Critical exponent ν for generalized surface fluctuations

M. J. Lowe and D. J. Wallace
Physics Department, University of Edinburgh, Edinburgh EH9 3JZ, Scotland
(Received 8 June 1982)

We study the field-theoretic model of a $(d-n)$-dimensional interface fluctuating in a bulk system of d dimensions. With the use of an extension of the effective potential method of Forster and Gabriunas, the critical exponent ν which controls the local thickness of the interface is obtained to third order in $(d-n)$. The implications for the interpretation of such surface models and for the description of systems containing such excitations are reviewed.

This paper is concerned with fluctuating (hyper) surfaces and their implications for the critical properties of thermodynamic systems which contain them. The most familiar type of fluctuating membrane is a $(d-1)$-dimensional surface fluctuating in a d-dimensional bulk and arising in Ising-type systems, where it represents the interface between adjacent regions of "up" and "down" spins or between liquid and vapor phases in a fluid. We are interested in this paper in the generalization to surfaces of $d-n$ dimensions which fluctuate into the remaining n dimensions of the d-dimensional bulk. In Ref. 1 the description of such surfaces is reviewed starting from topologically stable localized solutions of the Euler-Lagrange equations (such as vortices) in Landau-Ginzburg models; finite-energy solutions in n dimensions give rise to finite surface tension. Abrikosov vortices in superconductors[2] and Z_2 lattice gauge theories are examples of the case $n=2$. Surfaces corresponding to more general n arise in the generalized Z_2 gauge theories described by Wegner[3]; the Ising model is the particular case $n=1$. More recently, it has been suggested[4] that the statistical mechanics of vortex loops in a superconductor in three dimensions are dual to the loops arising in the high-temperature series expansion of the XY model,[5] and hence that these two systems exhibit the same kind of phase transition (with low- and high-temperature phases interchanged).

Our approach is to study in perturbation theory the fluctuations from a minimal flat surface. This is basically a low-temperature approximation. However, by means of the renormalization group it can be used to obtain a description of how surface fluctuations can build up an intrinsic local thickness in the surface as a critical temperature T_c is approached from below, provided that T_c is small, i.e., the system is close to its lower critical dimension. By this means, interface fluctuations can give a description of the Ising-model critical behavior in $1+\epsilon$ dimensions,[6] with $T_c=O(\epsilon)$; the critical exponent ν arises from the fractal effect of these surface fluctuations[7] and the exponent β is controlled by the fractal effect of droplets.[8] These calculations can be extended[9] to describe, in $n+\epsilon$ dimensions, critical behavior associated with generalized surfaces of dimension ϵ (i.e., of dimension n less than the bulk).

In this paper we report the results of using, for general n, the efficient techniques of Forster and Gabriunas,[10] who calculated to order ϵ^4 for the Ising case $n=1$. Our primary aim is to try to elucidate the interpretation of the critical points of such models. We first review briefly the calculation and results and return then to the question of interpretation.

If we parametrize a flat minimal surface by $d-n$ $(=\epsilon)$ coordinates x, then small transverse displacements of the surface from flat are described by n fields $f^a(x)$ $(a=1,\ldots,n)$. The effective energy for these fluctuations is governed by surface area

$$\mathcal{H}=\frac{1}{T_0}\int d^\epsilon x\,(\det g)^{1/2}\ , \tag{1}$$

where

$$g_{ij}=\delta_{ij}+\frac{\partial f^a}{\partial x_i}\frac{\partial f^a}{\partial x_j}\ .$$

The evaluation of correlation functions of f, etc., by perturbation expansion of $\mathcal{H}$ in powers of f gives a power series in the "bare" temperature parameter T_0. The critical behavior is obtained by introducing a dimensionless renormalized temperature T, which depends, for fixed T_0, on the length scale R at which it is defined according to[9]

$$R\frac{dT}{dR}=\beta(T)=-\epsilon T+nT^2+O(T^3)\ .$$

(The β function here is defined with opposite sign convention to that in Refs. 6, 9, and 10.) The fixed point $T_c=\epsilon/n+O(\epsilon^2)$ can be interpreted as a critical point at which the correlation length measuring the intrinsic width of the interface diverges with the critical exponent $\nu=1/\beta'(T_c)$.

Forster and Gabriunas[10] noted two points which considerably simplify this calculation. First, it is particularly useful to calculate the effective potential[11] for $\partial_i f(x) \equiv M_i$; this gives the generating functional for all vertex functions of the field $f(x)$ and, in order to study all necessary renormalizations, it is sufficient to work with a vector M_i which is independent of x. In the presence of a mass term $\frac{1}{2}m^2 f^2(x)$, a Ward identity[10] establishes the perturbative dependence of the effective potential on M_i in such a way that it is sufficient to perform the calculation at $M_i = 0$, i.e., only diagrams with no external legs need be calculated. Second, since at $M_i = 0$ all self-energy corrections to $\langle f^2 \rangle$ vanish as $\epsilon \rightarrow 0$, it follows that a derivative of the effective potential with respect to the mass must for $M_i = 0$ give *zero* to all orders T^p $(p > 1)$ in the limit $\epsilon \rightarrow 0$. This means that the singular part of the effective potential at order T^p is determined from the coefficients of lower order in T.

This approach can be extended to the case of general n as follows. One calculates the effective potential for $\partial_i f^a(x) = M_i^a$, taking M_i^a as x independent. A mass term is again necessary to control infrared divergences due to long wavelengths. In order to make the calculations more tractable, it is convenient to add to Eq. (1) a mass term of the form

$$\frac{1}{2}\frac{m^2}{T}\mu \int d^\epsilon x\, f^a(x)\Delta^{ab} f^b(x) \quad , \tag{2}$$

where

$$\mu = [\det(\delta_{ij} + M_i^a M_j^a)]^{1/2}$$

and

$$(\Delta^{-1})^{ab} = \delta^{ab} + M_i^a M_i^b \ .$$

With this choice of mass term it is tedious but practicable to calculate to two loops the singular part of the effective potential for any M_i^a and to obtain the coupling-constant (T) renormalization required.

The second observation of Forster and Gabriunas pertains also to the mass term (2) in the general case, and one may hence deduce the form of the singular part of the effective potential at third order from the known (including finite part) lower-order terms for $M = 0$. The result is

$$\beta(T) = -\epsilon T + nT^2 + \tfrac{1}{2}n^2 T^3 + \tfrac{1}{8}(3n-2)n^2 T^4 + O(T^5) \ .$$

This yields the critical exponent

$$\nu \equiv \frac{1}{\epsilon}\left[1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{2n} + O(\epsilon^3)\right] \ . \tag{3}$$

This result for ν is consistent with the leading order in $(1/n)$ expansion[9] and with Ref. 10 for the special case $n = 1$.

By exploiting the approach of Ref. 10 we are hence able to improve the understanding of the renormalizability of the model (1) and to extend previous results from it. We consider now the implications of the result (3).

For the case $n = 2$, (3) purports to give an expansion in $2 + \epsilon$ dimensions for the critical exponent ν for a thermodynamic system where geometrical excitations (vortex lines in three dimensions) are important. There are at least two potential candidates, the superconductor[2] and Z_2 gauge theory.[3] The prediction according to Eq. (3) of a continuous phase transition in $(2+\epsilon)$ dimensions does not contradict the first-order superconducting transition predicted[12,13] in $4-\epsilon$ dimensions; other examples of a change from second to first order between two and four dimensions are known.[14] The second and third terms in Eq. (3) are, however, incompatible with the ϵ expansion in $(2+\epsilon)$ dimensions obtained from the CP^{N-1} (complex projective) models.[15] Since by arguments based on their gauge symmetry and by explicit calculation in the $1/N$ expansion, the CP^{N-1} models are believed to be in the same universality class as U(1) gauge theories; this would rule out the possibility that (3) describes the phase transition to superconductivity. Apparently, the absence of any shadow of the U(1) gauge symmetry in (1) is important.

We are therefore restricted to the interpretation of (3) in terms of the generalized Z_2 models of Wegner[3] as suggested previously.[9] Since the Z_2 gauge theory corresponds to $n = 2$, and is dual in three dimensions to the lattice Ising model,[3] we are left with the amusing conjecture that Eq. (3) may provide an ϵ expansion in $2+\epsilon$ dimensions which describes the three-dimensional Ising model $(\epsilon = 1)$. The utilization of such a series for $\epsilon = 1$ is, of course, problematical. In particular, similar duality arguments would imply that the exponent ν for $\epsilon = 1$ should be for all n that of the Ising model in dimension $d = n + \epsilon \equiv n + 1$; we would be led therefore to expect the mean-field value $\nu = \frac{1}{2}$ for all $n > 3$. Naive resummations cannot be expected to reproduce both this value and a nontrivial ν for $n = 1$ or 2.

ACKNOWLEDGMENTS

We thank D. Forster for useful correspondence confirming the correctness of Eq. (3). This research was supported in part by the SERC under Grant No. GR/A81768.

[1]D. J. Wallace in *Proceedings of the Twenty First Scottish Universities Summer School in Physics,* edited by K. C. Bowler and D. G. Sutherland (SUSSP Publications, University of Edinburgh, Scotland, 1981).

[2]A. A. Abrikosov, Zh. Eksp. Teor. Fiz. 32, 1441 (1957) [Sov. Phys. JETP 33, 111 (1958)].

[3]F. J. Wegner, J. Math. Phys. (N.Y.) 12, 2259 (1971).

[4]T. Banks, R. J. Myerson, and J. Kogut, Nucl. Phys. B129, 493 (1977); P. R. Thomas and M. Stone, *ibid.* 144, 513 (1978); M. E. Peskin, Ann. Phys. (N.Y.) 113, 122 (1978).

[5]W. Helfrich and W. Müller, *Continuum Models of Discrete Systems* (University of Waterloo, Ontario, 1980); C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. 47, 1556 (1981).

[6]D. J. Wallace and R. K. P. Zia, Phys. Rev. Lett. 43, 808 (1979).

[7]F. David, Phys. Lett. 102B, 193 (1981).

[8]A. D. Bruce and D. J. Wallace, Phys. Rev. Lett. 47, 1743 (1981).

[9]M. J. Lowe and D. J. Wallace, Phys. Lett. 93B, 433 (1980).

[10]D. Forster and A. Gabriunas, Phys. Rev. A 24, 598 (1981).

[11]See, for example, D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (McGraw-Hill, New York, 1978), p. 91 *et seq.*

[12]B. I. Halperin, T. C. Lubensky, and S.-k. Ma, Phys. Rev. Lett. 32, 292 (1974).

[13]Duality arguments, in fact, support a continuous superconducting transition in three dimensions equivalent to that in the *XY* model (see Refs. 4 and 5).

[14]See, for example, A. J. Paterson, Nucl. Phys. B190 [FS3], 188 (1981), and references therein.

[15]S. Hikami, Prog. Theor. Phys. 62, 226 (1979).