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Comments

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Self-similarity and fractal dimension of the devil's staircase in the one-dimensional Ising model

R. Bruinsma
IBM T. J. Watson Research Center, Yorktown Heights, New York 10598

P. Bak
H. C. Ørsted Institute, DK-2100 Copenhagen Ø, Denmark
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The one-dimensional Ising model with long-range antiferromagnetic interaction in an applied field is known to exhibit a complete devil's staircase in its $T=0$ phase diagram. In this Comment we discuss its self-similar properties and determine the fractal dimension.

Recently,[1] Bak and Bruinsma determined the $T=0$ phase diagram of the one-dimensional Ising model with long-range antiferromagnetic interaction in an applied field H. This model has found applications in intercalated compounds, ferroelectrics, dipolar coupled antiferromagnets, and neutral-ionic transitions. As a function of the applied field the model passes through an infinity of commensurate phases with the periodicity assuming all commensurable values. It was the first time that such a structure, a devil's staircase, was proven to occur in a physical model although it had been speculated before for the Frenkel-Kontorova model.[2] In this Comment we will discuss the mathematical properties of the devil's staircase in view also of the recent attention[3] in applications of the concept of "fractal dimension"[4] in physical models.

The Hamiltonian is

$$\mathcal{H}=\sum_i HS_i+\frac{1}{2}\sum_{i,j}J(i-j)(S_i+1)(S_j+1) \quad , \qquad (1)$$

with $S_i=\pm 1$. For H large and negative, all spins are aligned and $S_i=1$. This phase becomes unstable at $H=-H_T$ where $H_T=4\sum_{i=1}^{\infty}J(i)$. For larger H, the magnetization $\langle S\rangle$ passes through an infinity of values: $\langle S\rangle=2q-1$ with q any rational number between 0 and 1 (Fig. 1). For H positive, $S_i=-1$. The stability interval for $q=m/n$ is[1]

$$\Delta H(q=m/n)=2\sum_{p=1}^{\infty}[pnJ(pn+1)+pnJ(pn-1)-2pnJ(pn)] \quad . \qquad (2)$$

If $J(i)$ is convex, then ΔH is positive and the phase diagram is a devil's staircase. The most stable intermediate phase is the antiferromagnet with $\langle S\rangle=0$. The importance of the other phases may be measured by the "fractal dimension" D_F of the collection of H values where a phase transition occurs. More precisely, if in an experiment one would measure a phase diagram such as Fig. 1 with a precision ΔH and observe $N_{\Delta H}$ phase transitions, then

$$D_F=\lim_{\Delta H\to 0}\ln(N_{\Delta H})/\ln(H_T/\Delta H) \quad . \qquad (3)$$

For a very-short-range interaction, only the $\langle S\rangle=0$ antiferromagnet occurs, and so $D_F=0$. For infinite-range interaction, e.g., $J(n)=1/n$, we would expect $D_F=1$. Another important characteristic of a devil's staircase is its self-similarity illustrated in Fig. 1. If we expand the scale of a small section ΔH of the phase diagram and reproduce the large-scale phase diagram exactly, then we would call it "self-similar," rather like scaling at a second-order phase transition. We will now calculate D_F and discuss the self-similarity assuming an interaction $J(|i|)=1/|i|^{\alpha}$.

The devil's staircase maps all rational numbers onto the real axis. We first discuss the subset of all integer multiples of $1/2^k$. The stability interval ΔH for a phase with $q=m/n$ is, from Eq. (2),

$$\Delta H(m/n)\simeq\gamma/n^{\alpha+1} \quad , \qquad (4)$$

where $\gamma=2\alpha(\alpha+1)\zeta(\alpha+1)$ and $\zeta(x)$ is the Riemann ζ function. With $n=2^k$ we calculate the stability intervals for $k=1,2,3$. For $k=1$, we cut out $\Delta H(\frac{1}{2})$ from $H_T(2)$. $H_T(2)$ is the combined stabili-

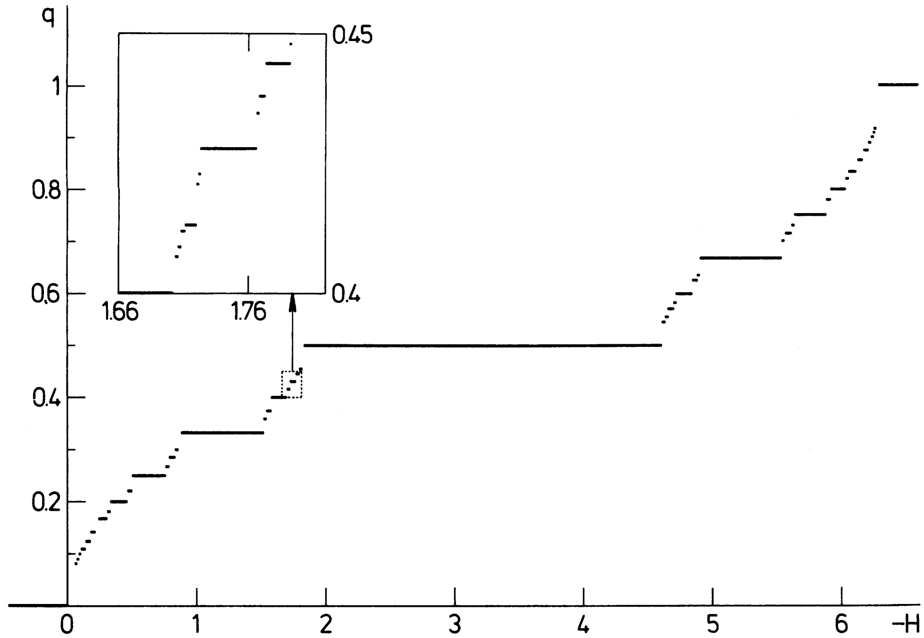

FIG. 1. Devil's staircase. The ratio of up spins over the total number of spins $q\,[=\frac{1}{2}(\langle S\rangle+1)]$ vs the applied field H for an interaction $J(i)=1/i^2$. Inset: the area in the square magnified ten times.

ty interval of all phase with q integral multiple of $1/2^k$. For $k=2$, we cut out $\Delta H(\frac{1}{4})$ two times from the remaining part of $H_T(2)$; and for $k=3$, we cut out $\Delta H(\frac{1}{8})$ four times from the remainder, etc. Since

$$\frac{\Delta H(\frac{1}{2})}{H_T(2)}=\frac{\Delta H(\frac{1}{4})}{\frac{1}{2}[H_T(2)-\Delta H(\frac{1}{2})]}=\frac{\Delta H(\frac{1}{8})}{\frac{1}{4}[H_T(2)-\Delta H(\frac{1}{2})-2\Delta H(\frac{1}{4})]}=1-(\tfrac{1}{2})^{\alpha}\ , \tag{5}$$

we always are cutting out the same fraction of the remaining part of $H_T(2)$ and the construction is self-similar. It is, in fact, a variety of Cantor's devil's staircase[4] and

$$D_F=\ln 2/\ln[\Delta H(\tfrac{1}{2})/\Delta H(\tfrac{1}{4})]=1/(1+\alpha) \tag{6}$$

from Eq. (3). Next we turn to integral multiples of $1/3^k$. This is again a self-similar construction with fractal dimension $D_F=1/(1+\alpha)$. In general, with every prime number p we can associate a devil's staircase with q assuming integral multiples of $1/p^k$ and fractal dimension $D_F=1/(1+\alpha)$. Each of these subsets is self-similar. However, since each requires a different scaling factor, their combination is not. To find the fractal dimension of the complete devil's staircase, we argue as follows: The number of rational numbers with denominator less than M is proportional to M^2. The precision scale needed to observe steps with $q=N/M$ is $M^{1+\alpha}$ from Eq. (4). Thus the fractal dimension is from Eq. (3):

$$D_F=\ln M^2/\ln M^{1+\alpha}=2/(1+\alpha)\ . \tag{7}$$

A numerical check on Eq. (3) using Eq. (2) reveals that D_F weakly depends on ΔH with a mean value in good agreement with Eq. (7). This was expected since the devil's staircase is not strictly self-similar.

In conclusion, the devil's staircase of the long-range antiferromagnetic Ising model in a uniform field is not self-similar but consists of an infinite collection of self-similar subsets. An "average" fractal dimension was found which gives a measure of the importance of the additional intermediate phases.

[1]P. Bak and R. Bruinsma, Phys. Rev. Lett. 49, 249 (1982).
[2]S. Aubry, in *Solitons and Condensed Matter Physics*, edited by A. R. Bishop and T. Schneider (Springer, London, 1979), p. 264.
[3]Y. Gefen, A. Aharony, B. Mandelbrot, and S. Kirkpatrick, Phys. Rev. Lett. 47, 1771 (1981), and references therein.
[4]B. Mandelbrot, *Fractals: Form, Chance and Dimension* (Freeman, San Francisco, 1977).