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Exact identities for both single- and two-component nonuniform Fermi systems

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The Ward-Pitayevski identities are extended to two-component fermion systems. Additional sum rules are derived for higher-order density-density response functions in both uniform and nonuniform single- and two-component systems. Possible applications of these relations are briefly considered.

Sum rules and other associated identities have played a major role in fermion systems by both providing exact results as well as a guide for appropriate approximations.[1,2] These identities, however, have been largely restricted to one fermion Green's function (via the Ward identities) or density-density pair correlation functions (via frequency-moment sum rules). The advent of more powerful probes with their corresponding multiexcitations,[3] the highly doped semiconductors with multivalley electrons,[4] and the recent interest in the electron-hole liquid[5] all make the extension of such identities particularly useful. In this paper we wish to list such exact relations for two-component single-particle Green's functions as well as provide sum rules for higher-order density-density correlation functions for single-component, as well as two-component, nonuniform fermion systems.

I. WARD IDENTITIES IN MULTICOMPONENT SYSTEMS

For itinerant ferromagnets (a *two*-component spin system) various relations between the one-particle self-energy M and the three-point vertex function Λ (so-called Ward-Takahashi[6,7] identities) have been discussed and applied elsewhere.[8] Here we present the extension to multicomponent single-particle Green's function of another set of relations, the Ward-Pitayevski[9] (WP) identities. We restrict our presentation to two-component systems; the generalization to a system of any number of components is straightforward and will be presented elsewhere.

Defining the two components as 1 and 2, the intraparticle interactions as v_{11} and v_{22} ($v_{11}=v_{22}$), and the interparticle interaction as v_{12} (either $v_{12}=\pm v_{11}$) (identities where the interactions are restricted to be of long range will be specified; see below), the following set of WP identities can be derived (see Fig. 1):

$$\Lambda^0_{11}(k,\omega)=1-\frac{\partial M_{11}(k,\omega)}{\partial\omega},\ \Lambda^0_{12}(k,\omega)=0, \tag{1}$$

$${}_{\alpha}\Lambda^{\infty}_{11}(k,\omega)=\frac{k\alpha}{m_1}+\frac{\partial M_{11}(k,\omega)}{\partial k\alpha},\ \ {}_{\alpha}\Lambda^{\infty}_{12}(k,\omega)=0, \tag{2}$$

$${}_{\alpha}\Lambda^{0}_{11}(k,\omega)=\frac{k\alpha}{m_1}+\frac{dM_{11}(k,\omega)}{dk\alpha},\ \ {}_{\alpha}\Lambda^{0}_{12}(k,\omega)=0, \tag{3}$$

and

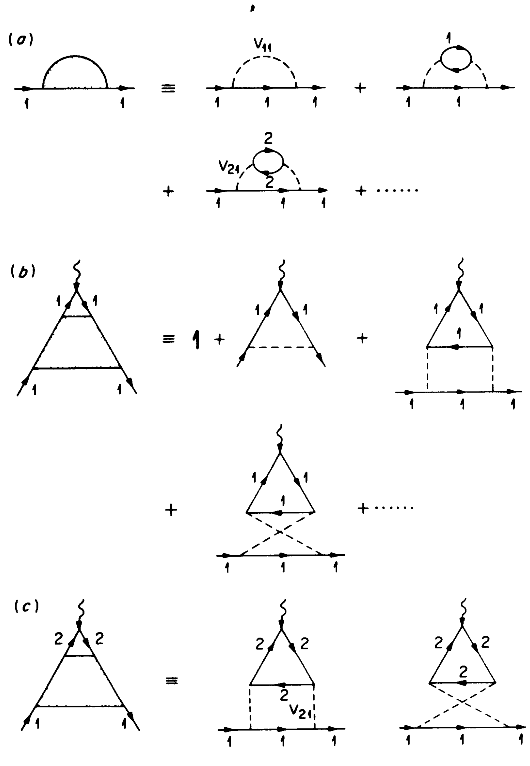

FIG. 1. Low-order contributions to (a) M_{11}, (b) Λ_{11}, (c) Λ_{12}.

$$\Lambda_{11}^{\infty}(k,\omega)=1-\frac{\partial M_{11}(k,\omega)}{\partial\omega}-\frac{1}{D}\left[\left[\frac{\partial\mu_1}{\partial n_1}\right]^2-\left[\frac{\partial\mu_1}{\partial n_2}\right]^2\right]\frac{\partial M_{11}(k,\omega)}{\partial\mu_1}-\frac{1}{D}\left[\frac{\partial\mu_2}{\partial n_1}\frac{\partial\mu_1}{\partial n_1}-\frac{\partial\mu_2}{\partial n_2}\frac{\partial\mu_1}{\partial n_2}\right]\frac{\partial M_{11}(k,\omega)}{\partial\mu_2}\,, \tag{4a}$$

$$\Lambda_{12}^{\infty}(k,\omega)=-\frac{1}{D}\left[\frac{\partial\mu_1}{\partial n_2}\frac{\partial\mu_2}{\partial n_2}-\frac{\partial\mu_1}{\partial n_1}\frac{\partial\mu_1}{\partial n_2}\right]\frac{\partial M_{11}(k,\omega)}{\partial\mu_1}-\frac{1}{D}\left[\left[\frac{\partial\mu_2}{\partial n_2}\right]^2-\left[\frac{\partial\mu_2}{\partial n_1}\right]^2\right]\frac{\partial M_{11}(k,\omega)}{\partial\mu_2}\,, \tag{4b}$$

where

$$D=\frac{\partial\mu_1}{\partial n_1}\frac{\partial\mu_2}{\partial n_2}-\left[\frac{\partial\mu_1}{\partial n_2}\right]^2 .$$

In Eqs. (1)—(4) μ_1, n_1, m_1, μ_2, n_2, and m_2 are the fully interacting chemical potentials, densities, and free masses of each component (more precisely μ_1 is the energy change upon the addition of *one* of the particles). The superscripts 0 and ∞ are taken from Ref. 1 (p. 268) and correspond to the order of the limit in q/ϵ. Finally, in Eq. (3) the full derivative is similar to Ref. 1 only now it includes both chemical potentials μ_1 and μ_2. Equations (1)—(3) follow immediately the analysis in Ref. 1. Equation (4), however, the most often used WP identity, is harder to derive. As in the case of the single component system, the above relations provide important constraints on the relations between the three-point vertex functions and the corresponding self-energy. Transport properties in, for example, the electron-hole liquid is one example where such relations become invaluable. A brief application will be considered shortly.

II. DENSITY-DENSITY RESPONSE FUNCTIONS

The one-component density-density response function (see Fig. 2) obeys several celebrated frequency and momentum sum rules.[2,10] Here we extend these relations to multicomponent systems. Define

$$\chi_{i,j}(\vec{q}_1,\vec{q}_2,\omega)$$

$$=\frac{-i}{\Omega}\int_{-\infty}^{+\infty}dt\,e^{i\omega t}\langle 0\,|\,T\hat{\rho}_i(\vec{q}_1,t)\hat{\rho}_j(\vec{q}_2,0)\,|\,0\rangle \tag{5}$$

with Ω the volume, $\hat{\rho}_i(\vec{q},t)$ the density operator of $i=1$ or 2, and T, the usual time-ordering operator. Following Ref. 10, we get the famous f sum rule[2] generalized to multicomponents:

$$\int_0^{\infty}\frac{d\omega}{\pi}\omega\,\mathrm{Im}\chi_{i,j}(\vec{q}_1,\vec{q}_2,\omega)$$

$$=\frac{\delta_{i,j}}{\Omega 2m_i}\vec{q}_1\cdot\vec{q}_2\rho_i(\vec{q}_1-\vec{q}_2), \tag{6}$$

where $\rho_i(\vec{q})$ is the Fourier transform of the ith component's density (we set $\hbar=1$). The result is valid for any *nonuniform* multicomponent fermion system with *arbitrary* velocity independent $v_{i,j}$ and for all $\vec{q}_1$ and $\vec{q}_2$.

For a *uniform system* $\chi_{i,j}$ can be written in terms of irreducible response functions $\pi_{i,j}$ (Ref. 11) (see Fig. 2). Equation (6) then implies

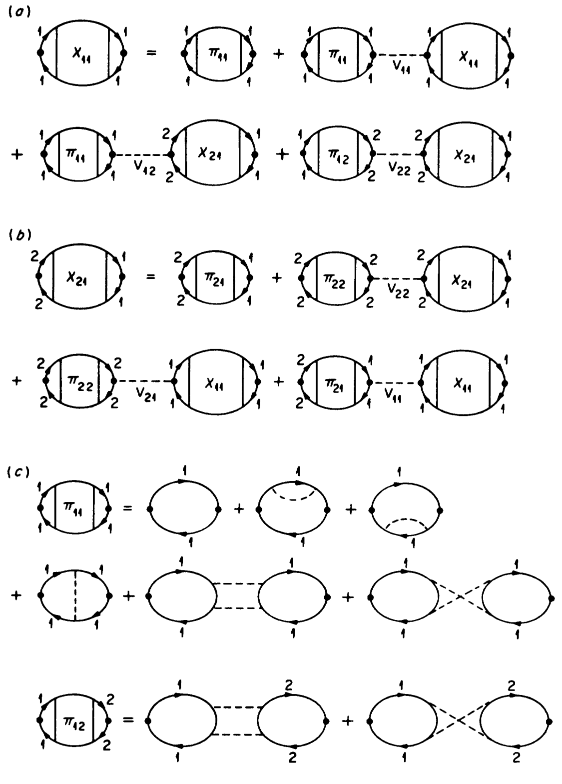

FIG. 2. (a) and (b) The graphical representation for the integral equation of the reducible susceptibility χ_{ij} in terms of its irreducible components π_{ij}. (c) Low-order contributions to π_{ij}.

$$\int_0^{\infty}\frac{d\omega}{\pi}\omega\chi_{i,j}(q,\omega)=\int_0^{\infty}\frac{d\omega}{\pi}\omega\,\mathrm{Im}\left[\frac{\pi_{i,j}(q,\omega)+v_{i,j}(q)\Delta(q,\omega)}{\epsilon(q,\omega)}\right]\quad\text{when } i=j$$

$$=\int_0^{\infty}\frac{d\omega}{\pi}\omega\,\mathrm{Im}\left[\frac{\pi_{i,j}(q,\omega)-v_{i,j}(q)\Delta(q,\omega)}{\epsilon(q,\omega)}\right]\quad\text{when } i\neq j$$

$$=\frac{\delta_{i,j}}{2m_i}q^2 n_i\ , \tag{7}$$

where $\Delta(q,\omega)=\pi_{11}(q,\omega)\pi_{22}(q,\omega)-\pi_{12}(q,\omega)\pi_{21}(q,\omega)$ and $\epsilon(q,\omega)=1+v_{11}(q)\pi_{11}(q,\omega)+v_{22}(q)\pi_{22}(q,\omega)+v_{12}(q)[\pi_{12}(q,\omega)+\pi_{21}(q,\omega)]$. A similar sum rule with anisotropic masses of the particles has been applied to Raman cross sections in highly doped semiconductors (Ref. 3, p. 298).

If we next restrict the interactions to long range [i.e., $v_{11}(q)=v_{22}(q)=4\pi e^2/\kappa q^2$ and $v_{12}=\pm 4\pi e^2/\kappa q^2$; the $\pm$ corresponds to equal or oppositely charged components and the κ scales the charge, e.g., in semiconductors], then the following sum rule can be derived for the structure factor $S_{\pm}(q)$:

$$\lim_{q\to 0}S_{\pm}(q)\equiv\lim_{q\to 0}[S_{11}(q)+S_{22}(q)\pm S_{12}(q)\pm S_{21}(q)]$$

$$\equiv\int_0^{\infty}\frac{d\omega}{\pi}\lim_{q\to 0}\mathrm{Im}[\chi_{11}(q,\omega)+\chi_{22}(q,\omega)\pm\chi_{12}(q,\omega)\pm\chi_{21}(q,\omega)]$$

$$=\lim_{q\to 0}\frac{\kappa q^2}{8\pi e^2}\omega_p^2\ , \tag{8}$$

where

$$\omega_p^2=\omega_{p,1}^2+\omega_{p,2}^2=\left[\frac{4\pi e^2}{\kappa}\right]\left[\frac{n_1}{m_1}+\frac{n_2}{m_2}\right].$$

[Equation (8) depends on $\epsilon(q,0)>0$ for all q.[2]] Note that the change of interparticle interaction from $+v_{12}$ to $-v_{12}$ not only changes the sign of χ_{12} but of course its internal structure as well.

There is no equivalent sum rule for the individual components S_{11}, S_{22}, or S_{12} since these can be seen [from Eq. (7)] not to be plasmon dominated at small q but are led by particle excitations (in fact, $\lim_{q\to 0}[S_{11}(q)$, $S_{22}(q)$, or $S_{12}(q)]\approx q$). Finally, Eq. (8) is not available for single- or multicomponent *nonuniform* systems as discussed in Ref. 12.

The static irreducible response functions of Eq. (7) satisfy individually compressibility sum rules.[2] Following the derivation of Ref. 2, we get

$$\lim_{q\to 0}\lim_{\omega\to 0}\pi_{11}(q,\omega)=\frac{1}{D}\frac{\partial\mu_1}{\partial n_1},\quad \pi_{12}(q,\omega)=\frac{-1}{D}\frac{\partial\mu_1}{\partial n_2}\ . \tag{9}$$

A similar sum rule has been discussed in multicomponent classical liquids.[13]

III. HIGHER-ORDER DENSITY CORRELATION FUNCTIONS

Unlike the fermion self-energy M or the density-density correlation function χ, sum rules for higher-order density response functions have not been as readily exploited, even in single-component systems. Exact identities have been almost exclusively confined to the static limit[14] (see below). The dynamic third-order correlation function has been studied largely via its spectral decomposition and corresponding symmetry and analytic properties.[15–17] Here we list some additional identities for these correlation functions in different region of frequency $\omega_1,\omega_2,\ldots,\omega_n$ (see Fig. 3). Consider the function $\Gamma_{\alpha,n-\alpha}$ given by

$$\Gamma_{\alpha,n-\alpha}(\vec{q}_1,\omega_1;\vec{q}_2,\omega_2,\ldots,\vec{q}_n,\omega_n)\delta(\omega_1+\omega_2+\cdots+\omega_n)$$

$$=\int_{-\infty}^{+\infty}dt_1e^{i\omega_1t_1}\int_{-\infty}^{+\infty}dt_2e^{i\omega_2t_2}\cdots\int_{-\infty}^{+\infty}dt_ne^{i\omega_nt_n}\frac{i^{n+1}}{\Omega}\left\langle 0\left|T\prod_{\beta_1=0}^{\alpha}\hat{\rho}_1(\vec{q}_{\beta_1},t_{\beta_1})\prod_{\beta_2=0}^{n-\alpha}\hat{\rho}_2(\vec{q}_{\beta_2},t_{\beta_2})\right|0\right\rangle \tag{10}$$

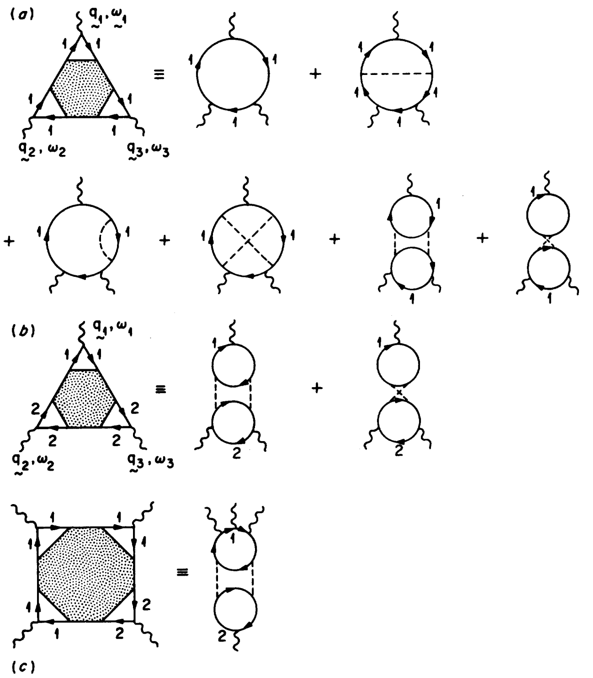

FIG. 3. (a) Examples of third-order Γ_{111} density response functions. The wavy lines are the screened external potentials without the last bare interactions, the multicomponent Fermi system need not be uniform. The reducible graph, with the wavy lines is the Γ_{111}; without the wavy lines is the irreducible vertex function $\tilde{\Lambda}_{111}$. (b) Example of third-order density response functions Γ_{122}. (c) Example of fourth-order (i.e. $n=4$) $\Gamma_{\alpha,n-\alpha}$ with $\alpha=1$.

(i.e., the usual nth-order time-ordered density correlation function with α first-component and $n-\alpha$ second-component contributions). We next center on the triple density-density correlation function [i.e., $n=3$ in Eq. (10); see Fig. 3]. Define a characteristic excitation of the system as $E_{\max}$, then for ω_1, ω_2, and ω_3 all $\gg E_{\max}$ time-reversal symmetry alone[16] gives

$$\Gamma_{\alpha,3-\alpha}=O\left[\frac{1}{\omega_1\omega_2\omega_3^2}+\frac{1}{\omega_1\omega_3\omega_2^2}+\frac{1}{\omega_3\omega_2\omega_1^2}\right].$$

But the more interesting ranges of frequencies ω_1, ω_2, and ω_3 also yield exact identities. For $\omega_1=0$ and $\omega_2 \gg E_{\max}$,

$$\Gamma_{111}(\vec{q}_1,0;\vec{q}_2,\omega_2;\vec{q}_3,\omega_3)=\frac{\vec{q}_2\cdot\vec{q}_3}{\omega_2^2 m_1}\chi_{11}(\vec{q}_2+\vec{q}_3,\vec{q}_1,0)+O\left[\frac{1}{\omega_2^4}\right], \tag{11a}$$

$$\Gamma_{122}(\vec{q}_1,0;\vec{q}_2,\omega_2;\vec{q}_3,\omega_3)=\frac{\vec{q}_2\cdot\vec{q}_3}{\omega_2^2 m_2}\chi_{12}(\vec{q}_2+\vec{q}_3,\vec{q}_1,0)+O\left[\frac{1}{\omega_2^4}\right], \tag{11b}$$

where χ_{ij} are defined in Eq. (5). [The notation $\Gamma_{111},\Gamma_{122}$ is a bit more explicit than in Eq. (10).] The result of Eq. (11) is in a loose way similar to the asymptotic behavior of the one-particle Green's function (see Ref. 1, Appendix C) and arises from the commutation relation of the Hamiltonian with $\hat{\rho}_i(\vec{q})$. It is valid for arbitrary *multi*component *nonuniform* fermion systems with velocity-independent v_{ij} and any finite $\vec{q}_1$, $\vec{q}_2$, and $\vec{q}_3$. Of course in a single-component system only Eq. (11a) survives. To our knowledge, such a form has not been considered for *either* single- or two-component systems. Possible importance of such identities to higher-order photon-phonon processes in both metals as well as doped semiconductors will be considered below.

For uniform multicomponent systems the irreducible function $\tilde{\Lambda}$ (see Fig. 3) follows from Eq. (11), i.e.,

$$\tilde{\Lambda}_{111}(\vec{q}_1,0;\vec{q}_2,\omega_2;\vec{q}_3,\omega_3)=\frac{\vec{q}_2\cdot\vec{q}_3}{\omega_2^2 m_1}\Pi_{11}(\vec{q}_1,0)+O\left[\frac{1}{\omega_2^4}\right], \tag{12a}$$

$$\tilde{\Lambda}_{122}(\vec{q}_1,0;\vec{q}_2,\omega_2;\vec{q}_3,\omega_3)=\frac{\vec{q}_2\cdot\vec{q}_3}{\omega_2^2 m_2}\Pi_{12}(\vec{q}_1,0)+O\left[\frac{1}{\omega_2^4}\right]. \tag{12b}$$

For long-range interactions v_{ij} another range of frequencies and momentum permits a rigorous identity; it is $\omega_2 \gg E_{\max}$, $\omega_3 \gg E_{\max}$, and ω_1 finite $\vec{q}_1 \to 0$. Then

$$\begin{aligned}\lim_{\vec{q}_1\to 0}\{&m_1[\tilde{\Lambda}_{111}(\vec{q}_1,\omega_1;\vec{q}_2,\omega_2;\vec{q}_3,\omega_3)\pm\tilde{\Lambda}_{211}(\vec{q}_1,\omega_1;\vec{q}_2,\omega_2;\vec{q}_3,\omega_3)]\\ &+m_2[\tilde{\Lambda}_{222}(\vec{q}_1,\omega_1;\vec{q}_2,\omega_2;\vec{q}_3,\omega_3)\pm\tilde{\Lambda}_{122}(\vec{q}_1,\omega_1;\vec{q}_2,\omega_2;\vec{q}_3,\omega_3)]\}\\ &=\lim_{\vec{q}_1\to 0}\left\{\frac{\vec{q}_2\cdot\vec{q}_3 2\omega_p}{\omega_1^2\omega_2^2}S_{\pm}(\vec{q}_1)+\left[\frac{\vec{q}_1\cdot\vec{q}_2}{\omega_2\omega_1^2}F(\vec{q}_2\cdot\vec{q}_3)+O\left[\frac{1}{\omega_2^2}\right]\right](\omega_1^2-\omega_p^2)\delta(\omega_1-\omega_p)\right\}\end{aligned} \tag{13}$$

with ω_p and $S_\pm(\vec{q}_1)$ defined in Eq. (8). The function $F(\vec{q}_2\cdot\vec{q}_3)$ is some complex nonsingular function of $\vec{q}_2\cdot\vec{q}_3$. It is interesting to note that Eq. (13) partially verifies the usual assumption (see, for example, Ref. 4) that even close to resonance (i.e., $\omega_1 \approx \omega_p$) $\tilde{\Lambda}(\omega_1)$ is a smooth function of ω_1, but we note that a weaker resonance still persists via the second term in Eq. (13). Separate identities for each of the $\tilde{\Lambda}_{ij}$ or for *nonuniform* systems are not available for the same reason discussed in Eq. (8).

To generalize the static relations of Ref. 14 [for $\Gamma_{\alpha,n-\alpha}$ in Eq. (10)] to multicomponent systems it is not easy to follow a direct graphical analysis.[14] Rather, we write the energy in the presence of two external potentials $V_1(\vec{q})$ and $V_2(\vec{q})$ (which couple to components 1 and 2, respectively) to all orders, i.e.,

$$E=\Omega\sum_{n=1}^{\infty}\left[\frac{1}{n}\sum_{\alpha=0}^{n}\left[\Gamma_{\alpha,n-\alpha}(\vec{q}_1,\vec{q}_2,\ldots,\vec{q}_n;n_1,n_2)\prod_{\beta_1=0}^{\alpha}V_1(\vec{q}_{\beta_1})\prod_{\beta_2=0}^{n-\alpha}V_2(\vec{q}_{\beta_2})\right]\right]. \tag{14}$$

In Eq. (14) the $1/n$ term arises from the usual coupling-constant integration. In $\Gamma_{\alpha,n-\alpha}$ we have set $\omega_1=\omega_2=\cdots=\omega_n=0$ and suppressed these variables and also explicitly indicated the dependence on the density of the two components n_1 and n_2. [Note that the Γ's in Eqs. (10), (14), or Fig. 3 are understood to correspond to a finite temperature calculation with $T\rightarrow 0$. This is relevant for $n\geq 4$ due to anomalous contributions from chemical potential shifts.[14,18]] We can now change each of the $V_1(\vec{q}_{\beta_1})$ to $V_1(\vec{q}_{\beta_1})+\Delta V_1(\vec{q})$. In the limit of $\vec{q}\rightarrow 0$ we have two equivalent ways for writing the change in E of Eq. (14) by changing n_1 and n_2 or by changing $V_1(\vec{q}_{\beta_1})$ and $V_2(\vec{q}_{\beta_2})$. Equating the two gives the following generalization of Ref. 14:

$$\begin{aligned}\lim_{\vec{q}\rightarrow 0}\,&(\alpha+1)\Gamma^{n+1}_{\alpha+1,n-\alpha}(\vec{q}_1,\ldots,\vec{q}_n,\vec{q};n_1,n_2)\\&=-\lim_{\vec{q}\rightarrow 0}\left[\frac{\partial\Gamma^{n}_{\alpha,n-\alpha}(\vec{q}_1,\ldots,\vec{q}_n;n_1,n_2)}{\partial n_1}\chi_{11}(\vec{q})+\frac{\partial\Gamma^{n}_{\alpha,n-\alpha}(\vec{q}_1,\ldots,\vec{q}_n;n_1,n_2,)}{\partial n_2}\chi_{21}(\vec{q})\right].\end{aligned} \tag{15}$$

In the functional-density formalism,[19] any component of the energy which can be demonstrated to be a functional of the density can be written like Eq. (14). For example, the exchange and correlation energy can be written as

$$\begin{aligned}E_{\text{xc}}=\Omega\sum_{n=1}^{\infty}\Bigg[\sum_{\alpha=0}^{n}\Bigg[&K_{\alpha,n-\alpha}(\vec{q}_1,\ldots,\vec{q}_n;n_1,n_2)\\&\times\prod_{\beta_1=0}^{\alpha}n_1(\vec{q}_{\beta_1})\prod_{\beta_2=0}^{n-\alpha}n_2(\vec{q}_{\beta_2})\Bigg]\Bigg].\end{aligned} \tag{16}$$

Similar argument then yields

$$\begin{aligned}\lim_{\vec{q}\rightarrow 0}&(\alpha+1)K^{n+1}_{\alpha+1,n-\alpha}(\vec{q}_1,\ldots,\vec{q}_n,\vec{q};n_1,n_2)\\&=\frac{\partial K^{n}_{\alpha,n-\alpha}(\vec{q}_1,\ldots,\vec{q}_n;n_1,n_2)}{\partial n_1}.\end{aligned} \tag{17}$$

The most common role for these identities has been to ensure that any approximations satisfy all of these relations in the appropriate limits.[20] For example, the three irreducible vertex functions of Eqs. (12) and (13) enter readily the perturbation expansion of the interacting electron gas (see, e.g., Fig. 3). Any approximation for $\tilde{\Lambda}$ must therefore satisfy the calculations of Eqs. (12) and (13). Equation (15) has been central in studying phonon frequencies at small $\vec{q}$.[14] The generalization to multicomponent systems is useful in examining the effect of doping carriers on the phonon spectrum in multivalleyed semiconductors,[21,22] intercalated graphites,[23] etc. For transport properties the role of the scattering potential, particularly $V_{k_F}(q)$ on the Fermi surface (FS),[24] is important in both thermal and electrical resistivity calculations.[25,26] It has been first demonstrated in Ref. 24 that

$$\begin{aligned}\lim_{q\rightarrow 0}V_{k_F}(q)&=\lim_{q\rightarrow 0}Z\,\frac{\Lambda^{\infty}(q,k_F)}{\epsilon(q)}V_{\text{app}}(q)\\&=-\frac{Z_e\pi^2}{m^*k_{F,1}},\end{aligned} \tag{18}$$

where Z is the quasiparticle renormalization constant,[1] $V_{\text{app}}(q)$ any nonscreened external potential of charge Z_e, and m^* the interacting effective mass on the FS.[1] Furthermore, an interesting result emerges that all approaches to this important quantity give the same $q\rightarrow 0$ limit.[24,26]

For multicomponent systems like metallic hydrogen,[27] electron-hole liquid,[5,28,29] doped degenerate-band semiconductors[4] we can use Eq. (4) to construct an equivalent quantity. After some algebra, we get

$$\lim_{q\to 0} V_{k_{F,1}}(q)=\lim_{q\to 0} Z_1\left[\frac{\Lambda_{11}^{\infty}(q,k_F)\pm\Lambda_{12}^{\infty}(q,k_F)}{\epsilon(q)}\right]V_{\mathrm{app}}(q)$$
$$=\left[1-\frac{(\partial\mu_1/\partial n_1\mp\partial\mu_1/\partial n_2)}{D}\left[\frac{\partial\mu_1}{\partial n_1}-\frac{\pi^2}{k_{F,1}m^*}\right]+\frac{(\partial\mu_1/\partial n_2\mp\partial\mu_2/\partial n_2)}{D}\frac{\partial\mu_1}{\partial n_2}\right]\lim_{q\to 0}\frac{V_{\mathrm{app}}(q)}{\epsilon(q)}\ . \tag{19}$$

[In Eq. (19) the + sign on the left-hand side goes with the − sign on the right-hand side and vice versa.] The ground-state description[30,26] for $V_{k_{F,1}}(q)$ in a multicomponent system gives

$$\lim_{q\to 0} V_{k_{F,1}}(q)=\lim_{q\to 0}\frac{[\pi_{11}(q)\pm\pi_{12}(q)]V_{\mathrm{app}}(q)}{\pi_{0,1}(q)\epsilon(q)}\ , \tag{20}$$

where $\pi_{0,1}(q)$ is the *noninteracting* static Lindhard function and $k_{F,1}$ the Fermi momentum of component 1. Equations (19) and (20) no longer yield the same forward scattering amplitude for a general two-component fermion system.[31]

Turning to relations (11)—(13) these can be useful in higher-order light,[3,4,32] phonon,[3,4] magnon[3,4] coupling with both single or multicomponent uniform or nonuniform fermion systems. For example, consider a doped semiconductor ($\sim 10^{18}$ cm^{-3}) with non-degenerate-band structure like GaAs. In four-photon processes (see Fig. 3) with an emission of a phonon and an energy range of ω_1 and $\omega_2 \gg \omega_p$, $\omega_1 \ll \omega_p$, or $\omega_1 \approx \omega_p$ the $(\rho A^2)^2$ dominates[32] [rather than $(\vec{J}\cdot\vec{A})^4$], and the cross section σ [according to Eqs. (11)—(13)] is proportional to

$$\sigma\approx|\Gamma_{111}(\vec{q}_1,\omega_1;\vec{q}_2,\omega_2;\vec{q}_3,\omega_3)|^2$$
$$\approx\left|\frac{\vec{q}_2\cdot\vec{q}_3}{\omega_2^2 m_1}\right|^2|\chi_{11}(\vec{q}_2+\vec{q}_3,\vec{q}_1,0)|^2\times|g(\vec{q}_1)|^2 D(\vec{q}_1,\omega_1)\text{ when }\omega_1\ll\omega_p \tag{21a}$$

and

$$\sigma\approx\lim_{q_1\to 0}\left[\frac{(\vec{q}_2\cdot\vec{q}_3)2\omega_p}{\omega_2^2}\frac{S(\vec{q}_1)}{(\omega_1+i\delta)^2-\omega_p^2}+\frac{\vec{q}_1\cdot\vec{q}_2}{\omega_2^2}\omega_1 F(\vec{q}_2\cdot\vec{q}_3)\delta(\omega_1-\omega_p)\right]^2$$
$$\times|g(\vec{q}_1)|^2 D(\vec{q}_1,\omega_1)\text{ when }\omega_1\approx\omega_p\ , \tag{21b}$$

which is the extension of the four-photon cross-section calculations of Ref. 32. In Eqs. (21) $D(\vec{q}_1,\omega_1)$ is the phonon propagator *corrected* for the free carriers and $g(\vec{q}_1)$ the bare-electron—phonon vertex function.[21] We recall that Eq. (21a) applies equally to nonuniform systems [not so Eq. (21b)]; the possibility of exploring such full response functions could be of interest. Also, such forms govern higher-order photon cross sections which are becoming more relevant[33] with higher laser power.

Finally, laser probes have been restricted, so far, to small momentum $\vec{q}_1$; however, high-energy electron scattering with the emission of two phonons (in the right energy range) or as a function of impurity concentration can extend the range of applicability of the above relations and provide some additional information for the fully interacting response functions.[34] Detailed considerations will be given elsewhere.

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