the bounded layered structure of Fig. 2 is still given by Eq. (2.20), but with α_1, α_2, and F depending now on the quantized values of k_3.

B. Semi-infinite layered geometry

We next consider the semi-infinite layered geometry illustrated in Fig. 3. In this case we no longer have perfect periodicity in the x_1 direction, the direction normal to the layering. As a result, Eqs. (2.11) and (2.13) are no longer valid.

Instead, we now look for surface wave solutions, i.e., solutions that are localized near the surface of the semi-infinite layered structure and which decay exponentially as one travels through the layers away from the surface. Thus, in the unprimed region $(nL<x_1<nL+d)$ we replace Eq. (2.11) for the displacement by

$$u_2(x_1)=e^{-\beta nL}(A_+e^{\alpha_1(x_1-nL)}+A_-e^{-\alpha_1(x_1-nL)})\ . \tag{2.28}$$

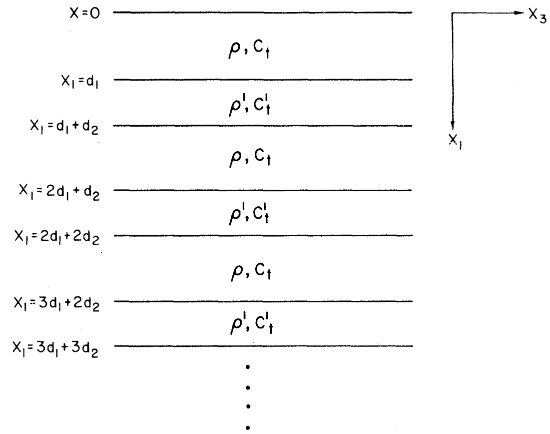

FIG. 3. Semi-infinite periodic layered structure. Propagation is in the x_3 direction, parallel to the surface.

In the primed region $[nL+d_1<x_1<(n+1)L]$ the displacement now has the form

$$u_2(x_1)=e^{-\beta nL}(B_+e^{\alpha_2(x_1-nL-d_1)}+B_-e^{-\alpha_2(x_1-nL-d_1)})\ . \tag{2.29}$$

The constant β governs the exponential decay of the displacement field as one penetrates into the stack. Exponential increase is excluded by requiring $\beta>0$.

It is readily seen that Eqs. (2.28) and (2.29) satisfy the differential equations of motion in the appropriate regions. We proceed by matching the expressions for the displacement in the various regions through the use of the boundary conditions.

To obtain the surface wave dispersion relation, it is sufficient to consider the boundary conditions along three interfaces: (1) $x_2=nL$, (2) $x_2=nL+d_1$, and (3) $x_2=0$. The application of the boundary conditions at $x_2=nL$ and $x_2=nL+d_1$ gives a set of four equations for A_+, A_-, B_+, and B_- which are identical to Eqs. (2.16)–(2.19) except that iq is replaced by $-\beta$ everywhere. If the coefficients B_+ and B_- are eliminated from this set of four equations, we obtain two equations for A_+ and A_-:

$$\begin{bmatrix}(1+F)(e^{\alpha_1d_1}-e^{-\beta L}e^{-\alpha_2d_2}) & (1-F)(e^{-\alpha_1d_1}-e^{-\beta L}e^{-\alpha_2d_2})\\ (1-F)(e^{\alpha_1d_1}-e^{-\beta L}e^{\alpha_2d_2}) & (1+F)(e^{-\alpha_1d_1}-e^{-\beta L}e^{-\alpha_2d_2})\end{bmatrix}\begin{bmatrix}A_+\\ A_-\end{bmatrix}=0\ . \tag{2.30}$$

The remaining boundary condition is that the surface $x_2=0$ be stress free. For the geometry considered here this condition is that $T_{12}=0$ at $x_2=0$. This leads to

$$\alpha_1A_+-\alpha_1A_-=0\ . \tag{2.31}$$

Equations (2.30) and (2.31) provide three equations for the three unknowns β, A_+, and A_-. Solving these equations we obtain an implicit dispersion relation

$$F\tanh(\alpha_1 d_1)+\tanh(\alpha_2 d_2)=0 \tag{2.32}$$

and the equation for the decay parameter β

$$e^{-\beta L}=\frac{\cosh(\alpha_1 d_1)}{\cosh(\alpha_2 d_2)} . \tag{2.33}$$

In general Eq. (2.32) must be solved numerically to obtain the dispersion curves. We will find that several different types of solutions are possible. For some solutions α_1 will be imaginary and α_2 will be real, and we can have a surface wave of the layered structure composed of surface waves in one film and bulk waves in the other film. Other solutions will have both α_1 and α_2 imaginary and we will then have a surface mode of the layered structure composed of bulk standing waves in each film.

In the limit that $\alpha_1 d_1$ and $\alpha_2 d_2$ are small, and to low order in $k_{||}$ one may obtain an analytic solution for the dispersion relation for the surface mode. We find that in this limit the bottom of the bulk band is given by the equation

$$\omega^2=c_b^2 k_{||}^2(1-\epsilon_b k_{||}^2) , \tag{2.34}$$

where

$$c_b^2=\frac{d_1 c_{44}+d_2 c'_{44}}{d_1\rho+d_2\rho'} \tag{2.35}$$

and

$$\epsilon_b=\frac{d_1^2 d_2^2 c_{44} c'_{44}}{12}\left[\frac{d_1 c'_{44}+d_2 c_{44}}{d_1 c_{44}+d_2 c'_{44}}\right]\times\frac{(1/c_t^2-1/c_t'^2)^2}{(d_1\rho+d_2\rho)^2} . \tag{2.36}$$

The frequency of the surface wave is given by

$$\omega_s^2=c_b^2 k_{||}^2(1-4\epsilon_b k_{||}^2) . \tag{2.37}$$

Comparing Eqs. (2.37) and (2.34) we see that this surface mode lies below the lowest bulk modes. From Eq. (2.33) we find that this mode exists only if $c_t<c_t'$.

C. A semi-infinite layered structure in contact with a semi-infinite homogeneous elastic medium

The geometry is illustrated in Fig. 4. The calculation is similar to that of the preceding section except that at $x_1=0$ one must match the displacement and stresses in the semi-infinite medium to those in the layered structure.

We present here only the results. We find that for

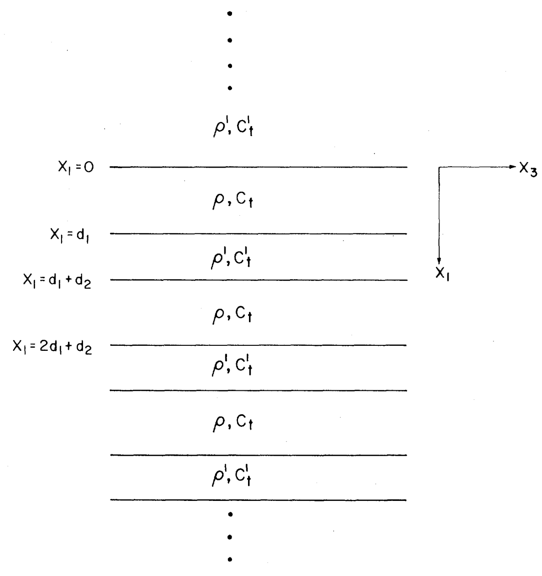

FIG. 4. Semi-infinite periodic layered structure in contact with an elastic half-space.

this structure there are localized elastic waves near the interface between the layered structure and the semi-infinte elastic medium. These waves correspond to bulk standing waves in the unprimed medium, and decaying solutions in the unprimed medium. A simple solution for the dispersion relation may be obtained:

$$\omega^2=c_t^2\left[k^2+\left[\frac{n\pi}{d_1}\right]^2\right] , \quad n=0,1,2,3,\ldots . \tag{2.38}$$

The parameter α_1 is given by

$$\alpha_1=in\pi/d_1 , \tag{2.39}$$

which corresponds to standing waves of differing orders. The decay parameter β is given by

$$\beta=\frac{\alpha_2 d_2}{L} \quad \text{for } n \text{ even} , \tag{2.40}$$

$$\beta=\frac{\alpha_2 d_2}{L}+\frac{i\pi}{L} \quad \text{for } n \text{ odd} . \tag{2.41}$$

In order that β have a real part, α_2 must be real. Since

$$\alpha_2=\left[k^2[1-(c_t^2/c_{t'}^2]-(c_t^2/c_{t'}^2)\left[\frac{n\pi}{d_1}\right]^2\right]^{1/2} , \tag{2.42}$$

this restricts the number n of standing modes allowed. We obtain

$$n < \frac{kd_1}{\pi}\left[\frac{c_t'^2}{c_t^2}-1\right]^{1/2} . \tag{2.43}$$

D. A semi-infinite layered structure in contact with an elastic film

The geometry under consideration is illustrated in Fig. 5. The calculation is similar to that of Sec. II B except that the free surface is now at $x_1 = d_s/2$, where d_s may range from $-d_1$ to $+d_1$. Thus the width of the last layer near the surface can now be different from that of the same kind of material in the bulk.

We find for this structure that the dispersion relation fo the surface waves is given by

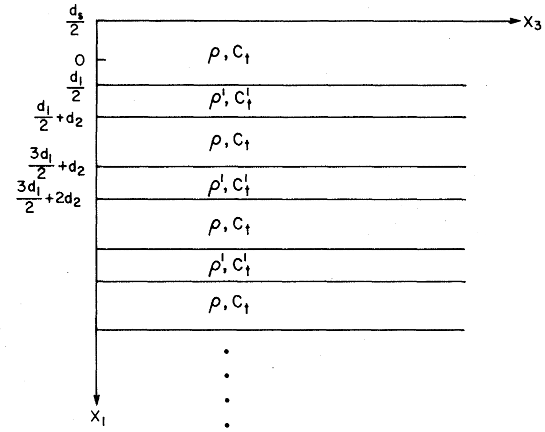

FIG. 5. Semi-infinite periodic layered structure in contact with an elastic film.

$$\sinh(\alpha_1 d_1)\cosh(\alpha_2 d_2)+\tfrac{1}{2}(F+F^{-1})\cosh(\alpha_1 d_1)\sinh(\alpha_2 d_2)-\tfrac{1}{2}(F-F^{-1})\cosh(\alpha_1 d_s)\sinh(\alpha_2 d_2)=0 , \tag{2.44}$$

together with the condition

$$\left|\frac{\cosh(\alpha_2 d_2)}{\cosh(\alpha_1 d_1)}+\frac{F-F^{-1}}{2}\frac{\sinh(a_2 d_2)}{\cosh(\alpha_1 d_1)}\sinh[\alpha_1(d_1+d_s)]\right| > 1 , \tag{2.45}$$

which ensures that one has exponentially decaying solutions inside the layered structure.

In the limit $k_3 \to 0$, one may obtain an explicit dispersion relation of the form

$$\omega_s^2 = c_b^2 k_3^2[1-(\epsilon_b+\epsilon_s)k_3^2] , \tag{2.46}$$

where

$$\epsilon_s = 3\epsilon_b d_s^2/d_1^2 . \tag{2.47}$$

The use of Eq. (45) then gives the existence condition for the surface wave with dispersion relation given by Eq. (2.46). If $c_t > c_t'$ the surface wave exists if $d_s > 0$ (thinner surface layer than in the bulk). If $c_t' > c_t$, one must have $d_s < 0$ (thicker surface layer than in the bulk).

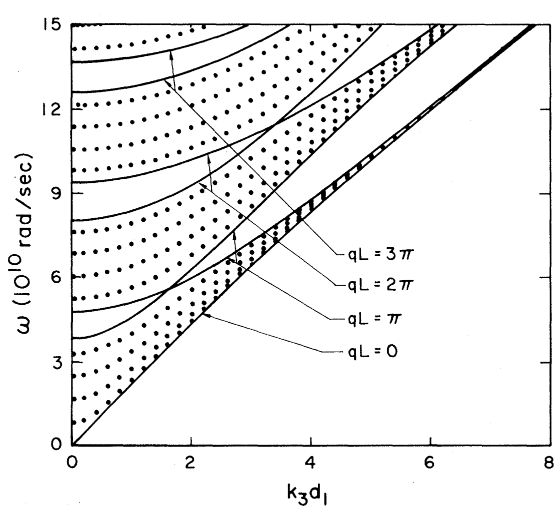

FIG. 6. Dispersion relation for transverse elastic waves propagating in the infinitely extended layered structure shown in Fig. 1. Calculations are performed with $d_1 = 1000$ Å of Nb and $d_2 = 500$ Å of Cu.

III. RESULTS

We now illustrate the theoretical results obtained in the preceding section by calculations for specific examples of the various geometries considered. Unless otherwise indicated the calculations here are performed using the following parameters, which are appropriate for a Nb-Cu layered structure:

	c_{44} (10^{11}dyn/cm^2)	ρ (g/cm^3)	c_t (10^5cm/sec)
Nb	2.87	8.57	1.83
Cu	7.53	8.92	2.905

The behavior of Rayleigh waves on a Nb-Cu superlattice has been studied recently both experimentally[18] and theoretically.[16]

We first consider the infinitely extended layered structure illustrated in Fig. 1. The implicit dispersion relation, Eq. (2.20) was solved numerically and

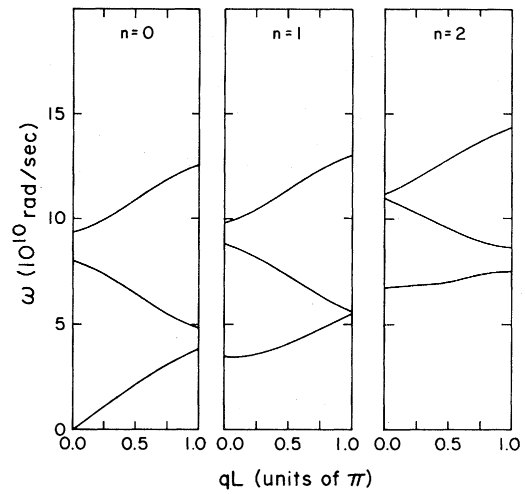

FIG. 7. Dispersion relations for the first three standing modes in the slab geometry of Fig. 2. Note the gaps at the zone edges. Calculations are performed with $d_1=1000$ Å of Nb and $d_2=500$ Å of Cu. The thickness of the slab is 2000 Å.

the results are shown in Fig. 6. Here we plot frequency versus $k_3 d_1$. We see in this figure that the dispersion curves essentially break up into different bands depending on the value of qL where q is the component of the wave vector perpendicular to the layering. The band edges occur at $qL=n\pi$, where $n=0,1,2,\ldots$. The boundaries of the bands are drawn with solid lines. The nearly horizontal dotted lines show the dispersion curves with qL/π increased incrementally by 0.2 as one moves up in frequency away from the band edges.

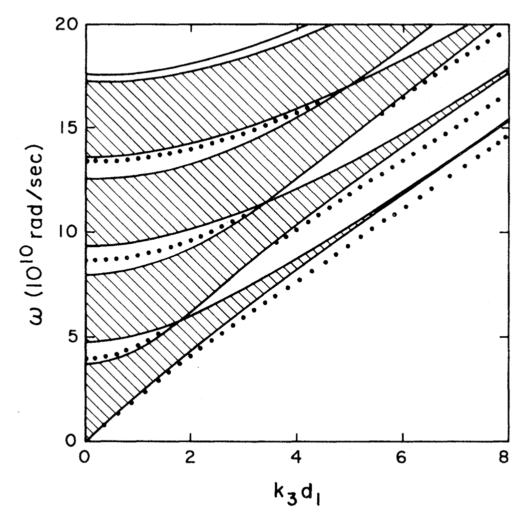

FIG. 8. Dispersion curves for surface modes in the semi-infinite layered structure shown in Fig. 3. Dotted lines show the surface modes. Shaded areas show regions where bulk modes propagate. The lowest-frequency surface mode merges with the bulk band at $k_3 d_1=0$. Calculations are performed with $d_1=1000$ Å of Nb and $d_2=500$ Å of Cu.

In Fig. 7 we plot the dispersion relation for the slab geometry illustrated in Fig. 2. As we saw in the preceding section, the wave vector perpendicular to the slab, k_3, is quantized by $k_3=n\pi/d$ where d is the thickness of the slab. Each value of n corresponds to a different standing mode. In Fig. 7 the dispersion relations for the first three standing modes are presented in the reduced zone scheme. We see that as the order of the mode increases, the frequency of the wave increases for a fixed qL. Also there are gaps at the zone edges ($qL=0,1$). These gaps correspond to frequency regions where propagation is not allowed. The position and width of these stop bands can be easily varied by changing the thickness of the slab.

We next turn to the results for surface waves on the semi-infinite layered structure. First, we consider the case $c_t < c_t'$. Equation (2.32) was solved numerically and the results are presented in Fig. 8. In this figure, the bulk bands are the shaded regions, and the surface modes are shown by dotted curves. The surface waves are found to exist below the lowest bulk band, and in the gaps between bulk bands. The surface waves in the gaps between the bulk bands were found earlier by Auld *et al.*[19] using a different method.

The surface mode lying below the lowest bulk band is similar to Love waves in several respects. First this surface mode of the layered structure has a sinusoidal variation through the thickness of the material with the lower transverse sound velocity,

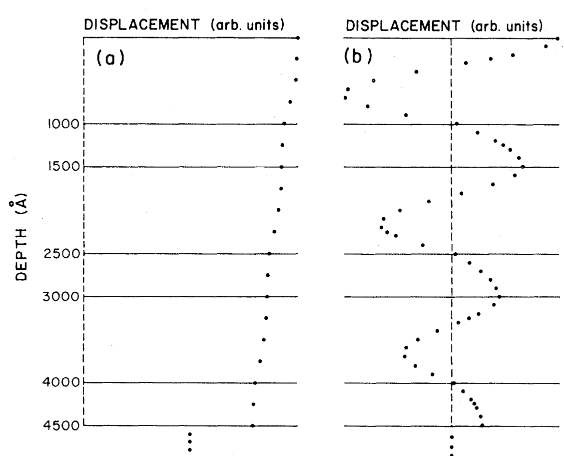

FIG. 9. Displacement patterns for surface modes. (a) shows the displacement of the surface wave at $k_3 d_1=0.5, \omega=1.12\times10^{10}$ rad/sec and (b) shows the displacement for the surface wave at $k_3 d_1=0.5$, $\omega=8.76\times10^{10}$ rad/sec.

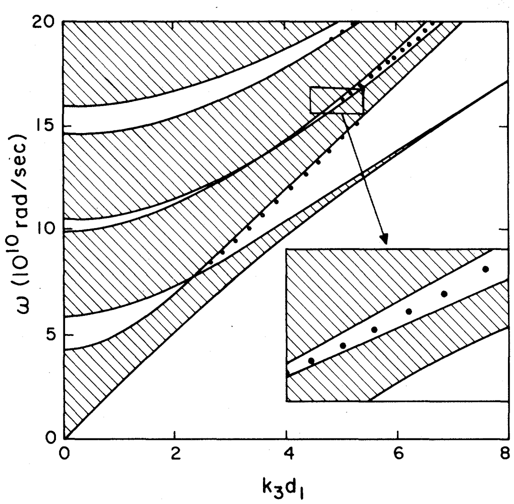

FIG. 10. Dispersion curves for the semi-infinite layered structure shown in Fig. 3 but with $c_t > c_t'$. Dotted lines show the surface modes. Shaded areas are the regions where bulk waves propagate. Calculations performed with $d_1 = 1000$ Å of Cu and $d_2 = 500$ Å of Nb.

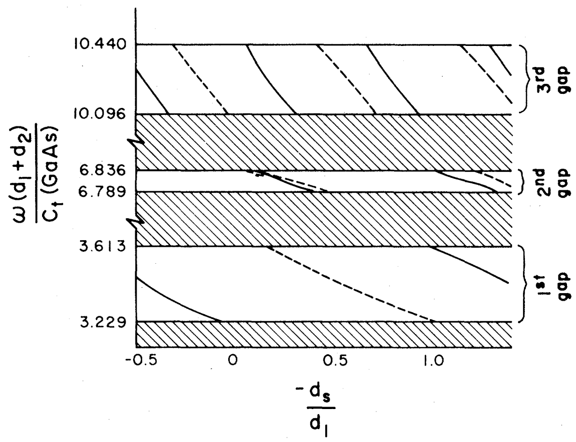

FIG. 11. Dispersion curves at $k_3 = 0$ for the structure shown in Fig. 5. Shaded regions show the frequency range for propagation of bulk waves. Solid lines show the surface waves for a GaAs/AlAs structure when GaAs is on the surface. Dashed lines show the surface modes of a GaAs/AlAs structure when AlAs is the surface layer.

and an exponential decay through the thickness of the material with the higher transverse sound velocity. Also this surface wave exists only when the outermost medium has the lower transverse sound velocity. In the large k_3 limit the velocity of this mode approaches the transverse sound velocity in the outermost (slower) medium.

In contrast, the surface modes that lie between the gaps of the bulk modes have sinusoidal variations through the thickness of both materials, and these surface modes may also exist even if the outermost medium has a higher transverse sound velocity. Sketches of the spatial variation of the displacement for the two types of surface waves are presented in Fig. 9.

In Fig. 10 we present dispersion curves for the case $c_t > c_t'$. Here there is no surface mode which lies below the lowest bulk band. There are, however, surface modes which lie in the gaps between bulk bands.

To this point, we have held the ratio $d_1/d_2 = 2$. We have also performed calculations for different values of d_1/d_2. The general features of the solutions—a set of bulk bands with surface modes lying in the gaps and below the bulk bands—do not change upon varying d_1/d_2. However, the position of the surface modes within the gaps can be strongly altered by changing d_1/d_2.

We do not give an example for the structure illustrated by Fig. 4 (a semi-infinite layered structure in contact with a semi-infinite elastic medium) since the equations are quite simple in this case.

We illustrate the solutions for the structure given in Fig. 5 (a semi-infinite layered structure in contact with an elastic film) below. We give in this section curves calculated for a layered structure of GaAs and AlAs. We assume that both materials have the same value of C_{44}, taken to be that of GaAs, and differ only by their densities. We take $\rho(\mathrm{AlAs})/\rho(\mathrm{GaAs}) = 0.7$. Figure 11 displays the different types of surface modes one obtains (in the limit $k = 0$) at the surface of a layered structure as a function of $-d_s/d_1$. Note that when $-d_s/d_1 = 1$, the free surface is at $x_1 = -d_1/2$, and when $-d_s/d_1 = -0.5$, the surface is at $x_1 = d_1/2$. Surface modes were found in the first three gaps for two cases: (1) when the surface layer is made of GaAs (solid lines) and (2) when the surface layer is made out of AlAs (dashed lines). Bulk waves lie within the shaded regions.

A more detailed study of these surface waves in GaAs and $\mathrm{Ga}_x\mathrm{Al}_{1-x}\mathrm{As}$ layered structures will appear elsewhere.[20]

IV. GREEN'S FUNCTIONS FOR THE SEMI-INFINITE LAYERED STRUCTURE

For many problems of physical interest, it is useful to have the Green's function for the system of interest, viz., the response function to an external perturbation. The Green's function may be used in a calculation of light scattering from thermally excited elastic waves,[21] of the mean square displacement,[22] and of nonlinear mixing of waves,[23] to name just a few examples. In addition, one may easily find the spectral density from the Green's function.

The spectral density function $S(k_3,\omega,x_1)$ gives a measure of the square of the amplitude of the displacement in a thermal fluctuation with wave vector k_3 and frequency ω at a position x_1. We will calculate the spectral density for our system, and show the relationship of the spectral density to the dispersion curves found earlier.

To find the response functions we start with the equation for the displacement $u_2(x_1,x_3,\omega)$ in the presence of an external force $F(x_1,x_3,\omega)$. We assume both the displacement and the external force have an $e^{-i\omega t}$ time variation. The equation of motion is then

$$\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{dx_3^2}+\frac{\omega^2}{c_t^2(x_1)}\right]u_2(x_1,x_3,\omega)=-\frac{F(x_1,x_3,\omega)}{c_t^2(x_1)} . \quad (4.1)$$

We have written $c_t^2(x_1)$ to indicate that the transverse sound velocity depends on the coordinate x_1 due to the layering. The solution of Eq. (4.1) may be written

$$u_2(x_1,x_3,\omega)=\int dx_1'\int dx_3' g(x_1,x_1',x_3,x_3';\omega)\times F(x_1',x_3',\omega) , \quad (4.2)$$

where the Green's function $g(x_1,x_1',x_3,x_3',\omega)$ satisfies

$$\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_3^2}+\frac{\omega^2}{c_t^2(x_1)}\right]g(x_1x_1'x_3x_3';\omega)=-\frac{\delta(x_1-x_1')\delta(x_3-x_3')}{c_t^2(x_1')} . \quad (4.3)$$

Since we have translational invariance parallel to the surface, we may perform a Fourier transform over x_3. Thus

$$g(x_1,x_1',x_3,x_3';\omega)=\int_{-\infty}^{\infty}\frac{dk_3}{2\pi}g(x_1x_1'k_3,\omega)e^{ik_3(x_3-x_3')} \quad (4.4)$$

and

$$\delta(x_3-x_3')=\int_{-\infty}^{\infty}\frac{dk_3}{2\pi}e^{ik_3(x_3-x_3')} . \quad (4.5)$$

The Green's function $g(x_1x_1',k_3,\omega)$ satisfies

$$\left[\frac{\partial^2}{\partial x_1^2}-k_3^2+\frac{\omega^2}{c_t^2(x_1)}\right]g(x_1,x_1',k_3,\omega)=-\frac{\delta(x_1-x_1')}{c_t^2(x_1)} . \quad (4.6)$$

We will now drop reference to k_3 and ω for ease of notation.

If $u^>(x_1)$ is the solution to the homogeneous version of Eq. (4.6) in the region $x_1>x_1'$ and $u^<(x_1)$ is the solution to the homogeneous version of Eq. (4.6) in the region $x_1<x_1'$, then the Green's function $g(x_1,x_1')$ is given by

$$g(x_1,x_1')=\frac{1}{W}[u^>(x_1)u^<(x_1')\theta(x_1-x_1')+u^<(x_1)u^>(x_1')\theta(x_1'-x_1)] , \quad (4.7)$$

where the Wronskian is

$$W=-c_t^2(x_1)\left[\frac{\partial}{\partial x_1}[u^>(x_1)]u^<(x_1)-\frac{\partial}{\partial x_1}[u^<(x_1)]u^>(x_1)\right] . \quad (4.8)$$

We will find later that the Wronskian is independent of the variable x_1.

To determine the Green's function completely, we need expressions for $u^>(x_1)$ and $u^<(x_1)$ which satisfy all the boundary conditions for the semi-infinite layered structure considered here. We have essentially found the appropriate expressions already in Sec. II. For $x_1<x_1'$, we want a wave propagating away from the delta function source at x_1' toward $+\infty$. For $x_1<x_1'$ we will have a wave propagating away from the delta function source toward the surface, and a wave reflected from the surface back toward the source. Thus we take

$$u^>(x_1)=e^{iqnL}(A_+^Te^{\alpha_1(x_1-nL)}+A_-^Te^{-\alpha_1(x_1-nL)}) \quad (4.9)$$

and

$$u^<(x_1)=e^{-iqnL}(A_+^Ie^{\alpha_1(x_1-nL)}+A_-^Ie^{-\alpha_1(x_1-nL)})+e^{iqnL}(A_+^Re^{\alpha_1(x_1-nL)}+A_-^Re^{-\alpha_1(x_1-nL)}) . \quad (4.10)$$

For a given k_3 and ω, q and α_1 may be obtained from Eqs. (2.20) and (2.5), respectively. In the region where surface waves propagate, q is imaginary so that for $x_1>x_1'$ we have exponential decay as $x_1\to+\infty$, and for $x_1<x_1'$ both exponential growth and exponential decay are allowed.

The equations for $u^>(x_1)$ and $u^<(x_1)$ given above are appropriate for the unprimed regions. Similar equations will hold in the primed regions as shown in Sec. II.

$A_+^{I,R,T}$ is related to $A_-^{I,R,T}$ by the boundary conditions in the bulk of the layered structure. From Eq. (30) (with β replaced by $-iq$) we find

$$A_+^{R,T}=C^R A_-^{R,T}\ , \tag{4.11}$$

where

$$C^R=\frac{(1+F)(e^{-\alpha_1 d_1}-e^{+iqL}e^{\alpha_2 d_2})}{(F-1)(e^{\alpha_1 d_1}-e^{iqL}e^{\alpha_2 d_2})}\ . \tag{4.12}$$

Similarly

$$A_+^I=C^I A_-^I\ , \tag{4.13}$$

where

$$C^I=\frac{(1+F)(e^{-\alpha_1 d_1}-e^{-iqL}e^{\alpha_2 d_2})}{(F-1)(e^{\alpha_1 d_1}-e^{-iqL}e^{\alpha_2 d_2})}\ . \tag{4.14}$$

Now $u^>(x_1)$ and $u^<(x_1)$ may be written in terms of A_-^I, A_-^R, and A_-^T only. To determine the Green's function completely, we need only to relate A_-^I to A_-^R. This is done by the boundary condition at $x_1=0$. In the outermost layer

$$\frac{\partial u^<(x_1)}{\partial x_1}=0 \text{ at } x_1=0,\quad n=0\ . \tag{4.15}$$

Using Eqs. (4.9), (4.11), (4.14), and (4.15) we obtain

$$A_-^R=DA_-^I\ , \tag{4.16}$$

where

$$D=\left[\frac{C^I-1}{1-C^R}\right]\ . \tag{4.17}$$

Thus $u^>(x_1)$ and $u^<(x_1)$ have the final forms

$$u^>(x_1)=e^{iqnL}(C^R e^{\alpha_1(x_1-nL)}+e^{-\alpha_1(x_1-nL)})A_-^T\ , \tag{4.18}$$

$$u^<(x_1)=e^{-iqnL}(C^I e^{\alpha_1(x_1-nL)}+e^{-\alpha_1(x_1-nL)})A_-^I + e^{iqnL}(C^R e^{\alpha_1(x_1-nL)} + e^{-\alpha_1(x_1-nL)})DA_-^I\ , \tag{4.19}$$

where

$$nL<x_1<nL+d_1\ . \tag{4.20}$$

For the primed regions $[nL+d_1<x_1<(n+1)L]$ one may find $u^>(x_1)$ and $u^<(x_1)$ by using the interior boundary conditions and Eqs. (2.13).

It is easily shown that the Wronskian may now be written as

$$W=2\alpha_1(C^R-C^I)A_-^T A_-^I c_t^2\ . \tag{4.21}$$

This form of the Wronskian is valid in both the unprimed and primed regions. It is also seen that the Wronskian is independent of position.

The general form for the Green's function may now be easily found by using Eq. (4.7) and Eqs. (4.18)–(4.21). The result is lengthy, and we do not reproduce it here. We do give the Green's function below for the special case that x_1 and x_1' are both in the outermost layer:

$$g(x_1,x_1';n=0,n'=0)=\frac{1}{2\alpha_1(C^R-C^I)}[(C^RC^I+DC^RC^R)e^{\alpha_1(x_1+x_1')}+(1+D)e^{-\alpha_1(x_1+x_1')} + C^R(1+D)e^{\alpha_1|x_1-x_1'|}+(C^I+C^RD)e^{-\alpha_1|x_1-x_1'|}]\ . \tag{4.22}$$

We demonstrate the utility of the Green's function calculated above by calculating the spectral density of thermal fluctuations in the outermost film. The spectral density contains more information than the dispersion curves given earlier, since it is a measure of the probability of a given mode to be thermally excited. It is easy to show that the spectral density $S(k_3,\omega,x_1)$ is proportional in this case to the imaginary part of the Green's function calculated above,[24] if one adds a small positive imaginary part to the frequency, and sets $x=x'$.

We examine the spectral density in the following way. We fix $x_1=x_1'=0$, so that we are looking at thermal fluctuations right at the surface of the semi-infinite layered structure. We then fix $k_3d_1=0.5$ and plot the spectral density [the imaginary part of $g(k_3,\omega+i\eta,x_1=x_1'=0)$] as a function of the frequency ω. The results are presented in Fig. 12 along with a portion of the dispersion curve for the same structure for comparison. It is clear in Fig. 12 that there are sharp peaks at the frequency of the surface waves with wave vector $k_3d_1=0.5$. The regions in the dispersion curves where bulk waves propagate are seen as broad bands in the spectral density plot. Thus in the outermost layer, it is the surface waves of the layered structure which are most likely to be thermally excited.

We may also examine the spatial variation of the spectral density by changing the value of x_1. The results are presented in Fig. 13. We see that the

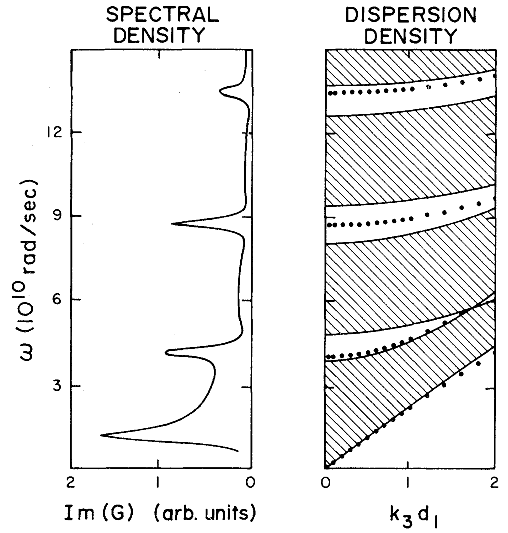

FIG. 12. The right-hand side shows a portion of the dispersion curves from Fig. 8. The left-hand side is a plot of the spectral density $[\mathrm{Im}(gk_3,\omega+i\eta,x_1-x_1'=0)]$ as a function of frequency. The peaks in the spectral density occur at frequencies where the surface modes exist with wave vector $k_3d_1=0.5$.

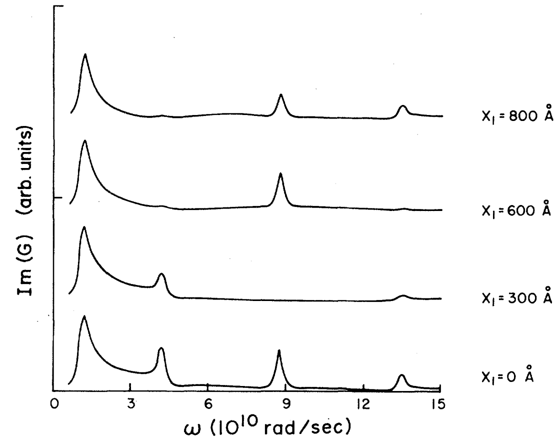

FIG. 13. Plot of the spectral density as a function of frequency for different depths.

peak at lowest frequency remains nearly constant in height as a function of the depth x_1. The peak near $\omega=9\times10^{10}$ rad/sec has a strong variation with depth. The behavior of these two peaks is consistent with the displacement patterns for the two surface modes displayed in Fig. 9.

V. CONCLUSIONS

We have derived the phonon dispersion equations for transverse waves in several different layered structures. The results show a variety of interesting features. There are bands in which the bulk elastic waves may propagate. In the case of layered structures with boundaries, surface waves propagate in the gaps between the bulk bands, and below the lowest bulk band. In addition, we have calculated the Green's function for a semi-infinite layered system and from this found the spectral density of thermal fluctuations in the outermost layer.

The results presented here were calculated within the continuum theory of elasticity. A microscopic approach[25] yields solutions which (in the long-wavelength limit) are identical to those developed here. We also note that the calculations presented here are limited to transverse waves where the displacement is parallel to the plane of the layering. An extension of the methods used here to Rayleigh waves on layered structures is in progress.[26]

ACKNOWLEDGMENT

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*Present address: Department of Physics, University of Colorado, Colorado Springs, Colorado 80917.

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