Finite-temperature properties of the planar ferromagnetic *xxz* chain

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We present analytic weak-coupling and numerical finite-chain results for the specific heat, the longitudinal zero-field susceptibility, and the longitudinal dynamic form factor of the planar ferromagnetic *xxz* chain at finite temperatures. It is shown that the "bound state" probed by the dynamic form factor $S_{zz}(q,\omega)$ within the Hartree-Fock approximation has to be interpreted as the first moment of S_{zz} of the states lying above the continuum appearing in the isotropic *xy* model.

I. INTRODUCTION

One-dimensional interacting systems have attracted the interest of many researchers because of their relative simplicity, allowing in some cases even an exact treatment of certain properties. A prominent example is the anisotropic nearest-neighbor Heisenberg chain of spin $\frac{1}{2}$ (*xyz* chain), which was proven to be an exactly integrable system. Starting with Bethe[1] and culminating in the works of Baxter[2] and Johnson *et al.*,[3] the whole low-lying energy spectrum of the *xyz* chain has been obtained in terms of explicit formulas.

Based on these ideas and a pioneering paper of Yang and Yang,[4] Takahaski and Suzuki[5] were then able to obtain some thermodynamic properties of the *xyz* chain exactly. These examples and their implications towards the understanding of the physics of many-particle systems make the *xyz* chain a candidate worth studying in its own right. But the recent discoveries of a number of quasi-one-dimensional magnetic compounds,[6] believed to be described by the spin-$\frac{1}{2}$ *xyz*-spin chain, have further added to the efforts of understanding this model. Exact results for the quantities needed to interpret the experiments performed with these materials, such as dynamic form factors (DFF), so far are available only for a few special cases of the *xyz* chain. Therefore one has to resort to approximate methods to treat the general case.

In this paper we employ two such methods in an attempt to understand $S_{zz}(q,\omega)$ (the DFF associated with out-of-plane spin fluctuations) and some static properties of the planar ferromagnetic *xxz* chain at finite temperatures. Firstly, after mapping the *xxz* chain onto an assembly of interacting fermions, we use the Hartree-Fock (HF) approximation first introduced by Bulaevski[7] to obtain analytic results expected to be valid for small uniaxial anisotropy. Secondly, we present results obtained by a numerical diagonalization of the Hamiltonian describing the *xxz* chain, which in principle allows us to calculate all the relevant properties of the system. However, the exponential growth of computing time with system size limited us to 12 spins at the most.

In Sec. II we introduce the model and describe the approximation used to obtain the analytic results. The internal energy, the specific heat, and the zero-field longitudinal susceptibility are then obtained in Sec. III; they are found to be in good agreement with known exact and, in the case of the internal energy, our finite-chain results. Section IV is devoted to the calculation of $S_{zz}(q,\omega)$, which is found to probe a temperature-renormalized continuum and a "bound state" whose energy dispersion is obtained analytically at low temperatures. This dispersion agrees very well with numerical finite-chain results, provided that we interpret it as the first energy moment of $S_{zz}(q,\omega)$ of all states whose energy lies above the continuum. The present work is a generalization of the results obtained by Schneider *et al.*[8] and Beck and Müller[9] of the $T=0$ dynamics of the planar *xxz* chain to finite temperatures.

II. MODEL AND APPROXIMATION

Our model describes a one-dimensional array of equally spaced spin-$\frac{1}{2}$ particles interacting only with nearest neighbors through their spin components. The magnitude of this exchange interaction is denoted by J, and we shall introduce an exchange anisotropy Δ in the z direction of the spins. Δ is assumed to be small but positive, thus making the model weakly ferromagnetic. The Hamiltonian can then be written as

$$\mathcal{H}=-J\sum_{l=1}^{N}(S_l^xS_{l+1}^x+S_l^yS_{l+1}^y+\Delta S_l^zS_{l+1}^z-hS_l^z)\ . \tag{2.1}$$

We also assume periodic boundary conditions, i.e.,

$$\vec{S}_{N+1}=\vec{S}_1 \ . \tag{2.2}$$

This model can be mapped onto an assembly of weakly interacting fermions,[10] which has the following Hamiltonian in momentum space:

$$\mathscr{H}=C+\sum_k \omega_k a_k^\dagger a_k+\sum V_q \rho_q \rho_{-q} \ , \tag{2.3}$$

where the density ρ_q is given by

$$\rho_q=\sum_k a_{k+q}^\dagger a_k=S^z(q) \ , \tag{2.4}$$

and the kinetic energy ω_k and the interaction potential V_q are

$$\omega_k=J(h+\Delta-\cos k) \ , \tag{2.5}$$

$$V_q=-J\Delta\cos q \ . \tag{2.6}$$

It should be noted that only the z component of the spin operator has a simple form in the corresponding fermion picture since it just becomes the density operator ρ_q.

For $\Delta=0$, the interaction V_q vanishes, and this case has been considered by Katsura *et al.*[11] and Niemeyer[12] using well-known methods for calculating thermodynamic properties of noninteracting fermions. This so-called xy model provides a good starting point for a perturbation calculation in the weakly interacting case $|\Delta| \ll 1$, which in Refs. 8 and 9 was exploited to study the excitation spectrum and the $T=0$ dynamics of (2.1). This was done by using retarded Green's functions defined by

$$\begin{aligned} G_{AB}(\omega)&=\langle\langle A \mid B \rangle\rangle \\ &=-i\int_{-\infty}^{+\infty} dt\, e^{i\omega(t-t')}\Theta(t-t') \\ &\quad\times\langle [A(t),B(t')]\rangle(q) \ , \end{aligned} \tag{2.7}$$

where A and B denote any two observables expressed in fermion operators and the angular brackets denote the canonical average. G_{AB} then obeys the following equation of motion:

$$\omega G_{AB}(\omega)=\frac{1}{2\pi}\langle [A,B]\rangle+\langle\langle [A,\mathscr{H}] \mid B \rangle\rangle \ . \tag{2.8}$$

In order to decouple (2.8), one introduces the HF approximation, which was first applied to (2.1) for $\Delta=-1$ by Bulaevski[7] and has since been used by other authors[13,14] to calculate static and dynamic properties of the isotropic Heisenberg antiferromagnet. The structure factor $S_{AB}(\omega)$ associated with the observables A and B is then given by

$$S_{AB}(\omega)=\frac{2}{1-e^{-\beta\omega}}\mathrm{Im}G_{AB}(\omega) \ . \tag{2.9}$$

III. STATIC QUANTITIES

The relevant Green's function is obtained by setting $A=a_k$ and $B=a_k^\dagger$. We then get within the HF approximation

$$G_{AB}(k,\omega)=\frac{1}{2\pi}\frac{1-2n_k}{\omega-\omega_k-\Sigma^*(k)} \ , \tag{3.1}$$

where $n_k=\langle a_k^\dagger a_k\rangle$ is the occupation number distribution and

$$\Sigma^*(k)=\frac{2J\Delta}{N}\sum_p n_p[1-\cos(k-p)] \tag{3.2}$$

is the proper self-energy.

With regard to (2.9), we also have

$$n_k=\frac{1}{e^{\beta\Omega_k}+1} \ , \tag{3.3}$$

where $\Omega_k=\omega_k+\Sigma^*(k)$. Thus the self-energy both determines and is determined by n_k, which is a general feature of the HF approximation for the one-particle Green's function. By setting the magnetic field $h=0$, a further condition on the distribution function n_k is given by

$$0=\left\langle\sum_l S_l^z\right\rangle=\sum_k n_k-\frac{N}{2} \ , \tag{3.4}$$

which must be satisfied at any finite temperature. Equation (3.4), together with (2.5) and (3.2), implies

$$\Omega_k=JA(\beta,\Delta)\cos k \ , \tag{3.5}$$

hence the single fermion energy suffers a frequency-independent renormalization due to the interaction in our fermion Hamiltonian (2.3), but the functional dependence on the wave vector remains the same as in the xy model.

By inserting (3.5) into (3.2),

$$A(\beta,\Delta)=1-\frac{\Delta}{\pi}\int_0^{2\pi}\cos y[1+\exp(-JA\beta\cos y)]^{-1}dy \ . \tag{3.6}$$

This can easily be solved for high and low temperatures with the results ($\tilde{T}=T/J$)

$$A=\begin{cases}1-\dfrac{2\Delta}{\pi}+\dfrac{\Delta\pi}{3(1-2\Delta/\pi)^2}\tilde{T}^2, & \tilde{T}\ll 1 \quad (3.7)\\ 1-\dfrac{\Delta}{4\tilde{T}}, & \tilde{T}\gg 1\ . \quad (3.8)\end{cases}$$

These results have already been derived in Ref. 7 for the special case $\Delta=-1$.

We can now easily calculate the internal energy in the HF approximation,

$$U_{\mathrm{HF}}=\frac{\langle\mathscr{H}\rangle}{JN}=-\frac{1}{4\Delta}(1-A)(1+A)\ , \quad (3.9)$$

which yields

$$U_{\mathrm{HF}}=\begin{cases}\dfrac{1}{\pi}\left[1-\dfrac{\Delta}{\pi}\right]+\dfrac{\pi\tilde{T}^2}{6(1-2\Delta/\pi)}, & \tilde{T}\ll 1 \quad (3.10)\\ -\dfrac{1}{8\tilde{T}}, & \tilde{T}\gg 1\ . \quad (3.11)\end{cases}$$

Note that U no longer depends on Δ in the high-temperature limit. Figure 1 shows a comparison of (3.10) to data obtained from a numerical diagonalization of finite rings of atoms for $\Delta=0.2$. Data were obtained for various temperatures and for system sizes N between 5 and 12. The extrapolation to the thermodynamic limit was then carried out by means of a method due originally to Van den Broeck and Schwartz[15] (VBS method). The data for even and odd N were extrapolated separately, and we found agreement of the two final values of about $\pm 0.1\%$ for $\tilde{T}>0.15$. We also noted that for $\tilde{T}>0.25$, the data for $N=12$ and $N=11$ came within about 1% of the extrapolated value. The temperature dependence of the internal energy is thus reasonably described by a T^2 behavior with small corrections of higher order in T at low temperatures. We can thus conclude that the specific heat at low temperatures is given by

$$C_V=\frac{\pi\tilde{T}}{3(1-2\Delta/\pi)}\ , \quad \tilde{T}\ll 1 \quad (3.12)$$

$$=\left[\frac{\pi}{3}+\frac{2\Delta}{3}+O(\Delta^2)\right]\tilde{T}\ . \quad (3.13)$$

This result agrees to first order in Δ with the exact low-temperature specific heat,

$$C_V=\frac{2\cos^{-1}(-\Delta)}{3(1-\Delta^2)^{1/2}}\tilde{T}\ , \quad (3.14)$$

obtained by Takahaski.[5]

By applying a field $h\neq 0$, Eq. (3.4) changes to

$$m=\frac{1}{N}\sum_k n_k-\tfrac{1}{2}\ , \quad (3.15)$$

where m is the magnetization of the system. The renormalized single-particle energy is then given by

$$\Omega_k=J(2\Delta m-A\cos k)+h\ . \quad (3.16)$$

From Eqs. (3.3), (3.15), and (3.16), we can calculate the zero-field magnetic susceptibility in the z direction,

$$\chi_z=\left.\frac{\partial m}{\partial h}\right|_{h=0}=\frac{1}{2J}\frac{J_0}{1-\Delta J_0}\ , \quad (3.17)$$

$$J_0=\frac{2\beta J}{\pi}\int_0^{\pi}dk\frac{e^{\beta p\cos k}}{(1+e^{\beta p\cos k})^2}\ , \quad (3.18)$$

which yields

$$J\chi_z=\begin{cases}\dfrac{1}{\pi-4\Delta}\left[1+\dfrac{\pi^2}{6(1-2\Delta/\pi)^2}\tilde{T}^2\right], & \tilde{T}\ll 1 \quad (3.19)\\ \dfrac{1}{4\tilde{T}}, & \tilde{T}\gg 1\ , \quad (3.20)\end{cases}$$

at $\tilde{T}=0$, this result agrees to $O(\Delta)$ with the exact zero-field susceptibility

$$J\chi_z=\frac{\mu}{\pi(\pi-\mu)\sin\mu} \quad (3.21)$$

obtained by Yang and Yang.[16]

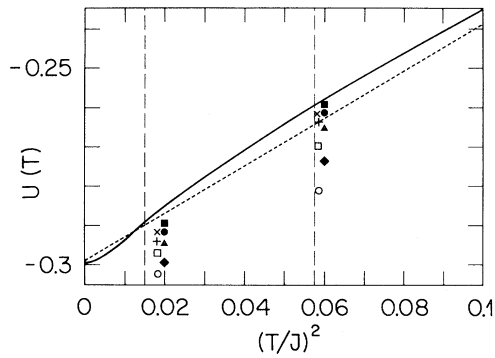

FIG. 1. Internal energy U vs $(T/J)^2$. Solid line denotes VBS extrapolated curve. (Ref. 15). Dashed line denotes theoretical values from (3.10). Vertical dashed lines at $(T/J)^2\approx 0.015$ and 0.057 include range where theory agrees well with numerical data. The symbols are results from the finite-chain calculations (○, $N=5$; ♦, $N=6$; □, $N=7$; ▲, $N=8$; +, $N=9$; ●, $N=10$; x, $N=11$; ■, $N=12$).

IV. ZERO-FIELD LONGITUDINAL DFF $S_{zz}(q,\omega)$

The longitudinal DFF is defined by

$$S_{zz}(q,\omega)=\frac{1}{Z}\sum_{\lambda,\lambda'}e^{-\beta\omega_\lambda}|\langle\lambda|S^z(q)|\lambda'\rangle|^2 \times\delta(\omega-\omega_\lambda+\omega_{\lambda'}) , \tag{4.1}$$

$$Z=\sum_\lambda e^{-\beta\omega_\lambda} , \tag{4.2}$$

$$S^z(q)=\frac{1}{\sqrt{N}}\sum_l e^{iql}S_l^z , \tag{4.3}$$

and λ is the eigenstate to the energy ω_λ of the Hamiltonian (2.1). This quantity measures the out-of-phase spin fluctuations in the z direction. We can obtain $S_{zz}(q,\omega)$ by setting $A=\rho(q)$ and $B=A^\dagger$ in the the definition of the Green's function (2.7), where the density is defined in terms of the fermion operators in (2.4). Besides some minor modifications, the calculation proceeds as in the $T=0$ case, which is well documented in Ref. 14, and we shall only state the final result,

$$G_{zz}(q,\omega)=\frac{1}{2\pi}\frac{\Gamma_0(q,\omega)}{1+\Delta f(q,\omega)\Gamma_0(q,\omega)} , \tag{4.4}$$

where Γ_0 is the noninteracting density Green's function with renormalized single-particle energies and $f(q,\omega)$ is an effective interaction obtained by summing over all exchange ladders and subsequently doing a random-phase approximation. Owing to the fact that $V_q\sim\cos q$ is separable, we can obtain an analytic expression for $f(q,\omega)$,

$$f(q,\omega)=\frac{\omega^2-2J^2A^2(1+B)\cos q(1-\cos q)}{JA^2(1+B)(1-\cos q)} , \tag{4.5}$$

$$B=\frac{2\Delta I}{2A\sin(q/2)} , \quad I=-\int_{-\pi/2}^{\pi/2}(n_{k+q/2}-n_{k-q/2})\sin k\,dk . \tag{4.6}$$

Defining $\Gamma_0=\Gamma_0'+i\Gamma_0''$, we obtain

$$\Gamma_0'(q,\omega)=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(n_{k+q/2}-n_{k-q/2})\frac{dk}{\omega+2JA\sin(q/2)\sin k} , \tag{4.7}$$

$$\Gamma_0''(q,\omega)=\begin{cases}0, & \omega>2JA\sin(q/2)\\ \dfrac{n_{k_0+q/2}-n_{k_0-q/2}}{\{[2JA\sin(q/2)]^2-\omega^2\}^{1/2}} , & k_0=\arcsin\left[\dfrac{\omega}{2JA\sin(q/2)}\right]\end{cases} . \tag{4.8}$$

At finite temperatures, the integrals in (4.6) and (4.7) can no longer be evaluated analytically because of the Fermi distribution n_k. We have thus done them numerically, taking special care of the removable singularity occurring in the integrand of (4.7) at $k=k_0$. Invoking (2.9), (4.4), and (4.8), we see that the resonance structure of $S_{zz}(q,\omega)$ is given by the following:

(i) A continuum bounded by

$$-2JA\sin(q/2)\le\omega\le 2JA\sin(q/2)=\omega_{TC}(q,T) , \tag{4.9}$$

so that, contrary to the $T=0$ case, the continuum now extends over all positive energies below ω_{TC}, negative energies also being included because of the thermally excited states at finite temperatures.

(ii) From (4.7) it can easily be seen that

$$\lim_{\omega\to\omega_{TC}+}\Gamma_0'(q,\omega)=-\infty , \tag{4.10}$$

which, together with (4.5), implies for $\Delta>0$ that there must be a zero of the denominator in (4.4) for an energy close to, but above, the continuum mentioned in (i). This implies that there must be a bound-state resonance at $\omega_B(q,T)>\omega_{TC}(q,T)$, which can be obtained by solving the equation

$$1+\Delta f(q,\omega)\Gamma_0(q,\omega)=0 \tag{4.11}$$

for ω.

We solved (4.11) analytically at low temperatures and obtained

$$\omega_B(q,T)=2J\sin(q/2)\left[D[1+2\Delta^2\sin(q/2)]+\frac{\Delta\pi}{3D^2}\{[1+2\Delta^2\sin^2(q/2)]-8D[1+\cos^2(q/2)]\Delta^2\}\tilde{T}^2\right] , \tag{4.12}$$

$$D=\left[1-\frac{2\Delta}{\pi}\right] . \tag{4.13}$$

The corresponding weight of the bound state is given by

$$S_{zz}(q,\omega_B) = 2\Delta \sin(q/2)\left[1-\frac{2\Delta\pi}{D^2}\frac{1+\cos^2(q/2)}{\sin^2 q/2)}\tilde{T}^2\right], \tag{4.14}$$

these results being valid for q not too small; see the Appendix for details of this calculation.

Figure 2 shows plots of $S_{zz}(q,\omega)$ at various temperatures and $\Delta=0.2$ at $q=\pi/2$. The question that now arises is, naturally, what is the significance of the bound state obtained in the HF approximation to $S_{zz}(q,\omega)$ at finite temperatures? To answer this question, we have looked at numerical finite-chain results of $S_{zz}(q,\omega)$, obtained direct from (4.1) after diagonalizing the Hamiltonian (2.1) in the $(2s+1)^N$-dimensional Hilbert space belonging to a system of N spins of magnitude s. Details of this procedure can be found in Refs. 17 and 18.

Obviously, because of the finite size of the system under consideration, there are only discrete lying contributions to $S_{zz}(q,\omega)$ in (q,ω) space, situated on the lines $q=2\pi l/N$ $(0\le l\le N-1)$. If we now define the finite-size bound-state frequency $\omega_B^N(q,T)$ to be

$$\omega_B^N(q,T)=\frac{\int_{\omega_{TC}(q,T)}^{\infty}\omega S_{zz}(q,\omega)d\omega}{\int_{\omega_{TC}(q,T)}^{\infty}S_{zz}(q,\omega)d\omega}, \tag{4.15}$$

we obtain good agreement with the HF value (4.12). This is shown in Figs. 3 and 4, where $\omega_B(q,T)$ [Eq. (4.12)] is plotted against $\tilde{T}^2$, starting with the exact known value at $\tilde{T}=0$,[4] together with $\omega_B^N(q,T)$ [Eq. (4.15)] for $N=10,12$ for various values of Δ, and $q=\pi$. Figures 5 and 6 show the same plot of

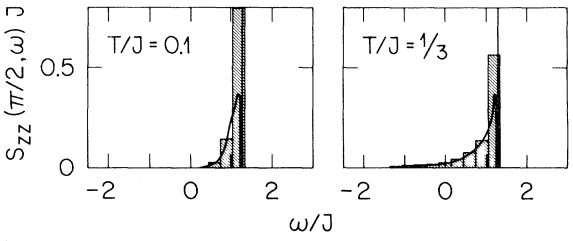

FIG. 2. $S_{zz}(\pi/4,\omega)J$ according to the HF calculation at temperatures indicated. The broad resonance is due to the continuum [Eq. (4.9)], while the sharp line denotes the bound state [Eqs. (4.12) and (4.14)]. The histograms were obtained from finite-chain ($N=12$) calculations by summing over all contributions within energy intervals $\Delta\omega/J=0.3$.

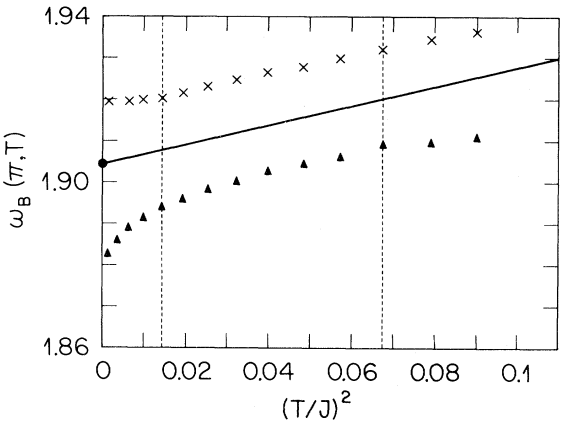

FIG. 3. Bound-state energies $\omega_B(\pi,T)$ vs $(T/J)^2$ as obtained from Eq. (4.12), (straight line), and from finite-chain results by the method explained in the text ($x,N=10$; ▲, $N=12$) for $\Delta=0.1$. ●, exact $T=0$ value. Vertical dashed lines see Fig. 1.

$q=\pi/2$ for $N=12$. Clearly, for reduced temperatures between 0.12 and 0.25, the curves $\omega_B^N(q,T)$ are almost straight lines, and their slope, corresponding to the magnitude of the $\tilde{T}^2$ shift of the bound state, agrees very well with (4.12). The deviations from a straight line for temperatures less than 0.12 have to be attributed to a finite-size effect, because they depend very sensitively on N, but not on the anisotropy Δ, as can be seen from Figs. 3 and 4. We can thus conclude that in the low-temperature limit, the HF approximation contracts nearly all the contributions to S_{zz} which lie above the continuum (4.9) into a bound state whose position is just the first energy moment of $S_{zz}(q,\omega)$ with respect to the states lying above $\omega_{TC}(q,T)$. The contributions from higher-lying excitations, not obtained within the HF theory, seem to cause only small corrections within the temperature range considered.

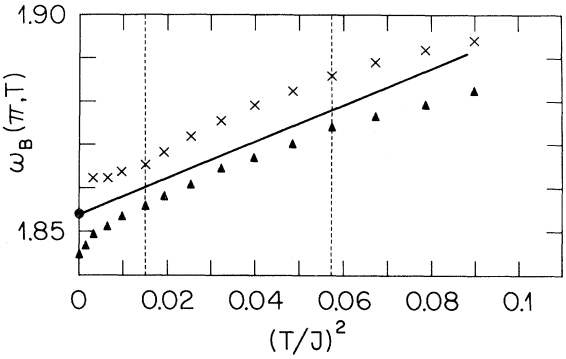

FIG. 4. Same as in Fig. 3, for $\Delta=0.2$.

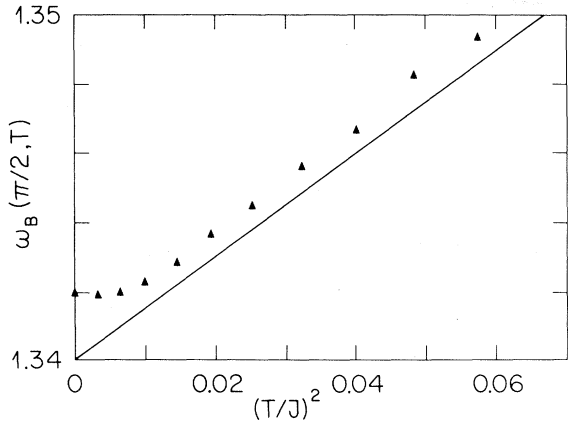

FIG. 5. Bound-state energies $\omega_B(\pi/2,T)$ vs $(T/J)^2$ as obtained from Eq. (4.12) (straight line), and from finite-chain results by the method explained in the text ($\triangle$,$N=12$) for $\Delta=0.1$.

The comparison of the spectral weight of $S^{zz}(q,\omega)$ at the bound-state frequency $\omega_B(q,T)$, given by (4.10) and the corresponding numerical quantity

$$S_{zz}(\omega_B^N(q,T))=\int_{\omega_{TC}}^{\infty} S_{zz}(q,\omega)d\omega \ , \tag{4.16}$$

shows that, although there is a $\tilde{T}^2$ dependence within the temperature range $0.12\leq\tilde{T}\leq 0.25$, there is no longer a quantitative agreement of Eqs. (4.6) and (4.14). Unfortunately, we were not able to use the VBS method to extrapolate (4.15) because reasonable agreement of theory and numerical data only sets in for $N\geq 8$. Moreover, we find a different scaling behavior for odd and even $N/2$, which is not surprising in view of the fact that we are considering a two-particle excitation.

V. CONCLUSIONS

In this work we have analyzed finite-temperature static and dynamic properties of the planar ferromagnetic Heisenberg chain with the use of the fermion representation. The fermion interaction was treated with the HF approximation, and low- and high-temperature expansions have been obtained analytically. We have found the following.

(i) The internal energy $U(T)$ agrees well with numerical finite-chain results, and the specific heat C_V and the longitudinal zero-field susceptibility are obtained exactly to leading order in Δ.

(ii) The longitudinal DFF probes a continuum and a bound state, the position of which is in good agreement with the finite-chain results at low-temperatures, provided that we interpret the finite-chain bound-state position as the expectation value of energy ω with respect to $S_{zz}(q,\omega)$ for $\omega\in[\omega_{TC},\infty]$. We can thus conclude that the HF approximation gives an accurate description of the first moment of the DFF with respect to the higher excited states above the continuum.

(iii) Besides the specific heat, all these quantities show a $\tilde{T}^2$ dependence at low temperatures, which is a consequence of the fermion description of the system. The same low-temperature behavior has been obtained in the planar case by using Bethe-ansatz thermodynamics.[5]

(iv) Recently, some papers[19,20] have been published dealing with the finite-temperature excitations of the xyz chain. It is not evident to us how to relate the exact results obtained in these works to the approach we have used here. Our information about the "finite-temperature excitation spectrum" has been obtained by approximating a dynamic form factor, which is an observable quantity. Since the energy spectrum of a Hamiltonian does not depend on temperature, it is not clear to us how one has to interpret the excitations obtained in Refs. 19 and 20.

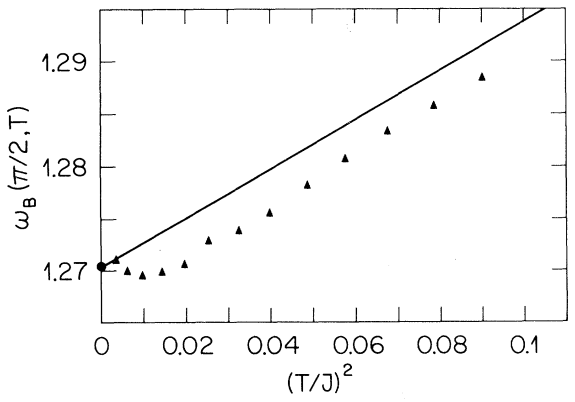

FIG. 6. Same as in Fig. 5, for $\Delta=0.2$.

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APPENDIX

We want to approximate

$$\Gamma_0'(q,\omega)=\frac{1}{\pi}\int_0^{\pi/2} f^q(x)g^{q,\omega}(x)dx \ , \tag{A1}$$

$$f^q(x)=n_{x+q/2}-n_{x-q/2} \ , \tag{A2}$$

$$g^{q,\omega}(x)=\frac{1}{\omega-\omega_{TC}\sin x}-\frac{1}{\omega+\omega_{TC}\sin x} \ . \tag{A3}$$

At low temperatures and q not too small, we can write

$$f^q(x)\approx\frac{-1}{\exp[JA\beta\cos(x+q/2)]+1} \ . \tag{A4}$$

Translating the integration by $q/2$ and substituting $t=\cos x$, we obtain

$$\Gamma_o'(q,\omega,T)=\int_{\cos(q/2)}^{-\sin(q/2)}\tilde{f}(t)\tilde{g}^{q,\omega}(t)dt\ , \tag{A5}$$

where

$$\tilde{f}(t)=\frac{1}{e^{JA\beta t}+1}\ , \tag{A6}$$

which changes rapidly only around $t=0$.

By applying now the well-known Sommerfield expansion to second order, (A1) becomes

$$\Gamma_0'(q,\omega,T)=\Gamma_0'(q,\omega,T=0)+\frac{\pi\omega_{TC}\sin(q/2)}{3D^2}\frac{[\omega^2+\omega_{Tc}^2\cos^2(q/2)]}{[\omega^2-\omega_{TC}^2\cos^2(q/2)]^2}\times\tilde{T}^2+O(\tilde{T}^4)\ . \tag{A7}$$

Together with (4.5), the solution $\omega_B(q,T)$ to Eq. (4.11) in leading order of $\omega_B-\omega_{TC}$ then leads to (4.12).

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