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Eigenvalues of the stability matrix for Parisi solution of the long-range spin-glass

C. De Dominicis and I. Kondor*

Service de Physique Théorique, Centre d'Etudes Nucléaires, Saclay, 91191 Gif-sur-Yvette, France

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We study, near T_c, the stability of Parisi's solution for the long-range spin-glass. In addition to the discrete, "longitudinal" spectrum found by Thouless, de Almeida, and Kosterlitz, we find "transverse" bands depending on one or two continuous parameters, and a host of zero modes occupying most of the parameter space. All eigenvalues are non-negative, proving that Parisi's solution is marginally stable.

In the spin-glass riddle the primary question to be settled is the stability of various "mean-field" solutions. The replica-independent Sherrington-Kirkpatrick[1] (SK) ansatz $q_{\alpha\beta}=q$, for the order-parameter matrix, was shown to be unstable by Almeida and Thouless[2] (AT), which led to a search of solutions breaking the replica symmetry. After several attempts[3–5] Parisi[6] proposed a most promising scheme where the order parameter q was replaced by a function $q(x)$ on the unit interval.[7] Its stability was investigated, near T_c, by Thouless, Almeida, and Kosterlitz[8] (TAK) who found it at best marginally stable. Their analysis was however confined to "longitudinal" fluctuations, whereas the space of fluctuations is vastly larger and besides, the most dangerous fluctuations lie in the "transverse" directions (i.e., into configurations with additional symmetry breaking). Indeed, the AT work has shown the existence of three families of eigenvalues: A unique $\lambda^{(1)}$ of order $\tau\equiv(T-T_c)/T_c$, a $(n-1)$ degenerate $\lambda^{(2)}$ (reducing to $\lambda^{(1)}$ as the replica number n vanishes), and a small *negative* $\lambda^{(3)}$, of order τ^2 and $n(n-3)/2$ degenerate. The corresponding eigenvectors $f_{\alpha\beta}$ [column vectors with $n(n-1)/2$ components, which can be conveniently thought of as real, symmetric matrices with zero diagonal elements] are $f^{(1)}_{\alpha\beta}$, a constant, i.e., a purely longitudinal vector; $f^{(2)}_{\alpha\beta}$, a two-valued matrix taking a constant value except on one arbitrarily distinguished row (and column) θ where it takes a different constant value; $f^{(3)}_{\alpha\beta}$, a three-valued matrix with two arbitrarily distinguished rows (columns) θ_1, θ_2. It turns out that TAK analysis amounts to a generalization of $f^{(1)}$, whereas AT work hints that the dangerous mode lies in the transverse direction ($f^{(3)}$ in particular).

In this work we follow the strategy used by Parisi who constructed the space of $q(x)$ by considering the limit of a discrete ansatz sequence $q^R_{\alpha\beta}$, $R=0,1,2,\ldots$. We construct equations for eigenvalues and eigenfunctions of the Hessian around the above sequence. We find that the generalization of $f^{(2)}$ ($f^{(3)}$) is a set of functions of two (three) variables. For simplicity we have kept to zero magnetic field and used the Parisi[9] approximation for the $q_{\alpha\beta}$ Lagrangian. The result exhibits *all eigenvalues as non-negative, which proves that the Parisi solution is marginally stable against all fluctuations,* longitudinal and transverse. The second family $\lambda^{(2)}(\kappa)$, $0\leqslant\kappa\leqslant 1$ is made of bands pinned on the TAK spectrum $\lambda^{(1)}\equiv\lambda^{(2)}(0)$. The third family is made of a zero mode on most of the parameter space, and of a continuum spanning the range $(0,2\tau^2)$.

(1) A very good representation, near T_c, of the SK free-energy functional, is given by the Parisi model

$$\frac{-f}{T}=\lim_{n\to 0}(2n)^{-1}\left[\tau\,\mathrm{tr}q^2+\tfrac{1}{3}\,\mathrm{tr}q^3+\left(\frac{y_p}{4}\right)\sum_{\alpha\neq\beta}q^4_{\alpha\beta}\right] , \quad (1)$$

where $\alpha,\beta=1,2\ldots n$. The eigenvalues of the Hessian of (1) are determined by

$$0=(\lambda+2\tau+3y_pq^2_{\alpha\beta})f_{\alpha\beta}+\sum_{\gamma\neq\alpha,\beta}(q_{\alpha\gamma}f_{\gamma\beta}+q_{\beta\gamma}f_{\gamma\alpha}) , \quad (2)$$

where in the following we keep the standard value $y_p=\frac{2}{3}$. The AT solution of (2) around the SK stationarity point now plays for us, in some loose sense, the role of an unperturbed problem in the search for a solution around the Parisi stationarity point. Switching in the "perturbation" (i.e., the replica symmetry breaking) lifts the high degeneracy, and rotates the AT eigenvectors, but these can still be constructed in close analogy with AT. The simplest matrices $f^{(1)}_{\alpha\beta}$ solving (2) have the same hierarchical structure as Parisi's $q^R_{\alpha\beta}$. Substituting this ansatz into (2) yields a system of linear equations which, in the continuous limit $R\to\infty$, goes over into precisely the TAK integral equation.[10] Its solutions have the same break-point $x_1=2\tau+O(\tau^2)$ as $q(x)$ [beyond which $q(x)=x_1/2$], i.e., $f(x)=-\sin\omega x$ for $x<x_1$ and $f(x)=$ constant for $x\geqslant x_1$, corresponding to the discrete set of eigenvalues $\lambda^{(1)}\equiv\omega^{-2}$ obtained from [11]

$$\cot\omega x_1=\omega(1-x_1) . \quad (3)$$

This spectrum contains one "large mass"

$\lambda^{(1)}=2\tau+\cdots$ and a sequence of small (but stable) masses

$$\lambda_m^{(1)}=\frac{4\tau^2}{m^2\pi^2}+\cdots, \quad m=1,2,\ldots \ . \tag{4}$$

(2) We need now enter deeper into Parisi's hierarchical block procedure.[6,10] Consider, for example, the α row index ($\alpha=1,2\ldots n$). It may be replaced by the sequence of hierarchical block numbers $(j_0,j_1,j_2\ldots j_R;a)$, where $j_0=1$, $j_1=1,2,\ldots n/m_1$, $j_2=1,2\ldots m_1/m_2,\ldots, j_R=1,2\ldots m_{R-1}/m_R$ and where $a=1,2\ldots m_R$ labels replicas in the smallest block R (m_l is the number of replicas in each of the m_{l-1}/m_l blocks l). Coordinates of a matrix element $q_{\alpha\beta}$ are now written with the two sequences $(j_0,j_1\ldots j_R;a)$ and $(l_0,l_1,\ldots l_R;b)$. Let us call *overlap* the *uninterrupted* sequence $j_0\equiv l_0$, $j_1=l_1,\ldots j_i=l_i$ but $j_{i+1}\neq l_{i+1}$, or rather the number i, $\alpha \bigcap_0 \beta=i$. We then have $q_{\alpha\beta}=q_i$. In the continuous limit where $m_i=i/(R+1)$, $R\to\infty$, and $m_i<x<m_i+1$, then $q_i\to q(x)$. Consider now an eigenvector $f_{\alpha\beta}^{(\theta)}$ of the 2nd family with one distinguished replica θ. We need now know, besides i (or x in the continuous limit), the overlaps $\theta \bigcap_0 \alpha=k^\alpha$, $\theta \bigcap_0 \beta=k^\beta$. It is easily observed that if those two numbers are equal $k^\alpha=k^\beta=k$, $k\leq i$, but if $k^\alpha\neq k^\beta$, $i=\min(k^\alpha,k^\beta)$, in which case the other number is $\max(k^\alpha,k^\beta)\equiv k$. The eigenvector is thus now dependent upon two variables i (or x, a measure of the distance to the diagonal) and k (or z, a measure of the distance to θ); $f(x;z)$. This type of ansatz leads to

$$0=-\lambda^{(2)}f(x;z)+2q(x)\int_x^1 dt\, f(t;z)+2\int_0^x dt\, q(t)f(t;z)-2q(z)\int_z^1 t\,dt\frac{\partial}{\partial t}f(z;t) \tag{5}$$

for $z<x$, and for $x<z$

$$\begin{aligned}0=&-\left[\lambda^{(2)}+xq(x)+\int_x^z dt\, q(t)\right]f(x;z)+q(x)\int_x^1 dt[f(t;x)+f(t;z)]\\&+\int_0^x dt\, q(t)[f(t;z)+f(t;x)]-\int_z^1 t\,dt\frac{\partial}{\partial t}[q(z)f(x;t)+q(x)f(z,t)]+xq(x)f(x;x)+\int_x^z dt\, q(t)f(x;t) \ .\end{aligned} \tag{6}$$

The TAK family is obviously included as a particular solution independent of z. A study of the discrete equations behind (5) and (6) shows that solutions $f(x;z)$ can be adequately parametrized by a breakpoint value κ on z, beyond which $f_\kappa(x;z\geq\kappa)$ is z independent, and that the eigenvalues can be obtained from the one-dimensional integral equation for $(\partial/\partial z)f_\kappa(x;z)|_{z=\kappa-0}\equiv F_\kappa(x)$.

For example, in the region $\kappa<x_1$, $F_\kappa(x)$ is a Gegenbauer function[12] for $0\leq x<\kappa$, a sine function with a phase shift for $\kappa\leq x<x_1$, and a constant for $x\geq x_1$ (note that F_κ has a jump as x crosses κ). Matching boundary conditions gives the eigenvalues spectrum $\lambda^{(2)}\equiv\lambda_\kappa$.

If $\kappa<x_1$, again, with $\lambda^{(2)}\equiv\omega^{-2}$, one has

$$L(\omega\kappa)=\frac{\tan\omega(x_1-\kappa)+\omega(1-x_1)}{1-\omega(1-x_1)\tan\omega(x_1-\kappa)} \ , \tag{7}$$

where

$$L(y)\equiv|4-y^2|^{-1/2}C'(\xi)/C(\xi)$$

and

$$\xi\equiv y|4-y^2|^{-1/2} \ ,$$

and $C(\xi)$ satisfies

$$(\xi^2-\epsilon)C''+4(\xi C'+C)=0 \ . \tag{8}$$

Here $\epsilon=\mathrm{sgn}(4-y^2)$, and the boundary conditions are $C(0)=0$, $C'(0)=1$. For $\epsilon=+1$ (8) is a Gegenbauer equation.[12] Solutions of (7) for eigenvalues $\lambda_m^{(2)}(\kappa)$ can be exhibited, for $\kappa\ll x_1/m$, as

$$\lambda_m^{(2)}(\kappa)=\lambda_m^{(1)}\left[1+\left(\tfrac{2}{3}\right)\pi^2m^2\left(\frac{\kappa}{x_1}\right)^3+\cdots\right] \ , \tag{9}$$

where $\lambda_m^{(1)}$ is given by (4), thus displaying bands pinned at $\lambda_m^{(1)}$. In the other extreme $\kappa\gg(x_1/m)$ (but always keeping $\kappa<x_1$) one gets

$$\lambda_m^{(2)}(\kappa)=\kappa^2/(3m\pi)^{2/3}+\cdots \ . \tag{10}$$

As κ increases and reaches the value x_1, (7) becomes

$$L(\omega x_1)=\omega(1-x_1) \ . \tag{11}$$

It keeps to that form for $\kappa>x_1$, and (11) only adds the large mass $\lambda^{(2)}\sim2\tau+\cdots$ to the 2nd family spectrum.

Note that *no negative eigenvalue* $\lambda^{(2)}=-\omega^{-2}$ is compatible with (7) or (11). Indeed, in that case, the right- and left-hand sides of (7) (11) remain of *opposite* signs. Since the original matrix is real symmetric no complex solution exists either.

Having solved for $\lambda^{(2)}(\kappa)$ and $F_\kappa(x)$, one can return to (5) and (6) and solve for $(\partial/\partial z)f_\kappa(x;z)$ [and $f_\kappa(x;z)$] thus obtaining the eigenvectors.

(3) Turning to the third family, we need now a complete information on the eigenvector $f_{\alpha\beta}^{(\theta_1\theta_2)}$ with two distinguished replicas θ_1, θ_2. This includes the following: (i) As above the $\alpha \bigcap_0 \beta$ overlap i; (ii) the overlaps with θ_1, $\theta_1 \bigcap_0 \alpha=k_1^\alpha$, $\theta_1 \bigcap_0 \beta=k_1^\beta$ and with θ_2, $\theta_2 \bigcap_0 \alpha=k_2^\alpha$, $\theta_2 \bigcap_0 \beta=k_2^\beta$; and (iii)

the overlap $\theta_1 \cap_0 \theta_2 = r$, an external parameter. Again $\max(k_j^\alpha;k_j^\beta) = k_j$, $j = 1,2$ and the domain of i is determined by $\min(k_j^\alpha,k_j^\beta)$ when $k_j^\alpha \neq k_j^\beta$, and/or by $i \geqslant k_j^\alpha = k_j^\beta$ when the k_j's are equal. Altogether, taking into account constraints of the hierarchical blocks geometry, and writing x, z_j, ρ for the continuous version of i, k_j, r one is left with only the following sectors:

(i) $f(x;z,z)$, for $z_1 = z_2 \equiv z < \rho$,

(ii) $f(x,z_1,\rho)$ and $f(x;\rho,z_2)$, for $\rho < z_1,z_2$,

where x varies between 0 and 1, and

(iii) $f(\rho;z_1,z_2)$, for $\rho < z_1,z_2$.

For these eigenvectors $f^{(3)}$ one can write a discrete set of coupled linear equations that, in the continuum limit, goes over into six coupled integral equations on sections of the unit cube. The system is fully symmetric in (z_1,z_2) and, as a whole, depends upon the external parameter $0 < \rho < 1$. It shrinks back onto Eqs. (5) and (6) whenever $f^{(3)}$ becomes independent of z_1 (or z_2). The gist of this "dangerous" family lies in sector (iii).

As above, solutions for the eigenvectors can be parametrized by a breakpoint κ (beyond which f is z independent). If $\kappa < \rho$, the system reduces to the second family (it is blind to any distinction between θ_1 and θ_2). If the breakpoint happens above ρ we could in principle introduce κ_1 and κ_2 (for z_1 and z_2, respectively). Assuming for the moment that the solutions are symmetric in (z_1,z_2) we are left with a single breakpoint. Letting $F_\kappa(z_1,z_2) \equiv (\partial^2/\partial z_1 \partial z_2) \times f_\kappa^{(3)}(\rho;z_1,z_2)$ one obtains an equation decoupled from the other five,

$$0 = [-\lambda^{(3)}(\kappa;\rho) + q^2(z_1) + q^2(z_2) - 2q^2(\rho)]F_\kappa(z_1,z_2) + \dot{q}(z_1)\int_{z_1}^1 t\,dt\,F_\kappa(t;z_2) + \dot{q}(z_2)\int_{z_2}^1 t\,dt\,F_\kappa(z_1;t) \ . \tag{12}$$

If we let $z_1 = z_2 = \kappa - 0$, then (12) yields either (i) $F_\kappa(\kappa-0;\kappa-0) = 0$ or (ii) $\lambda^{(3)}(\kappa;\rho) = 2[q^2(\kappa) - q^2(\rho)]$. In this last case one obtains, depending upon the relative position of x_1 with respect to ρ and κ,

$$\lambda^{(3)} = \begin{cases} 0, & x_1 < \rho < \kappa & (13) \\ \frac{1}{2}(x_1^2 - \rho^2), & \rho < x_1 < \kappa \ , & (14) \\ \frac{1}{2}(\kappa^2 - \rho^2), & \rho < \kappa < x_1 \ . & (15) \end{cases}$$

A study of the system of six coupled integral equations shows that, if $\lambda \neq \lambda^{(3)}$, the system folds back onto the second family [Eqs. (5) and (6)]. It also shows that (15) is not an admissible solution (it corresponds to an identically vanishing eigenvector). This is not a loss of any eigenvalues since (14) and (15) span the same interval $(0, 2\tau^2)$. The same remark applies to possible solutions with asymmetric breakpoints κ_1, κ_2 in sectors (ii) and (iii). Note that the soft mode (13) occupies most of the parameter space (except for a piece of order τ^2) and corresponds to fluctuations disturbing the flat core of $q(x)$. Although we did not mention it, such soft modes also exist in the two previous families.

(4) Equations (3), (7), (11), (13), and (14) give the complete spectrum of eigenvalues. This spectrum in *wholly non-negative,* proving that the Parisi long-range spin-glass solution (at least for the approximate Parisi model used here for simplicity) is *marginally stable.*

In addition to extracting the eigenvalue spectrum from (2) there are good reasons to solve it in full for the eigenvectors. Indeed, that would allow one to construct the one-loop correction to the tree approximation, that is the first correction in the reciprocal of the coordination number. That would reveal both the effect of short-range corrections, and presumably the lower critical dimension that has been the subject of much speculation.

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[*]On leave of absence from Institute for Theoretical Physics, Eötvös University, Budapest, Hungary.

[1]D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. 35, 1792 (1975).

[2]J. R. L. de Almeida and D. J. Thouless, J. Phys. A 11, 983 (1978).

[3]A. J. Bray and M. A. Moore, Phys. Rev. Lett. 41, 1068 (1978); A. Blandin, J. Phys. (Paris) 39, C6-1499 (1978).

[4]A. Blandin, M. Gabay, and T. Garel, J. Phys. C 13, 403 (1980).

[5]C. De Dominicis and T. Garel, J. Phys. Lett. 40, L575 (1979); A. J. Bray and M. A. Moore, J. Phys. 13, 419 (1980).

[6]G. Parisi, J. Phys. A 13, L115 (1980); 13, 1101 (1980); 13, 1887 (1980).

[7]More recent work of H. Sompolinsky [Phys. Rev. Lett. 47, 935 (1981)] parametrizes the solution to SK equations with, besides $q(x)$, an "anomaly" $\Delta(x)$. This formulation is, however, at least at the level of the tree approximation, equivalent to Parisi's [C. De Dominicis, M. Gabay, and B. Duplantier, J. Phys. A 15, L47 (1982); H. J. Sommers (unpublished)].

[8]D. J. Thouless, J. R. L. de Almeida, and J. M. Kosterlitz, J. Phys. C 13, 3272 (1980), Eq. (19).

[9]G. Parisi, Phys. Rev. Lett. 43, 1754 (1979).

[10]Equation (19) of Ref. 8.

[11]J. R. L. de Almeida, Ph.D. thesis (Birmingham, 1980) (unpublished).

[12]See, e.g., W. Magnus, F. Oberhettinger, and R. P. Soni, *Special Functions of Mathematical Physics* (Springer, New York, 1966).