First-order phase transitions in the Potts model with trilinear symmetry breaking

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It is shown that a trilinear symmetry breaking which destroys the equivalence of the states in a continuum generalization of the p-state Potts model yields first-order phase transitions for all $p>1$, in contrast to the results of the symmetric theory where there is a second-order transition for $p<2$ and a first-order transition for $p>2$, in $d=6-\epsilon$ dimensions, and in spite of mean-field—theory predictions. A calculation in renormalized perturbation theory, to one-loop order, on a three-trilinear-coupling theory yields two new accessible and partly stable asymmetric fixed points beyond the symmetric one, except for the three-state Potts model, one for $2.2^- \leq p < \infty$ and the other for $1<p<\frac{13}{3}$. There is a fixed-point runaway for the first one when $p<2.2^-$ and for the second when $p>\frac{13}{3}$, which are interpreted as the usual first-order transitions. For the indicated ranges of p the transitions are of first order near a spinodal point, with uniaxial ordering in the first case and transverse ordering in the second. Critical exponents that could describe the approach to the spinodal points are explicitly calculated.

I. INTRODUCTION

The study of the nature of the phase transition in the p-state Potts model[1] is of great current interest. The critical value $p_c(d)$, as a function of dimensionality d at which the transition changes from second order when $p<p_c(d)$ to first order, when $p>p_c(d)$, has been determined in recent momentum-space renormalization-group (RG), works by Pytte and Aharony in both $d=6-\epsilon$ and in $d=4-\epsilon'$ dimensions.[2–4] It has been suggested by Pytte[3] that despite the presence of a stable and accessible fixed point for $2<p<\frac{10}{3}$,[5,6] there is a first-order transition near a spinodal point in $d=6-\epsilon$ dimensions. For $p>\frac{10}{3}$ it is assumed that the fixed-point runaway corresponds to the usual first-order transition, whereas it seems definitely established that for $p<2$ there is a second-order transition.[3,5]

Critical exponents have also been calculated and although it is clear that for $p<2$ these are the exponents obtained earlier for the second-order transition, it is not certain that the exponents are at all adequate to describe the spinodal point when $2<p<\frac{10}{3}$ (Ref. 3) because of the role that instantons or critical droplets play in a first-order transition.[7–9]

The works of Pytte and Aharony, either in $d=6-\epsilon$ or in $d=4-\epsilon'$ dimensions, are restricted to the neighborhood of $p\simeq 2$ as far as the ordered state which distinguishes between a first- and a second-order transition is concerned. Indeed, $(p-2)$ must be taken as a small expansion parameter in the Landau-Ginzburg free-energy functional if this is to be used to describe a first-order phase transition by means of an expansion in a few low powers in the continuum fields. Physical implications for the interesting three-state Potts model are then obtained by extrapolation of $(p-2)$ to 1.

The nature of the phase transition in the three-state Potts model has been a matter of controversy for some time.[10] Assuming the transition to be of first order, it has been shown by Blankschtein and Aharony that this can be changed into a second-order transition, either at a critical or a tricritical point, by means of linear and quadratic symmetry-breaking perturbations in $d=4-\epsilon'$ dimensions, which can easily be introduced experimentally in the numerous physical realizations of the three-state Potts model,[11(a)] and in the fewer ones of the four-state model.[11(b)]

Theoretically, quadratic symmetry-breaking perturbations in the continuum Potts model can be shown to follow from an anisotropic coupling between the components of the Potts vectors in the discrete model[12] and this should reflect, for example, the effect of anisotropic stresses in the magnetic transition of certain cubic ferromagnets.[11] An anisotropic coupling in the discrete model also generates trilinear symmetry-breaking perturbations in the continuum Potts model.

The work of Ref. 11(a) is carried out assuming that the underlying symmetry is that of the rotationally invariant (two-component) XY model and that the trilinear coupling $w \ll (\epsilon')^{1/2}$. As a consequence, although w changes under iterations of the renormalization group, it need not be fully renormalized, and the critical exponents determined by the fixed-point value of the quartic coupling are those of the XY model.

The continuum p-state $(p=n+1)$ Potts model with n-component "Potts vectors" belongs, in general, to a different universality class than the continuum n-vector model. It is only for the two-state Potts model—the Ising model—that there is an exact cancelation of the tensorial coefficients in the trilinear terms. These coefficients reflect the discrete symmetry of the permutation group S_{n+1} under which the Potts Hamiltonian is invariant.[13] For all other $p \neq 2$ these tensorial coefficients play an important role, except in the "restricted" continuum Potts model of Zia and Wallace,[14] where the underlying symmetry is in the tensorial coefficients of the quartic terms.

It is well known that unless the trilinear terms are artifically suppressed or taken to be vanishingly small, as in Refs. 11(a) and 14, the continuum Potts ϕ^3 field theory has to be studied in $d=6-\epsilon$ dimensions.[15,16] We argue that it is only then that one can work out reliably the consequences of the underlying discrete symmetry. The quartic terms that are usually needed to stabilize the ϕ^3 theory may, eventually, have to be renormalized as well as suggested in recent work by Fucito and Parisi.[17] Unfortunately, this seems to demand renormalization at an intermediate dimension which can only be done approximately.

The importance of trilinear symmetry breaking for the continuum one-state Potts model has been discussed recently,[18,19] but the general-state model has not been considered so far. There is the possibility that new fixed points in a RG approach may lead to new critical behavior and eventually change the nature of the phase transition. The purpose of this paper is to explore this possibility, which cannot be ruled out *a priori*. Indeed, as will be discussed below, mean-field theory with trilinear symmetry breaking predicts a quite different behavior from the symmetric theory. In the limit of very weak anisotropies that may always be present experimentally, this is relevant to the phase transition in the physical realizations of the three- and four-state Potts model.

To study the effects of trilinear symmetry breaking we resort to a RG calculation in renormalized perturbation theory with dimensional regularization and minimal subtraction,[20] to one-loop order, in $d=6-\epsilon$ dimensions.

In Sec. II we explain how the effective Hamiltonian is obtained for trilinear symmetry breaking into m "longitudinal" and $(n-m)$ "transverse" field components, with $n=p-1$. This yields a three-trilinear-coupling theory and the further work is restricted to a single longitudinal component. In Sec. III we follow Pytte[3] along lines initiated by Priest and Lubensky,[5] and analyze the stability of the fluctuations around mean-field theory, assuming a uniaxial ordering. In addition to the dependence on $(p-2)$ that one has in the symmetric theory, the sign of the ratio v/u, in which v is the trilinear coupling between one longitudinal and two transverse field components and u between pure longitudinal components, is shown to be crucial to the stability analysis. The coupling between pure transverse components does not appear in the fluctuations about mean-field theory.

One of the main conclusions of mean-field theory is that if $v/u<0$ and $|v/u|$ is not too small, trilinear symmetry breaking is relevant to change the nature of the phase transition, leading to a continuous second-order transition for $p>2$. However, there is nothing intrinsic in mean-field theory that enables one to decide on the sign of v/u, and for that purpose, we argue that one has to resort to a RG study of the fluctuations. These turn out to exclude, in a nontrivial way, a negative v/u.

In Sec. IV it is shown that there are three nontrivial fixed points: the symmetric one and two new asymmetric fixed points, one for $2.2^- < p < \infty$ and the other for $1 < p < \frac{13}{3}$. In the first one, which has two nonzero fixed-point couplings, such that $v^*/u^* > 0$, there is a fixed-point runaway for $p < 2.2^-$ which is one of the intriguing results of this work, and we interpret this to correspond to the usual first-order transition. Note that this is in contrast to the symmetric theory, described by our first fixed point, in which there is a continuous second-order transition with a stable fixed point for $p<2$, as in Ref. 3. The fixed-point ratio v^*/u^* favors a first-order transition near a spinodal point for $p \gtrsim 2.2^-$. Our second asymmetric fixed point also turns out to describe a first-order transition near a spinodal point for $p < \frac{13}{3}$, whereas there is a fixed-point runaway for $p \geq \frac{13}{3}$ that we interpret as the usual first-order transition. Whenever this third fixed point is stable we argue that this corresponds to ordering in the transverse components. The critical exponents associated with the fixed points are also calculated in Sec. IV. Some implications of this work and possible extensions are discussed in Sec. V.

II. MODEL

We follow recent authors[3,4] taking the effective Hamiltonian of Priest and Lubensky[5] for the p-state Potts model,

$$\begin{aligned}\mathscr{H}=&-\tfrac{1}{4}\int(r+k^2)\sum Q_{ii}(k)Q_{ii}(-k)+w\int\sum Q_{ii}(k)Q_{ii}(k')Q_{ii}(-k-k')\\&-u_4\int\sum Q_{ii}(k)Q_{ii}(k')Q_{jj}(k'')Q_{jj}(-k-k'-k'')\\&-v_4\int\sum Q_{ii}(k)Q_{ii}(k')Q_{ii}(k'')Q_{ii}(-k-k'-k''),\end{aligned}\tag{2.1}$$

where Q_{ii} are the diagonal elements of a $p\times p$ dimensional traceless tensor. With the model renormalized by dimensional regularization, the integrations are extended over all momenta k in dimension $d=6-\epsilon$, and the summations are over the p Potts states.

The bare propagator takes the form

$$\langle Q_{ii}(k)Q_{jj}(-k)\rangle=\left[\delta_{ij}-\frac{1}{p}\right]\frac{2}{r+k^2},\tag{2.2}$$

with the tensorial coefficient coming from a particular representation of the Q_{ii}, in terms of the components A_α ($\alpha=1,\ldots,p-1$) of the p-state Potts model,

$$H=-J\sum_{\langle xx'\rangle}\vec{A}(x)\cdot\vec{A}(x'),\tag{2.3}$$

with $p(p-1)$-dimensional vectors $\vec{A}(x)$. This representation is

$$Q_{ii}=\sum_{\alpha=1}^{p-1}A_\alpha a_{ii}^\alpha,\quad i=1,\ldots,p\tag{2.4}$$

where

$$a_{ii}^\alpha=\left[\frac{p-\alpha}{p-\alpha+1}\right]^{1/2}\times\begin{cases}0 & \text{if } i<\alpha\\ 1 & \text{if } i=\alpha\\ -1/(p-\alpha) & \text{if } i>\alpha\end{cases}\tag{2.5}$$

with the normalization such that

$$\sum_i Q_{ii}(k)Q_{ii}(-k)=\sum_\alpha A_\alpha(k)A_\alpha(-k).\tag{2.6}$$

The $A_\alpha(k)$ then become the components of the continuous field $A(k)$.

The trilinear and the cubic terms become

$$\sum_i Q_{ii}Q_{ii}Q_{ii}=\sum_{\alpha,\beta,\gamma}D_{\alpha\beta\gamma}A_\alpha A_\beta A_\gamma,\tag{2.7}$$

$$\sum_i Q_{ii}Q_{ii}Q_{ii}Q_{ii}=\sum_{\alpha,\beta,\gamma,\delta}E_{\alpha\beta\gamma\delta}A_\alpha A_\beta A_\gamma A_\delta,\tag{2.8}$$

with the tensorial coefficients

$$D_{\alpha\beta\gamma}\equiv\sum_i a_{ii}^\alpha a_{ii}^\beta a_{ii}^\gamma=\frac{1}{[(p-\alpha)(p-\alpha+1)]^{1/2}}\times\begin{cases}1 & \text{if } \beta=\gamma>\alpha\\ (p-\alpha-1) & \text{if } \beta=\gamma=\alpha\\ 0, & \text{otherwise}\end{cases}\tag{2.9}$$

and

$$E_{\alpha\beta\gamma\delta}\equiv\sum_i a_{ii}^\alpha a_{ii}^\beta a_{ii}^\gamma a_{ii}^\delta,\tag{2.10}$$

which is nonzero only for certain symmetrical components, in particular,

$$E_{\alpha\alpha\alpha\alpha}=\frac{(p-\alpha)^3+1}{(p-\alpha+1)^2(p-\alpha)}.\tag{2.11}$$

It also turns out that combinations of $D_{\alpha\beta\gamma}$ and $E_{\alpha\beta\gamma\delta}$ yield specific sum rules that can serve as a check on the tensorial coefficients. In what follows we only make use of

$$\sum_{\beta,\gamma}D_{\alpha\beta\gamma}D_{\beta\gamma\epsilon}=\left[1-\frac{2}{p}\right]\delta_{\alpha\epsilon},\tag{2.12}$$

$$\sum_{\mu,\nu,\eta}D_{\alpha\mu\eta}D_{\beta\mu\nu}D_{\gamma\eta\nu}=\left[1-\frac{3}{p}\right]D_{\alpha\beta\gamma},\tag{2.13}$$

which appear in the one-loop diagrams for the two- and three-point irreducible vertex functions, respectively.

With a trilinear symmetry breaking into m longitudinal and $(n-m)$ transverse components A_μ and A_q ($n\equiv p-1$), we have

$$\sum_\alpha A_\alpha^2=\sum_{\mu=1}^m A_\mu^2+\sum_{q>m}^n A_q^2,\tag{2.14}$$

and the effective Hamiltonian is

$$\mathscr{H}=-\tfrac{1}{4}\int (r+k^2)\sum_\alpha A_\alpha^2+\int\left[u\sum_{\mu,\nu,\eta} D_{\mu\nu\eta}A_\mu A_\nu A_\eta+3v\sum_{\mu,q,r} D_{\mu q r}A_\mu A_q A_r+w\sum_{q,r,s} D_{qrs}A_q A_r A_s\right]$$
$$-\int\left[u_4\sum_{\alpha,\beta} A_\alpha^2 A_\beta^2+v_4\sum_{\alpha,\beta,\gamma,\delta} E_{\alpha\beta\gamma\delta}A_\alpha A_\beta A_\gamma A_\delta\right], \quad (2.15)$$

where $(\mu,\nu,\eta)\leq m$ indicates longitudinal and $(q,r,s)>m$ indicates transverse components, while the other indices run over all components. Note there is no trilinear coupling between two longitudinal and one transverse component because $D_{\mu\nu q}\equiv 0$ from Eq. (2.9). We neglect symmetry breaking in the quartic part of the Hamiltonian which is irrelevant to our discussion. Also, an external-field term that would be needed for a detailed study of the equation of state has been omitted. Below the critical temperature one may assume a spontaneous symmetry breaking that favors ordering along a longitudinal component, and this will be done in the following section. To single out the longitudinal component as the critical components in the disordered phase, one may add a quadratic symmetric-breaking term[21]

$$\mathscr{H}_g=-\tfrac{1}{4}g\int\left[(n-m)\sum A_\mu^2-m\sum A_q^2\right], \quad (2.16)$$

with $g<0$, and let g be vanishingly small to ensure a dominant trilinear symmetry breaking.

For simplicity, it will be assumed that $m=1$ since this is sufficient for the uniaxial ordering that we have in mind. A meaningful trilinear symmetry breaking requires that $n>1$, i.e., $p>2$ but our results can presumably be continued to $p<2$.

III. ORDERED PHASE IN MEAN-FIELD THEORY

We assume, following Priest and Lubensky,[5] that uniaxial ordering takes place with

$$A_\alpha(k)=Q+\mathscr{L}_\alpha(k),\ \alpha=1$$
$$=\mathscr{L}_\alpha(k),\ \alpha>1 \quad (3.1)$$

where the order parameter Q is the thermal average $\langle A_\alpha\rangle$, for $\alpha=1$, and $\mathscr{L}_\alpha(k)$ is the fluctuation part. Equation (2.15) yields then the mean-field Hamiltonian

$$\mathscr{H}_{\rm MF}=-\tfrac{1}{4}rQ^2+(p-2)cuQ^3-(u_4+bv_4)Q^4, \quad (3.2)$$

where

$$c=[p(p-1)]^{-1/2},$$
$$b=c^2(p^2-3p+3) \quad (3.3)$$

in the notation of Ref. 3.

Following Pytte[3] we find that the coefficients of the quadratic parts in the longitudinal and transverse fluctuations at zero momentum in Eq. (2.15) that yield the corresponding inverse susceptibilities are

$$r_L=r-12(p-2)cuQ+24(u_4+bv_4)Q^2,$$
$$r_T=r+12cvQ+8(u_4+3c^2v_4)Q^2, \quad (3.4)$$

and stability of the mean-field solutions requires that both r_L and r_T be positive. This decides when a first-order transition is favored against a continuous second-order transition. Note that $\mathscr{H}_{\rm MF}$ and r_L are independent of the trilinear terms in v and w and that w, which couples transverse fluctuations, does not appear so far. In contrast to the symmetric theory,[3] Eqs. (3.4) involve different couplings in the term linear in Q.

The extrema of the mean-field free energy, determined by $\partial\mathscr{H}_{\rm MF}/\partial Q=0$, yield a small-$Q$ and a large-Q solution. The first one, which develops continuously from $Q=0$ with $r\lesssim 0$, yields

$$r_L=|r|,$$
$$r_T=\frac{1}{p-2}\left[2\left(1-\frac{v}{u}\right)-p\right]|r|. \quad (3.5)$$

Before discussing this solution, note first that the sign of u or v is irrelevant in Eqs. (3.2) and (3.4). These equations are invariant under the simultaneous change $u\to -u$, $Q\to -Q$, *and* $v\to -v$. They change, however, when the sign of v/u is changed.

For $p>2$, Eq. (3.5) yeilds

$$r_T>0\ \text{if}\ \frac{v}{u}<-\tfrac{1}{2}(p-2),$$
$$r_T<0\ \text{if}\ \frac{v}{u}>-\tfrac{1}{2}(p-2), \quad (3.6a)$$

and, for the continuation to $p<2$,

$$r_T>0\ \text{if}\ \frac{v}{u}>\tfrac{1}{2}(2-p),$$
$$r_T<0\ \text{if}\ \frac{v}{u}<\tfrac{1}{2}(2-p). \quad (3.6b)$$

Given a $v/u>0$, r_T is always negative for $p>2$ and, if v/u is not too small, r_T will be positive when continued to $p<2$. This results in the same unstable (stable) local minimum for $p>2$ $(p<2)$ that was found before when $v = u$.[3]

Consider next $v/u < 0$. Then r_T is always negative for $p < 2$ but, if $|v/u|$ is large enough ($|v/u| > 1$ is sufficient for the four-state Potts model), r_T will be positive for $p > 2$, and the small-Q local minimum is stable allowing a continuous second-order transition for $p > 2$. This is a new result not anticipated by the symmetric theory.

The large-Q solution to the mean-field free energy yields

$$r_L = \frac{(p-2)^2 c^2 u^2}{u_4 + b v_4} ,$$
$$r_T = \frac{6(p-2)c^2 uv}{u_4 + b v_4} + \frac{(p-2)^2 c^2 u^2}{(u_4 + b v_4)^2}[3u_4 + (b + 6c^2) v_4] , \tag{3.7}$$

together with

$$r_c = \frac{(p-2)^2 c^2 u^2}{u_4 + b v_4} , \quad Q_c = \frac{1}{2} \frac{(p-2)cu}{u_4 + b v_4} , \tag{3.8}$$

in which r_c is the value of r where the large-Q minimum crosses over from a metastable point when $r > r_c$ to an absolute minimum for $r < r_c$, and Q_c is the discontinuity of the order parameter at the first-order transition. At the same time, the small-Q minimum changes from at absolute minimum when $r > r_c$ and becomes metastable for $0 < r < r_c$. In distinction to previous work, the sign of the first term in r_T is not only determined by $(p-2)$ but by the sign of v/u as well.

When $v/u > 0$, r_T is positive for $p > 2$ with a stable large-Q minimum that favors a first-order transition at $r = r_c$, similar to earlier work.[3] However, if $v/u < 0$, r_T may become negative for small positive $(p-2)$, and the large-Q minimum is unstable. The continuation to small negative $(p-2)$ would again yield a stable large-Q minimum and a first-order transition.

To summarize, if $v/u > 0$ trilinear symmetry breaking does not seem to be relevant in determining the nature of the transition and there is the boundary between second-order ($p < 2$) and first-order transitions ($p > 2$) at $p = 2$ that was found before. If, instead, $v/u < 0$ but $|v/u| > \frac{1}{2}(p-2)$, the break in trilinear symmetry is relevant to the order of the transition and, for $p > 2$, a second-order transition may be expected. For small positive $(p-2)$ this prediction is consistent with the exclusion of a first-order transition from the stability of the large-Q minimum in Eq. (3.7), when $v/u < 0$.

The results for $p < 2$ should be taken with some care because it is not clear that one has a meaningful trilinear symmetry breaking when the number of components $n < 1$, with a fixed longitudinal component. Nevertheless, if a second-order transition due to trilinear symmetry breaking occurs for $p > 2$, that transition, if at all possible, should remain a second-order one, with trilinear symmetry breaking becoming irrelevant, when $p < 2$. A possible mechanism by means of which this could take place, but which cannot be accounted for in mean-field theory where u and v are fixed parameters, is a change of sign of v/u with the sign of $(p-2)$ or the vanishing of v.

To decide if the second-order transition suggested by the stability argument around mean-field theory when $p > 2$ can actually be expected, we resort to a RG calculation in the following section. As a preliminary to a one-loop calculation in dimension $d = 6 - \epsilon$, we argue that one can take Pytte's work as a guide to relate the renormalized inverse transverse susceptibility χ_T^{-1} to the mean-field expression for the large-Q r_T in Eq. (3.7). It is shown there[3] that the same sign of $(p-2)$, which determines the nature of the transition in mean-field theory, also determines the transition in a RG calculation. The renormalized χ_T^{-1} at the first-order transition when $p \gtrsim 2$ was found to be

$$2\chi_T^{-1} \simeq \frac{6(p-2)c^2}{u_4 + b v_4} \omega^2(l^*) , \tag{3.9}$$

where $\omega(l^*)$ is the renormalized trilinear coupling, which includes a further nonlinear dependence in $(p-2)$. The quartic couplings u_4 and v_4 do not have to be renormalized when $p \sim 2$, and it is implicitly assumed that Eq. (3.9) is the dominant sign-dependent [with $(p-2)$] term in χ_T^{-1} for larger p. From the fact that there is a stable positive susceptibility for $p \sim 2$, it is inferred that this is the case for all $p < \frac{10}{3}$, for which a fixed point exists.[3]

We have to be somewhat more cautious in the presence of trilinear symmetry breaking. Although the trilinear symmetry breaking term in r_T, Eq. (3.7), is the dominant term when $p \sim 2$, it is possible that a RG calculation does not yield a finite r_T unless p is larger than a threshold above 2. It will be shown in the next section that this is precisely the case. Nevertheless, we argue that there will be a sign-dependent trilinear symmetry-breaking term, basically similar to

$$(\chi_T^{-1})_{\rm as} \simeq \frac{6(p-2)c^2 u^{*2}}{u_4 + b v_4} \frac{v^*}{u^*} , \tag{3.10}$$

above threshold, as in the first part of Eq. (3.7), which has to be positive for a stable first-order transition to take place, and this is only possible if $v^*/u^* > 0$, with u^* and v^* the fixed-point values of the couplings at p near threshold. We assume that the renormalization of the quartic couplings u_4 and v_4 is unnecessary for the main effect of trilinear symmetry breaking near the threshold for p. We expect $(\chi_T^{-1})_{\rm as}$ to be the dominant contribution to χ_T^{-1} due to trilinear symmetry breaking when p is close to 2. The possibility of a negative $(\chi_T^{-1})_{\rm as}$ with $v^*/u^* < 0$ could be warning that the first-order transition is not stable.

A RG study of the $(p-2)$ dependence of the fixed-point ratio v^*/u^*, and of w^*/u^* in a calculation to higher-loop order, should be crucial to decide the role of trilinear symmetry breaking on the nature of the phase transition in the p-state Potts model.

IV. RENORMALIZATION-GROUP APPROACH

For a study of the fixed-point equations and of the disordered phase the quartic terms in Eq. (2.15) are not needed. It will be assumed that there is only one longitudinal component that becomes critical as $r \to 0$. To ensure that the remaining $(p-2)$ components are noncritical, an auxiliary squared noncritical mass $\tilde{m}^2$ may be introduced in the Hamiltonian via a term $-\frac{1}{4}\tilde{m}^2 A_q A_q$, summed over the $(p-2)$ q components. Since we are mainly interested in trilinear symmetry breaking, $\tilde{m}^2$ will be taken to be vanishingly small. It is also convenient to express u, v, and w in terms of the bare dimensionless couplings u_0, v_0, and w_0 by means of $u = \kappa^{\epsilon/2} u_0$, $v = \kappa^{\epsilon/2} v_0$, and $w = \kappa^{\epsilon/2} w_0$, where κ is an arbitrary momentum-scale parameter and $\epsilon = 6 - d$.

The renormalization is done by means of dimensional regularization and minimal subtraction at the critical point, for fixed $\tilde{m}^2$.[2,15,20] The bare one-particle irreducible (1PI) two- and three-point vertex functions $\Gamma_{ij}^{(2)}$ and $\Gamma_{ijk}^{(3)}$ transform as the tensors δ_{ij} and D_{ijk}, and are given by

$$\Gamma_{11}^{(2)}(k) = k^2 - [A_1(p) u_0^2 I_0(k) + A_2(p) v_0^2 I_2(k)] \tag{4.1a}$$

for the longitudinal components, where

$$A_1(p) = \tfrac{1}{2} D_{111}^2 = \tfrac{1}{2}(p-2)^2 c^2 \,, \qquad A_2(p) = \tfrac{1}{2}\sum_{q>1}^{n} D_{1qq}^2 = \tfrac{1}{2}(p-2)c^2 \,, \tag{4.1b}$$

and

$$\Gamma_{qq}^{(2)}(k) = k^2 + \tilde{m}^2 - [B_1(p) v_0^2 I_1(k) + B_2(p) w_0^2 I_2(k)] \tag{4.2a}$$

for the transverse components, where

$$B_1(p) = D_{1qq}^2 = c^2 \,, \qquad B_2(p) = \frac{1}{2}\left[D_{qqq}^2 + \sum_{r>q}^{n} D_{qrr}^2 + 2\sum_{r>1}^{q-1} D_{rqq}^2 \right] = \tfrac{1}{2} c^2 p(p-3) \,, \tag{4.2b}$$

$I_i(k)$ being the one-loop diagram with $i = 0,1,2$ transverse internal lines shown in Fig. 1, in which the $\tilde{m}^2$ dependence is supressed since ultimately $\tilde{m}^2 \to 0$,

$$\Gamma_{111}^{(3)}(k) = \kappa^{\epsilon/2} D_{111} \{ u_0 + [C_{11}(p) u_0^3 L_0(k) + C_{12}(p) v_0^3 L_3(k)] \} \,, \tag{4.3a}$$

where

$$C_{11}(p) = D_{111}^2 = (p-2)^2 c^2 \,, \qquad C_{12}(p) = \frac{1}{D_{111}} \sum_{q>1}^{n} D_{1qq}^3 = -c^2 \,, \tag{4.3b}$$

$$\Gamma_{1qq}^{(3)}(k) = \kappa^{\epsilon/2} D_{1qq} \{ v_0 + [C_{21}(p) u_0 v_0^2 L_1(k) + v_0^3 C_{22}(p) L_2(k) + v_0 w_0^2 C_{23}(p) L_3(k)] \} \,, \tag{4.4a}$$

where

$$C_{21}(p) = D_{111} D_{1qq} = -(p-2)c^2 \,, \qquad C_{22}(p) = D_{1qq}^2 = c^2 \,, \tag{4.4b}$$

$$C_{23}(p) = D_{qqq}^2 + 2\sum_{r>1}^{q-1} D_{rqq}^2 + \sum_{s>q}^{n} D_{qss}^2 = c^2 p(p-3) \,, \tag{4.5a}$$

$$\Gamma_{qqq}^{(3)}(k) = \kappa^{\epsilon/2} D_{qqq} \{ w_0 + [C_{31}(p) w_0 v_0^2 L_2(k) + C_{32}(p) w_0^3 L_3(k)] \} \,,$$

for $2 \le q \le p-2$, and zero for $q = p-1$, where

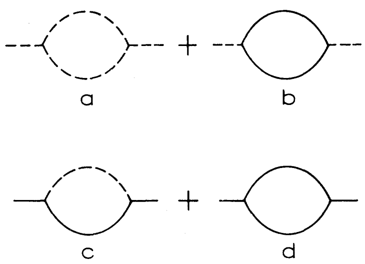

FIG. 1. One-loop contributions to the self-energy parts of the longitudinal (a and b) and transverse (c and d) one-particle irreducible (1PI) two-point vertex functions in Eqs. (4.1) and (4.2). Dashed and solid lines represent longitudinal and transverse free-field propagators, respectively. Summation over the indices of internal lines is given by Eqs. (4.1b) and (4.2b).

$$C_{31}(p)=3D_{1qq}^2=3c^2 \ ,$$

$$C_{32}(p)=D_{qqq}^2+3\sum_{r>1}^{q-1} D_{rqq}^2+\frac{1}{D_{qqq}}\sum_{s>q}^{n} D_{qss}^3 \qquad (4.5b)$$

$$=c^2p(p-4) \ .$$

The same expression as Eq. (4.5), including $q=p-1$, is obtained for $\Gamma_{rqq}^{(3)}(k)$ when $r<q$, with the factor D_{qqq} being replaced by D_{rqq}. It follows from Eq. (2.9) that all other three-point vertex functions vanish. In these equations, k is a shorthand notation for the external momenta k_1, k_2, and k_3 of the one-loop triangular diagrams $L_i(k)$ with $i=0,1,2$ transverse internal lines shown in Fig. 2. Equations (4.1)–(4.5) check in the symmetric case with the unrestricted summations in Eqs. (2.12) and (2.13) being satisfied.

The longitudinal 1PI two-point vertex function with an A^2 insertion at momentum q, $A^2=\sum_\alpha A_\alpha^2$ summed over all α, is given by

$$\Gamma_{11}^{(2,1)}(k,q)=1+[2A_1(p)u_0^2L_0(k)$$
$$+2A_2(p)v_0^2L_3(k)] \ , \qquad (4.6)$$

where $A_1(p)$ and $A_2(p)$ are the coefficients in Eq. (4.1) and the one-loop diagrams are those in Eq. (4.3), as shown in Fig. 3.

It can easily be checked that the vertex functions are consistent with the Ising one-component ($p=2$) limit, with no contribution to $\Gamma_{11}^{(2)}(k)$ and $\Gamma_{11}^{(2,1)}(k,q)$ from one-loop diagrams with trilinear couplings. Furthermore, there is no question of trilinear coupling-constant renormalization for $p=2$, since $\Gamma_{111}^{(3)}=0$, to one-loop order, as can be seen from the vanishing of the tensorial coefficient D_{111} and the

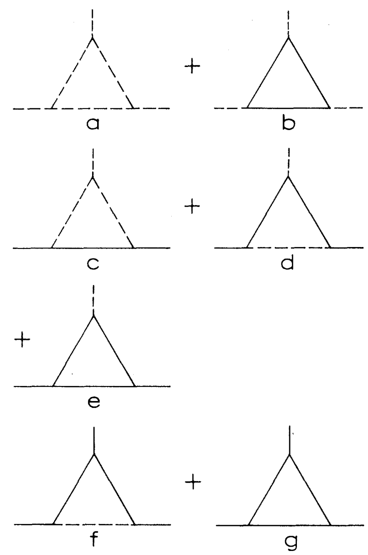

FIG. 2. One-loop contributions to the 1PI three-point vertex functions $\Gamma_{111}^{(3)}$ (a and b) given by Eq. (4.3), to $\Gamma_{1qq}^{(3)}$ ($c-e$) given by Eq. (4.4), and to $\Gamma_{qqq}^{(3)}$ (f and g) given by Eq. (4.5).

summation $\sum_{q>1}^{n} D_{1qq}^3$ in that limit. In what follows, the renormalization is done for a fixed $p>2$, although ultimately the critical exponents for the Gaussian model are recovered when the results are continued to $p=2$.

The bare vertex functions are renormalized eliminating the dimensional poles in ϵ that appear in the one-loop diagrams by means of dimensionless renormalization functions $Z_\phi^{(i)}(u,v,w;\epsilon)$ and $Z_{\phi^2}^{(i)}(u,v,w;\epsilon)$, where ϕ and ϕ^2 denote the fields A and A^2 in the notation of this paper, while u,v, and

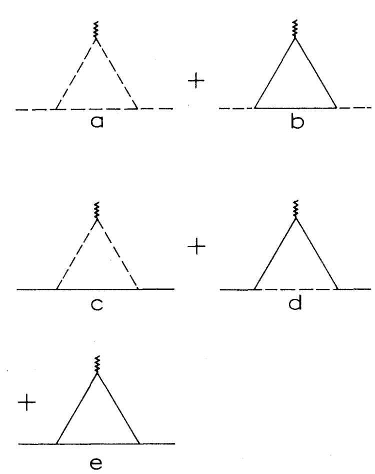

FIG. 3. One-loop contributions to the 1PI two-point vertex functions with one A^2 insertion $\Gamma_{11}^{(2,1)}$ (a and b) given by Eq. (4.6) and $\Gamma_{qq}^{(2,1)}$ ($c-e$). The wavy line indicates the insertion point.

w stand for the dimensionless renormalized couplings, not to be confused with those in Eq. (2.15). At the same time, each u_0, v_0, and w_0 is expanded in powers of u, v, and w, and the renormalized vertex functions follow from

$$\begin{aligned} &Z_\phi^{(1)}\Gamma_{11}^{(2)}=\Gamma_{R11}^{(2)},\ \ Z_\phi^{(2)}\hat{\Gamma}_{qq}^{(2)}=\hat{\Gamma}_{Rqq}^{(2)}\ ,\\ &Z_{\phi^2}^{(1)}\Gamma_{11}^{(2,1)}=\Gamma_{R11}^{(2,1)},\ \ Z_{\phi^2}^{(2)}\hat{\Gamma}_{qq}^{(2,1)}=\hat{\Gamma}_{Rqq}^{(2,1)}\ ,\\ &(Z_\phi^{(1)})^{3/2}\Gamma_{111}^{(3)}=\Gamma_{R111}^{(3)},\ \ (Z_\phi^{(1)})^{1/2}Z_\phi^{(2)}\Gamma_{1qq}^{(3)}=\Gamma_{R1qq}^{(3)}\ ,\\ &(Z_\phi^{(2)})^{3/2}\Gamma_{qqq}^{(3)}=\Gamma_{Rqqq}^{(3)}\ , \end{aligned} \tag{4.7}$$

where in the case of the two-point transverse vertex functions what is renormalized is $\hat{\Gamma}_{qq}^{(2)}=\Gamma_{qq}^{(2)}-m^2$.

The functions needed to make the renormalized vertices finite are

$$\begin{aligned} Z_\phi^{(1)}&=1-\frac{1}{6\epsilon}(p-2)^2c^2u^2-\frac{1}{6\epsilon}(p-2)c^2v^2\ ,\\ Z_\phi^{(2)}&=1-\frac{1}{3\epsilon}c^2v^2-\frac{1}{6\epsilon}c^2p(p-3)w^2\ ,\\ Z_{\phi^2}^{(1)}&=1-\frac{1}{\epsilon}(p-2)^2c^2u^2-\frac{1}{\epsilon}(p-2)c^2v^2\ , \end{aligned} \tag{4.8}$$

and the expansions for the couplings are given by

$$\begin{aligned} u_0&=u-\frac{1}{4\epsilon}3(p-2)^2c^2u^3\\ &\quad+\frac{1}{4\epsilon}(p-2)c^2uv^2+\frac{1}{\epsilon}c^2v^3\ ,\\ v_0&=v+\frac{1}{12\epsilon}(p-2)c^2vu^2\\ &\quad-\frac{1}{3\epsilon}c^2[2-\tfrac{1}{4}(p-2)]v^3\\ &\quad-\frac{1}{6\epsilon}5c^2p(p-3)vw^2+\frac{1}{\epsilon}(p-2)c^2uv^2\ ,\\ w_0&=w-\frac{1}{2\epsilon}5c^2wv^2+\frac{1}{\epsilon}c^2[\tfrac{3}{12}(p-3)-(p-4)]pw^3\ , \end{aligned} \tag{4.9}$$

to one-loop order.

A. Fixed points

Equations (4.9) can be used to study the fixed points, which are the roots of the Wilson β functions defined as

$$\begin{aligned} \beta_u(u,v,w;\epsilon)&=\left[\kappa\frac{\partial}{\partial\kappa}u\right]_{g_i}\ ,\\ \beta_v(u,v,w;\epsilon)&=\left[\kappa\frac{\partial}{\partial\kappa}v\right]_{g_i}\ ,\\ \beta_w(u,v,w;\epsilon)&=\left[\kappa\frac{\partial}{\partial\kappa}w\right]_{g_i}\ , \end{aligned} \tag{4.10}$$

in which the derivatives are taken at fixed bare dimensional couplings,

$$g_u=\kappa^{\epsilon/2}u_0,\ \ g_v=\kappa^{\epsilon/2}v_0,\ \ g_w=\kappa^{\epsilon/2}w_0\ , \tag{4.11}$$

and we find,

$$\begin{aligned} \beta_u&=-\frac{\epsilon}{2}u-\tfrac{3}{4}(p-2)^2c^2u^3\\ &\quad+\tfrac{1}{4}(p-2)c^2uv^2+c^2v^3\ , \end{aligned} \tag{4.12}$$

$$\begin{aligned} \beta_v&=-\frac{\epsilon}{2}v+\tfrac{1}{12}(p-2)^2c^2vu^2\\ &\quad-\tfrac{1}{3}c^2[2-\tfrac{1}{4}(p-2)]v^3\\ &\quad-\tfrac{5}{6}c^2p(p-3)vw^2+(p-2)c^2uv^2\ , \end{aligned} \tag{4.13}$$

$$\beta_w=-\frac{\epsilon}{2}w-\tfrac{5}{2}c^2v^2w+\tfrac{1}{4}c^2p(13-3p)w^3\ . \tag{4.14}$$

If one sets $u=v=w$, the symmetric coupling to start with, one readily verifies that the root of $\beta_u=0$, $\beta_v=0$, and $\beta_w=0$ is given by

$$u_{\rm sym}^{*2}=\frac{2p}{10-3p}\epsilon\ , \tag{4.15}$$

in accordance with Priest and Lubensky.[5] Alternatively, solving for the nonzero roots of Eqs. (4.12)–(4.14) as functions of p, we find that the symmetric fixed point (FP 1)

$$u^*=v^*=w^*=u_{\rm sym}^* \tag{4.16}$$

is the only one in which none of the couplings is zero. This follows from the nonzero solution of $\beta_w=0$,

$$w^{*2}=\left[2+10c^2\frac{v^{*2}}{\epsilon}\right]\epsilon\Big/c^2p(13-3p)\ , \tag{4.17}$$

and from the detailed solution of the two remaining equations $\beta_u=0$ and $\beta_v=0$. The symmetric fixed point is a solution when $p<\frac{10}{3}$ and, for $2<p<\frac{10}{3}$, it has been suggested to describe the spinodal point.[3]

Next, we find the asymmetric fixed point (FP 2)

$$u^*\neq0,\ \ v^*\neq0,\ \ w^*=0\ , \tag{4.18}$$

shown in Fig. 4 as a function of p. This fixed point has two remarkable features: First, it has a runaway for $p<2.2^-$, and second, it persists for any large but finite p. In the limit of asymptotically large p,

$$u^{*2}\simeq0.0243\epsilon,\ \ v^{*2}\simeq2.073p\epsilon\ . \tag{4.19}$$

From Eqs. (4.12) and (4.13) it also follows that for

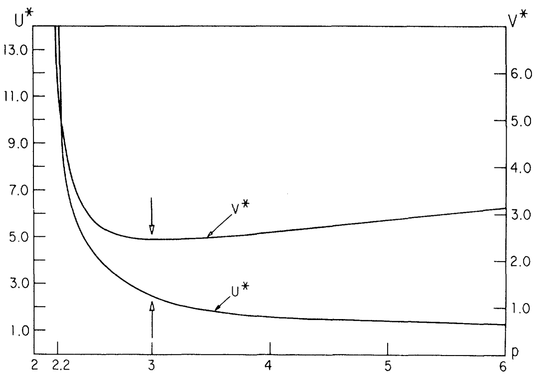

FIG. 4. Dependence of the dimensionless renormalized fixed-point couplings u^* and v^* (with different scales) in units of $\epsilon^{1/2}$ for fixed point FP 2, Eq. (4.18), with the number of Potts states p. The arrows indicate the symmetric couplings for the three-state Potts model. Note the fixed-point runaway for $p \lesssim 2.2^-$.

the three-state Potts model, the nonzero u^* and v^* are symmetric,

$$u^{*2} = v^{*2} = 6.0\bar{\epsilon} = u^{*2}_{\text{sym}}(p=3) , \tag{4.20}$$

and this is shown in Fig. 4. This is reasonable because the two components for $p=3$ are equivalent and the fixed-point couplings for the renormalized vertices $\Gamma^{(3)}_{R111}$ and $\Gamma^{(3)}_{R122}$ should be the same, while $\Gamma^{(3)}_{R222} \equiv 0$ from Eq. (2.9). There is an exact symmetry for the three-state Potts model manifest by the tensorial coefficient $D_{222} \equiv 0$, which implies that there is no trilinear symmetry breaking in the fixed-point couplings, even if the initial u and v in Eq. (2.15) are taken to be different.

The other fixed point for $p > 1$ (FP 3), excluding $p=3$,

$$u^* = 0, \quad v^* = 0, \quad w^{*2} = \frac{2(p-1)\epsilon}{13-3p} , \tag{4.21}$$

is real whenever $p < \frac{13}{3}$. As usual, there is the trivial fixed point $u^* = v^* = w^* = 0$, and all the remaining fixed points are nonreal.

The stability of the fixed points is determined as usual[15,16] by the eigenvalues λ_i of the 3×3 matrix Λ with elements

$$\Lambda_{\rho\sigma} = \left[\frac{\partial \beta_\rho}{\partial \sigma} \right]_{\text{FP}} , \quad \rho,\sigma = u,v,w \tag{4.22}$$

calculated at the fixed points, in which a direction of stability corresponds to a positive eigenvalue. The eigenvectors $R(u,v,w)$ that solve

$$\Lambda R = \lambda R \tag{4.23}$$

give the directions of stability in the parameter space (u,v,w). The boundaries between sectors of stable and unstable directions about a given fixed point, which are determined by $\lambda = 0$, unless the eigenvectors are along u, v, and w, provide a local picture of the possible crossover flows between the various fixed points. Note that there are now three nontrivial fixed points which coexist when $2.2^- < p < \frac{10}{3}$.

First, we determine the eigenvalues and obtain (FP 1)

$$\lambda_1 \simeq 2.08\epsilon, \quad \lambda_2 \simeq -0.67\epsilon, \quad \lambda_3 \simeq 1.0\epsilon \tag{4.24}$$

for $p = 2.2$, with increasing λ_1 and decreasing λ_2 as p is increased, (FP 2)

$$\begin{aligned} &\lambda_1 \simeq -33.61\epsilon , \\ &\lambda_2 \simeq -22.1\epsilon , \\ &\lambda_3 \simeq 1.00\epsilon \end{aligned} \tag{4.25}$$

for $p = 2.2$, the neighborhood of the point of interest in Eq. (3.10); and finally, for the same value of p (FP 3),

$$\lambda_1 = -0.5\epsilon , \quad \lambda_2 = -0.29\epsilon, \quad \lambda_3 = 1.0\epsilon . \tag{4.26}$$

Qualitatively, the same feature of one or two negative eigenvalues persists for larger p.

The sectors of local stability are shown in Fig. 5, where it can be seen that FP 2 can be reached through the unstable directions of FP 1, the symmetric fixed point. We find that FP 1 is unstable to perturbations in u and v within the range $-1/4.84 \le v/u \le 1/2.98$.

The fixed points that are relevant to the discussion of the preceding section are FP 1 and FP 2. For $p < 2.2^-$, because of the fixed-point runaway, there is only the usual first-order transition. For $p \gtrsim 2.2^-$, FP 2 competes with FP 1. However, since $v^*/u^* > 0$, trilinear symmetry breaking will not change the sign of the overall inverse susceptibility,

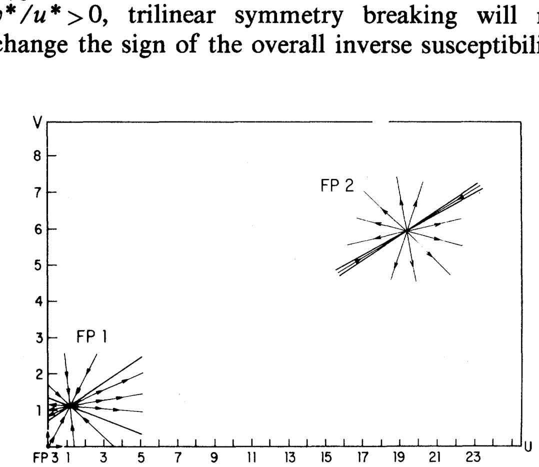

FIG. 5. Sectors of local fixed-point stability in the (u,v) plane determined as explained in Sec. IV.

according to Eq. (3.10), and the transition should remain of first order, with a spinodal point, for all finite p.

The discussion of the RG equations in the disordered phase does not assume that the model has to order along the longitudinal field component. If the ordering is along the transverse components, the mean-field argument of the Appendix shows that the only relevant trilinear coupling is w and this suggests that the transition, determined in this case by FP 3, is of first order for all $p>1$. When $1<p<\frac{13}{3}$, the stable (in one direction) fixed point indicates that the first-order transition takes place near a spinodal point, whereas the runaway for $p>\frac{13}{3}$ is interpreted as the usual first-order transition. The boundary between these two transitions occurs at $p_c'=\frac{13}{3}$, in contrast to the $p_c=\frac{10}{3}$ of the symmetric theory.

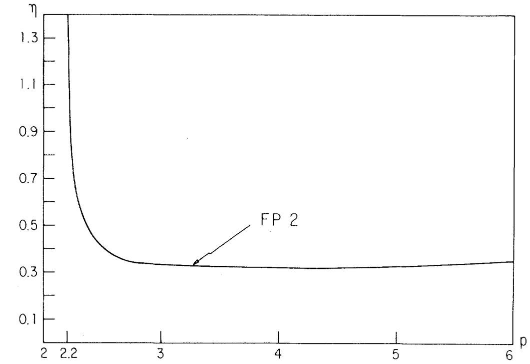

FIG. 6. Dependence with p of the critical exponent η given by Eqs. (4.27) and (4.28) for the new FP 2. The p dependence of the correlation length exponent ν can be read off by means of the relationship $\nu^{-1}-2=5\eta$, Eq. (4.34), discussed in the text.

B. Critical exponents

For FP 1 the critical exponents are known to describe a second-order transition when $p\lesssim 2$, and possibly a spinodal point for $2<p<\frac{10}{3}$. With η defined as

$$\eta\equiv\gamma_\phi^{(1)}(u^*,v^*,w^*)\ , \tag{4.27}$$

where [15,16]

$$\gamma_\phi^{(1)}(u,v,w)=\kappa\frac{\partial \ln Z_\phi^{(1)}}{\partial\kappa}$$
$$=2\beta_u(u,v,w)b_1^{(1)}u+2\beta_v(u,v,w)b_2^{(1)}v\ , \tag{4.28}$$

in which $b_1^{(1)}$ and $b_2^{(1)}$ are the coefficients in $Z_\phi^{(1)}=1+b_1^{(1)}u^2+b_2^{(1)}v^2$, Eq. (4.8), we find

$$\gamma_\phi^{(1)}(u^*,v^*)=\tfrac{1}{6}c^2(p-2)[(p-2)u^{*2}+v^{*2}]\ . \tag{4.29}$$

When $p=2$ this yields the $\eta=0$ that one expects to have for the Ising model in $d=6-\epsilon$ dimensions. For FP 1, we have the known[3,5,6]

$$\eta=-\tfrac{1}{3}(2-p)\epsilon/(10-3p)\ , \tag{4.30}$$

while for FP 2 we find the result shown in Fig. 6. With FP 3 describing the transverse components becoming critical, we have

$$\eta_T=\gamma_\phi^{(2)}(u^*,v^*,w^*) \tag{4.31}$$

for the k dependence of $\Gamma_{Rqq}^{(2)}(k,\tilde{m})-\tilde{m}^2\sim k^{2-\eta_T}$, which can be calculated from

$$\gamma_\phi^{(2)}(u,v,w)=\kappa\frac{\partial \ln Z_\phi^{(2)}}{\partial\kappa}$$
$$=2\beta_v(u,v,w)b_1^{(2)}v+2\beta_w(u,v,w)b_2^{(2)}w\ , \tag{4.32}$$

when in Eq. (4.8),

$$Z_\phi^{(2)}\simeq 1+b_1^{(2)}v^2+b_2^{(2)}w^2\ .$$

This yields, at FP 3,

$$\gamma_\phi^{(2)}(w^*)=\frac{1}{3}\frac{p-3}{13-3p}\epsilon\ , \tag{4.33}$$

which gives the $\eta=0$ that one expects for the two-component model in $d>4$.

The correlation-length exponent ν is given by [15,16]

$$\nu^{-1}-2=\gamma_{\phi^2}^{(1)}(u^*,v^*,w^*)-\gamma_\phi^{(1)}(u^*,v^*,w^*)$$
$$=5\gamma_\phi^{(1)}(u^*,v^*,w^*) \tag{4.34}$$

for critical longitudinal components, which follows from the fact that the diagrams in the expansion for $\Gamma_{\phi^2}^{(2,1)}$ are those in $\Gamma_{111}^{(3)}$, while the tensorial coefficients are those in $\Gamma_{11}^{(2)}$. Then

$$\gamma_{\phi^2}^{(1)}(u,v,w)=\kappa\frac{\partial \ln Z_{\phi^2}^{(1)}}{\partial\kappa}$$
$$=2\beta_u(u,v,w)\tilde{b}_1u$$
$$+2\beta_v(u,v,w)\tilde{b}_2v=6\gamma_\phi^{(1)}\ , \tag{4.35}$$

in which $Z_{\phi^2}^{(1)}\simeq 1+\tilde{b}_1u^2+\tilde{b}_2v^2$ for the renormalization of the ϕ^2 insertion in Eq. (4.8). This yields the known[3,5,6]

$$\nu^{-1}=2-\tfrac{5}{3}(2-p)\epsilon/(10-3p) \tag{4.36}$$

for FP 1 and the $\nu(p)$ for FP 2 that is shown in Fig. 6. Accordingly, with $\gamma_{\phi^2}^{(2)}=6\gamma_\phi^{(2)}$ and Eq. (4.33) we get

$$\nu_T^{-1}-2=\frac{5}{3}\,\frac{p-3}{13-3p}\epsilon \tag{4.37}$$

for the transverse components.

In contrast to the symmetric theory, where there is a range of p ($p \le 2$) for which the transition is of second order, as predicted by mean-field theory and confirmed by the RG calculation, the present work suggests that the transition is always of first order when there is trilinear symmetry breaking ruled by FP 2 or FP 3. For the latter, this is shown in the Appendix. In these cases, the critical exponents may describe the approach to the spinodal point, subject to the reservations pointed out in the Introduction.

V. CONCLUDING REMARKS

We have shown that in a three-trilinear-coupling theory for the p-state Potts model there are three nontrivial fixed points: the symmetric one and two new asymmetric fixed points that are accessible and partially stable; one for $2.2^- \le p < \infty$ and the other for $1 < p < \frac{13}{3}$. Analysis of the stability of the mean-field free energy suggests that these two fixed points do not correspond to second-order transitions but to first-order transitions near a spinodal point. In the present study we first assumed, following previous authors, that the ordering takes place along one of the field components A_α of the p Potts vectors $\vec{A}(x)$ and, accordingly, we introduced trilinear symmetry breaking in the components of the fields. This, in turn, breaks the equivalence of the p Potts states, even in the disordered phase.

A word of caution about our interpretation of FP 2, Eq. (4.18), is appropriate at this point. From the close relationship between the renormalized χ_T^{-1} of Pytte, Eq. (3.9), and the leading term in the fluctuation correction to mean-field theory, when $p \sim 2$, it is reasonable to infer that $(\chi_T^{-1})_{\text{as}}$ of Eq. (3.10) should be the dominant contribution to χ_T^{-1} with trilinear symmetry breaking when $p \sim 2$, based on Eq. (3.7), if it were not that the threshold for FP 2 is definitely above $p \sim 2$, where Eq. (3.10) may be considerably modified in a way for which we cannot provide a reliable estimate. The calculation of the first-order transition becomes considerably more complicated when $(p-2)$ is not a small parameter, and this is why earlier works have been restricted to this range.[3,4] In view of the role of instantons in a first-order transition, which presumably is not fully accounted for in the kind of perturbation expansions used here and in the works of previous authors, it does not seem worthwhile at present to determine the precise way in which Eq. (3.10) is modified for larger p. Since it is only the crudest feature of $(\chi_T^{-1})_{\text{as}}$—the dependence on v^*/u^*—that has been used to interpret FP 2, we do not expect this to be a severe limitation of our work.

Our main conclusions are the following. First, that the RG treatment of fluctuations in the continuum Potts model with trilinear symmetry breaking results in two further first-order phase transitions with stable and accessible fixed points, within $2.2^- \le p \le \frac{13}{3}$—one continued to all larger p—a kind of phase transition first discussed by Pytte. For $p \gtrsim \frac{13}{3}$ there is a fixed-point runaway in one of these, which corresponds to the usual first-order phase transition. The important point in the case of FP 2 is that the fluctuations exclude a negative v^*/u^* that would favor a second-order transition when $p > 2$, according to the mean-field prediction. In the case of the three-state Potts model the only nontrivial fixed point is the symmetric one. These results suggest that one should not expect a second-order phase transition in the physically interesting realizations of the three- and four-state Potts model in the presence of weak anisotropic stresses. Although this is a reasonable expectation, the theoretical bases for it are part of the calculations of the present work.

Our second main result is that the second-order phase transition that one expects from mean-field theory in the case of trilinear symmetry breaking with $v/u > 0$ and $p < 2$ is destroyed due to fluctuations. This takes place with uniaxial ordering. Although physical realizations of the p-state Potts model have only been proposed so far for integer p, we believe, nevertheless, that this is a result of theoretical interest.

The nature of the transition may be changed further with the addition of quadratic symmetry breaking. A preliminary mean-field calculation shows that the r_T for the small-Q solution in Eq. (3.5) has an additional positive term so that r_T could be positive for $p > 2$, indicating a stable second-order transition.[22] We do not know at present, however, how the fixed-point picture with trilinear symmetry breaking is modified by a break in quadratic symmetry, and whether the runaway threshold of p is lowered enough for the second-order transition to become stable.

Another way of changing the transition could be by the addition of an external field. It is quite possible that quadratic symmetry breaking may induce, in general, a further break in trilinear symmetry. Preliminary calculations on the three-state Potts model show that this is the case. This and the above extension will be explored in future work.[22]

One can also go beyond a three-trilinear-coupling theory and carry the symmetry breaking to all components. In the case of the four-state Potts model with one longitudinal component, this amounts to a five-coupling theory. It can be shown, however, that the simpler three-coupling theory worked out here is a solution of the latter. This and the results of further work in progress will be reported elsewhere.[22]

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APPENDIX

In the case of transverse ordering we write

$$(p-2)^{1/2}A_\alpha(k)=Q_T+\mathcal{T}_\alpha(k),\quad \alpha=2,\ldots,n$$
$$A_1(k)=\mathcal{T}_1(k)\ , \tag{A1}$$

since $n-1=p-2$, where the order parameter Q_T is the thermal average $(p-2)^{1/2}\langle A_\alpha\rangle$, for $\alpha>1$, and $\mathcal{T}(k)$ is the fluctuation part. The mean-field Hamiltonian becomes then

$$\mathcal{H}_{\rm MF}=-\tfrac{1}{4}rQ_T^2-cwQ_T^3-(u_4+\tfrac{1}{2}v_4)Q_T^4\ , \tag{A2}$$

and the coefficients of the quadratic terms in $\mathcal{T}_1(k)$ and $\mathcal{T}_\alpha(k)$, for $\alpha\geq 2$, that follow from Eq. (2.15) are

$$r_L=r+8(u_4+\tfrac{1}{4}v_4)Q_T^2\ ,$$
$$r_T=r+12cwQ_T+24(u_4+\tfrac{1}{2}v_4)Q_T^2\ . \tag{A3}$$

The absence of a linear term in Q_T for r_L is due to $D_{\alpha\beta\beta}=0$ when $\alpha>\beta$. Note that the only trilinear couplings that will determine the stability of the mean-field solution is w. The extrema for small and large Q_T are again determined by $\partial\mathcal{H}_{\rm MF}/\partial Q_T=0$ and, for the Q_T that develops continuously from $Q_T=0$ with $r\lesssim 0$, we find

$$r_L=-|r|\ ,$$
$$r_T=|r|\ , \tag{A4}$$

independently of p, indicating that the small-Q_T extremun is always unstable.

To study the stability of the large-Q_T solution we make the free energy the same for $Q_T=0$ and $Q_T\neq 0$. When combined with $\partial\mathcal{H}_{\rm MF}/\partial Q_T=0$ we find

$$r_L=(3u_4+v_4)\frac{w^2c^2}{(u_4+\frac{1}{2}v_4)^2}\ ,$$
$$r_T=\frac{w^2c^2}{(u_4+\frac{1}{2}v_4)}\ , \tag{A5}$$

which is always stable. The discontinuity of the order parameter and the value of r for which the large-Q_T minimum crosses over from a metastable point $[r>r_c(T)]$ to an absolute minimum $[r<r_c(T)]$ are given by

$$Q_c(T)=\frac{1}{2}\frac{wc}{(u_4+\frac{1}{2}v_4)}\ ,$$
$$r_c(T)=\frac{w^2c^2}{(u_4+\frac{1}{2}v_4)}\ . \tag{A6}$$

Thus whenever w is finite the mean-field Hamiltonian (A2) has a first-order transition near a spinodal point.

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