Gauge-invariant frustrated Potts spin-glass

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The Ising model of a spin-glass is generalized to the Potts model. Our model possesses a frustration and a gauge symmetry similar to those of the random Ising model. The gauge symmetry enables us to obtain exact results for the internal energy, specific heat, and correlation functions. The infinite-range version of this model is discussed at the same level of accuracy as the Sherrington-Kirkpatrick solution of the Ising spin-glass and the phase diagram is obtained. The model is solved exactly on the Bethe lattice and a nontrivial phase diagram is presented.

I. INTRODUCTION

The existence of a spin-glass phase was first proposed by Edwards and Anderson[1] to explain the magnetic properties of certain dilute magnetic alloys. In these materials it is believed that the exchange interaction between localized spins is randomly distributed around zero due to the oscillatory nature of the Ruderman-Kittel-Kasuya-Yosida interaction.[2] This has led to the presentation and study of a variety of spin-glass models. Most attention has been directed at the Ising model in which the exchange interaction is assumed to be a quenched random variable and may be positive or negative. The low-temperature state of such a random spin system is probably quite different from conventional ones like the ferromagnetic or antiferromagnetic state and is called a spin-glass state. An important feature of the randomness in spin-glasses was pointed out by Toulouse[3] and called frustration. Closely related to frustration is the existence of a gauge transformation and some rigorous results on Ising spin-glasses have been obtained by one of the authors[4] by making use of the gauge transformation.

In this paper we generalize the Ising model of a spin-glass to a Potts model. Our model is also frustrated and possesses the same type of gauge symmetry as in the Ising model. This enables us to obtain an exact value for the internal energy, to show that the specific heat is finite, and to obtain certain equalities for the correlation functions in a certain region of phase diagrams. In Sec. III we discuss the infinite-range version of this model at the same level of accuracy as the Sherrington-Kirkpatrick solution[5] of the infinite-ranged Ising spin-glass. This leads to a phase diagram and it is shown that when the number of components in the Potts model exceeds six then the transitions between paramagnetic, ferromagnetic, and spin-glass phases are all of first order. We solve the model exactly on the Bethe lattice in Sec. IV. Although the free energy turns out to be a trivial function of the temperature and randomness, the susceptibility has a divergence as in the nonrandom Potts model on the Bethe lattice.[6] The resulting phase diagram therefore has a nontrivial structure. A short comment is given in Sec. V.

II. MODEL AND EXACT RESULTS

The q-state nonrandom Potts model is defined by the pair interaction

$$H_{ij}=-J\delta_{\lambda_i,\lambda_j}\ , \tag{2.1}$$

where the λ_i are q component spins. If we use a representation $\lambda_i=\omega^{k_i}$ where $\omega=e^{2\pi i/q}$ and $k_i=0,1,\ldots,-1$, the Kronecker symbol in (2.1) can be expressed as[7]

$$H_{ij}=-\frac{J}{q}\left[1+\sum_{r=1}^{q-1}\lambda_i^r\lambda_j^{q-r}\right]. \tag{2.2}$$

A natural generalization of this model to a random system is therefore (omitting a constant)

$$H_{ij}=-\frac{1}{q}\sum_{r=1}^{q-1}J_{ij}^{(r)}\lambda_i^r\lambda_j^{q-r}\ , \tag{2.3}$$

where the $J_{ij}^{(r)}$ are random. As a generalization of the $\pm J$ distribution for an Ising spin-glass,[8] we may consider a discrete distribution defined by

$$J_{ij}^{(r)}=J\tau_{ij}^r \tag{2.4}$$

where τ_{ij} is 1 with probability p and is one of the other powers of ω we defined above $(\omega,\omega^1,\ldots,\omega^{q-1})$ with probability $(1-p)/(q-1)$. If $J_{ij}^{(r)}=J\omega^{kr}$, (2.3) reads

$$H_{ij}=-J\delta_{\lambda_i,\omega^k\lambda_j}+\text{const}\ .$$

A generalization of the Gaussian distribution for J in the Ising case[5] is provided by

$$P\{J_{ij}^{(r)}\}=(2\pi J^2)^{-(q-1)/2}\times\exp\left[-\sum_{r=1}^{q-1}(J_{ij}^{(r)}-J_0)(J_{ij}^{(q-r)}-J_0)/2J^2\right] \tag{2.5}$$

with $J_{ij}^{(q-r)}=J_{ij}^{(r)*}$.

In both the distributions (2.4) and (2.5) the random variable $J_{ij}^{(r)}$ assumes complex values. However, the realness of the Hamiltonian is assured by its definition (2.3) and the condition $J_{ij}^{(q-r)}=J_{ij}^{(r)*}$. The random interactions are taken to be quenched.

It is possible to derive some exact results for the internal energy, specific heat, and correlation functions for the above models by the method of gauge transformations.[4] First we consider the internal energy for the discrete distribution (2.4) of random interactions. For this distribution the configurational average, denoted by square brackets with subscript av, of a function f of the $J_{ij}^{(r)}$ ($\equiv J\tau_{ij}^r$) may be written as

$$[f]_{\text{av}}=A^B\text{Tr}_{(\tau)}\times\exp\left[K_p\sum_{(ij)}\sum_{r=1}^{q-1}\tau_{ij}^r\right]f\{J\tau_{ij}^r\}\ , \tag{2.6}$$

where $A=p^{1/q}[(1-p)/(q-1)]^{(q-1)/q}$ is the normalization factor, B is the total number of bonds, K_p is defined by $e^{qK_p}=p(q-1)/(1-p)$ and the trace runs over the q roots of unity. K_p has been chosen so that the factor $\exp\left[K_p\sum_r\tau_{ij}^r\right]$ gives the correct weight assigned in (2.4) to each configuration of τ_{ij}. Specifically the internal energy is

$$[\langle H\rangle]_{\text{av}}=A^B\text{Tr}_{(\tau)}\exp\left[K_p\sum_{(ij)}\sum_{r}\tau_{ij}^r\right]\frac{\text{Tr}_{(\lambda)}\sum_{(ij)}H_{ij}\exp\left[-\beta\sum_{(ij)}H_{ij}\right]}{\text{Tr}_{(\lambda)}\exp\left[-\beta\sum_{(ij)}H_{ij}\right]}\ , \tag{2.7}$$

where the angular brackets denote a thermal average and $\beta=1/k_BT$. We now make a gauge transformation of variables

$$\lambda_i\to\lambda_i\mu_i,\ \ J_{ij}^{(r)}\to J_{ij}^{(r)}\mu_i^{q-r}\mu_j^r\ , \tag{2.8}$$

where μ_i is arbitrarily chosen to be one of the q roots of unity at each site. This transformation leaves the traces in (2.7) invariant. Thus the same argument as in the Ising case[4] applies and we are led to the exact value of the internal energy if $K_p=K$ $(=\beta J/q)$,

$$[\langle H\rangle]_{\text{av}}=-BJ(pq-1)/q. \tag{2.9}$$

As in the Ising model,[4] the condition $K_p=K$ confines us to a subspace of the phase diagram. The result (2.9) shows that the internal energy has no singularities in this subspace defined by $K_p=K$ even if this subspace intersects the phase boundaries.

The same gauge transformation (2.8) yields an upper bound on the specific heat. A straightforward generalization of the argument for the Ising model[4] proves

$$k_BT^2[\langle C\rangle]_{\text{av}}\le J^2Bp(1-p)\ , \tag{2.10}$$

when $K_p=K$. This inequality shows that the specific heat is always finite in the subspace $K_p=K$.

The magnetization of the random Potts model may be defined by

$$m^2=\lim_{|i-j|\to\infty}[\langle\lambda_i^r\lambda_j^{q-r}\rangle]_{\text{av}}\ . \tag{2.11}$$

Correspondingly, the spin-glass order parameter is

$$Q^2=\lim_{|i-j|\to\infty}[\langle\lambda_i^r\lambda_j^{q-r}\rangle\langle\lambda_i^{q-r}\lambda_j^r\rangle]_{\text{av}}\ . \tag{2.12}$$

The order parameters m and Q are related in the subspace $K_p=K$; using the gauge transformation (2.8), one can easily show that

$$[\langle\lambda_i^r\lambda_j^{q-r}\rangle]_{\text{av}}=[\langle\lambda_i^r\lambda_j^{q-r}\rangle\langle\lambda_i^{q-r}\lambda_j^r\rangle]_{\text{av}}\ , \tag{2.13}$$

when $K_p=K$, for any pair of sites i,j. From this relation it follows that $m=Q$. Hence, the subspace $K_p=K$ does not have an intersection with the spin-glass phase defined by $m=0$ and $Q\neq 0$.

Similar exact results can be derived also for the Gaussian distribution (2.5). The definition of the gauge transformation is the same as in (2.8). The argument parallels that of the random Ising model[4] with a Gaussian distribution, and we just quote the results:

$$[\langle H\rangle]_{\text{av}}=-Bq^{-1}(q-1)J_0\ , \tag{2.14}$$

$$k_BT^2[\langle C\rangle]_{\text{av}}\le Bq^{-2}J^2\ , \tag{2.15}$$

and

$$[\langle\lambda_i^r\lambda_j^{q-r}\rangle]_{\text{av}}=[\langle\lambda_i^r\lambda_j^{q-r}\rangle\langle\lambda_i^{q-r}\lambda_j^r\rangle]_{\text{av}}\ . \tag{2.16}$$

All of the above relations are valid only if $qJ_0=\beta J^2$ which again limits us to a subspace of the phase diagram. The last relation (2.16) proves $m=Q$.

Since we later discuss the infinite-ranged model, it is useful to know the infinite-range limit of the exact results. As to the Gaussian distribution function, we have to scale the variables in the following way:

$$J=\tilde{J}/\sqrt{N}\,,\quad J_0=\tilde{J}_0/N, \quad \text{and } B=N^2/2\ . \tag{2.17}$$

The condition $qJ_0=\beta J^2$, under which (2.14) through (2.16) were proved, is now $q\tilde{J}_0=\beta\tilde{J}^2$. The exact results read

$$[\langle H\rangle]_{\rm av}=-N(q-1)\tilde{J}_0/2q\ , \tag{2.18}$$

$$k_B T^2[\langle C\rangle]_{\rm av}\le N\tilde{J}/2q^2, \tag{2.19}$$

and

$$m=Q \tag{2.20}$$

when $q\tilde{J}_0=\beta\tilde{J}^2$. The discrete distribution (2.4) yields the same results in the infinite-range limit if the variables are appropriately scaled. We define

$$K=\beta J/q=\beta\tilde{J}/q\sqrt{N} \tag{2.21}$$

and

$$\tilde{J}_0=\epsilon\tilde{J}/(q-1)\ , \tag{2.22}$$

where ϵ represents the deviation of p from $1/q$:

$$p=(1+\epsilon/\sqrt{N}\,)/q\ . \tag{2.23}$$

B is $N^2/2$. Then the condition $K_p=K$ $(=\beta J/q)$, under which (2.9) and (2.10) were derived, becomes $q\tilde{J}_0=\beta\tilde{J}^2$, and the expressions (2.9) and (2.10) reduce to the same ones as (2.18) and (2.19).

III. INFINITE-RANGED MODEL

The infinite-ranged model often provides us with useful information about the phase diagram mainly because the mean-field theory is exact for the infinite-ranged model in certain circumstances.[9] Sherrington and Kirkpatrick[5] therefore studied the infinite-ranged random Ising model and showed that three phases, ferromagnetic, paramagnetic, and spin-glass phases, exist. Although their method involved some methematically unjustifiable techniques, they provided a basis for more detailed analysis of spin-glasses.[10] In this respect, it is worthwhile to solve the infinite-ranged random Potts model at the same level of accuracy as the Sherrington-Kirkpatrick solution of the random Ising model.

In the infinite-ranged model, the random interaction $J_{ij}^{(r)}$ obeys the same distribution function for any pair of sites (i,j) in the system. The distribution function may be discrete (2.4) or Gaussian (2.5). It is not difficult to verify that these two distributions give the same answer in the thermodynamic limit if an appropriate scaling of variables is chosen [see (2.17) and (2.21) through (2.23)]. Following Sherrington and Kirkpatrick,[5] we use the replica method and assume symmetry between different replicas. The calculation itself is a simple generalization of the method of Sherrington and Kirkpatrick, and the resulting expression of the free energy is

$$-\frac{\beta F}{N}=-\tfrac{1}{2}\tilde{K}_0(q-1)m^2+\tfrac{1}{4}\tilde{K}^2(q-1)(Q-1)^2$$
$$+\frac{1}{(2\pi)^{(q-1)/2}}\int\left[\prod_r dz_r\right]\exp\left[-\tfrac{1}{2}\sum_r z_r z_{q-r}\right]\ln \mathrm{Tr}_{(\lambda)}\exp\left[\sum_r \Omega(z_r)\lambda^{-r}\right] \tag{3.1}$$

where $\Omega(z_r)=\tilde{K}_0 m+\tilde{K}Q^{1/2}z_r$, $\tilde{K}_0=\beta\tilde{J}_0/q$, $\tilde{K}=\beta\tilde{J}/q$, and $z_{q-r}=z_r^*$. The order parameters m and Q satisfy the extremum conditions

$$\delta F/\delta m=0,\ \ \delta F/\delta Q=0\ . \tag{3.2}$$

From (3.1) it can be verified that

$$m=[\langle\lambda_i^r\rangle]_{\rm av}$$

and

$$Q=[\langle\lambda_i^r\rangle\langle\lambda_i^{q-r}\rangle]_{\rm av}\ ,$$

as expected.

As in the Ising case the extrema obtained from (3.2) are not true minima of the free energy (3.1) as a function of m and Q. In fact F is minimum with respect to m but maximum with respect to Q at the solution of (3.2). This odd behavior is related to the assumption of replica symmetry,[10] but a further investigation of this problem is beyond our present scope.

If q is small, it is easy to solve the equations (3.2) numerically. The result for $q=3$ is shown in Fig. 1. The transition between paramagnetic and spin-glass phases takes place at $\tilde{K}^{-1}$ $(=q/\beta\tilde{J})=1$, and it is of second order. The paramagnetic state changes into

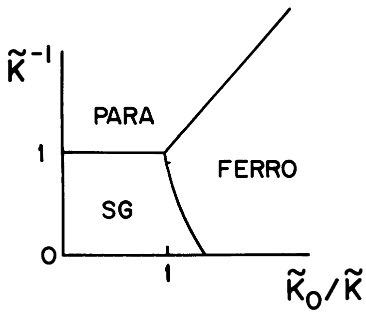

FIG. 1. Phase diagram of the three-state case. Only the para-SG transition is of second order.

the ferromagnetic one if $\tilde{K}_0/\tilde{K}>1$ through a first-order transition. The transition between ferromagnetic and spin-glass states is also of first order. These two first-order phase boundaries terminate at the tricritical point located at $\tilde{K}^{-1}=1$ and $\tilde{K}_0/\tilde{K}=1$ where three phases merge. It should be noticed that the tricritical point is on the exactly solvable line $q\tilde{J}_0=\beta\tilde{J}^2$ or $\tilde{K}^2=\tilde{K}_0$.

It is possible to calculate some quantities analytically even if q is not necessarily small. The internal energy is, from (3.1) and (3.2),

$$-E/N=\beta\tilde{J}^2(q-1)/2q^2+(q-1)(\tilde{J}_0m^2-\beta\tilde{J}^2Q^2/q)/2q. \quad (3.3)$$

If $q\tilde{J}_0=\beta\tilde{J}^2$, an analysis of (3.2) proves $m=Q$, in agreement with the exact prediction. With these two relations, $q\tilde{J}_0=\beta\tilde{J}^2$ and $m=Q$, the internal energy (3.3) reduces to

$$-E/N=(q-1)\tilde{J}_0/2q\ ,$$

which has already been proved in (2.18). Hence the assumption of symmetric replicas gives the correct answer at least on the line $q\tilde{J}_0=\beta\tilde{J}^2$(or $\tilde{K}_0=\tilde{K}^2$).

The ground-state entropy in the case $\tilde{J}_0=0$ is found to be

$$S=-\tfrac{1}{4}k_BN(q-1)C_q^2\ , \quad (3.4)$$

where

$$C_q=(2\pi)^{-q/2}q^{3/2}\times\int_{-\infty}^{\infty}dx_0e^{-x_0^2}\left[\int_{-\infty}^{x_0}dx\,e^{-x^2/2}\right]^{q-2}.$$

Especially in the limit $q\to\infty$, [see (A7) in the Appendix]

$$C_q\to4e^{-2}\sqrt{2\pi(\ln q)/q}\ ,$$

and thus

$$S\to-8\pi k_BNe^{-4}\ln q\ . \quad (3.5)$$

The ground-state entropy is negative for all q, which manifests the existence of fundamental errors in the present mathematical treatment.

In the limit of large q, the free energy can be evaluated explicitly as detailed in the Appendix. It turns out that the variables should be scaled as

$$\bar{K}=\sqrt{q/\ln q}\,\tilde{K}$$

and

$$\bar{K}_0=(q/\ln q)\tilde{K}_0\ , \quad (3.6)$$

so that transition temperatures are of the order of 1 in terms of $\bar{K}$ and $\bar{K}_0$.

We first investigate the spin-glass phase transition by assuming $m=0$. The free energy is

$$-\frac{\beta F}{N}=[\tfrac{1}{4}\bar{K}^2(Q^2+1)+1]\ln q\quad\text{if } \bar{K}\sqrt{Q}<\frac{1}{\sqrt{2}} \quad (3.7)$$

and

$$-\frac{\beta F}{N}=[\tfrac{1}{4}\bar{K}^2(Q-1)^2+\bar{K}\sqrt{Q}\,4\sqrt{2\pi}e^{-2}]\ln q\quad\text{if } \bar{K}\sqrt{Q}>\sqrt{2}. \quad (3.8)$$

Let us look for extrema of F as a function of Q (≤1). If $\bar{K}<1/\sqrt{2}$, $\bar{K}\sqrt{Q}<1/\sqrt{2}$ is satisfied by all Q between 0 and 1. The expression (3.7) is valid for any Q ($0\leq Q<1$) and the only extremum is at $Q=0$. The system is thus paramagnetic [Fig. 2(a)]. When $\bar{K}>\sqrt{2}$, $\bar{K}\sqrt{Q}<1/\sqrt{2}$ is satisfied in the range $0\leq Q<\bar{K}^{-2}/2$, and $\bar{K}\sqrt{Q}>\sqrt{2}$ is satisfied if

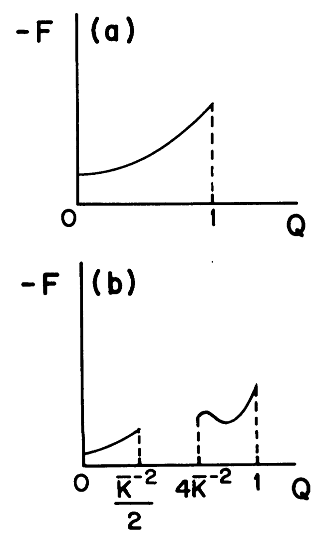

FIG. 2. (a) Free energy has an extremum only at $Q=0$ when $\bar{K}<1/\sqrt{2}$. (b) There exist two minima of $-F$ if $\bar{K}>\sqrt{2}$.

$4\bar{K}^{-2} < Q \le 1$. Hence the expressions (3.7) and (3.8) are valid in respective regions [Fig. 2(b)]. There are two minima of $-F$, one at $Q=0$ and the other in $4\bar{K}^{-2} < Q < 1$. Matching the values of the free energy at these two minima, we find a first-order transition at $\bar{K}_c = 3.6477$ and $Q_c = 0.4378$. We assume that no other phase transitions exist in the intermediate region $1/\sqrt{2} < \bar{K} < \sqrt{2}$.

Transition from the paramagnetic to ferromagnetic state can be studied in a similar manner. With m finite, the free energy is

$$-\frac{\beta F}{N} = \left[-\frac{\bar{K}_0}{2} m^2 + \frac{\bar{K}^2}{4}(Q^2+1) + 1 \right] \ln q \quad (3.9)$$

if

$$\tfrac{1}{2}\bar{K}^2 Q + 1 > \bar{K}_0 m \ , \quad (3.10)$$

and

$$-\frac{\beta F}{N} = \left[-\frac{\bar{K}_0}{2}(m-1)^2 + \tfrac{1}{2}\bar{K}_0 + \tfrac{1}{4}\bar{K}^2 (Q-1)^2 \right] \ln q \quad (3.11)$$

if

$$\tfrac{1}{2}\bar{K}^2 Q + 1 < \bar{K}_0 m \ . \quad (3.12)$$

Both (3.9) and (3.11) are derived under the assumption $\bar{K}\sqrt{Q} < 1/\sqrt{2}$. If $\bar{K} < 1/\sqrt{2}$, the condition $\bar{K}\sqrt{Q} < 1/\sqrt{2}$ is always satisfied ($0 \le Q < 1$). Thus we assume $\bar{K} < 1/\sqrt{2}$ for the moment. Apparently there are two extrema of the free energy. One is at $m=0$, $Q=0$ from (3.9) and the other at $m=1$, $Q=1$ from (3.11). The paramagnetic extremum ($m=Q=0$) always exists since the condition (3.10) is satisfied by $m=Q=0$ for any $\bar{K}$ and $\bar{K}_0$. The ferromagnetic extremum exists only if $\bar{K}^2/2 + 1 < \bar{K}_0$ as seen from (3.12). Therefore the system is definitely in the paramagnetic phase if $\bar{K}^2/2 + 1 > \bar{K}_0$. When $\bar{K}^2/2+1 < \bar{K}_0$, the extremum condition (3.2) yields two solutions, $m=Q=0$ and $m=Q=1$. Transition points, or the phase boundary, can be calculated by matching the values of free energy at these two extrema. The phase boundary is found to be

$$\bar{K}_0/\bar{K} = 2/\bar{K} + \bar{K}/2 \ . \quad (3.13)$$

Although the boundary (3.13) of ferromagnetic and paramagnetic phases was derived under the condition $\bar{K} < 1/\sqrt{2}$, it would be reasonable to assume that the boundary persists until it meets another phase boundary (see Fig. 3).

Transition from ferromagnetic to spin-glass phase

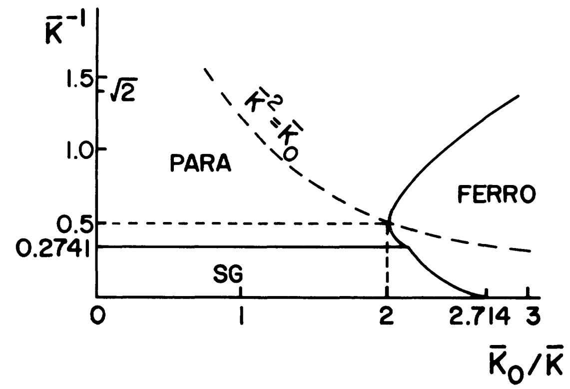

FIG. 3. Phase diagram in the large q limit. All transitions are of first order.

occurs when the values of free energy of these two states match:

$$-\frac{\beta F_{SG}}{N} = [\tfrac{1}{4}\bar{K}^2(Q_s-1)^2 + \bar{K}\sqrt{Q_s}\, 4\sqrt{2\pi}e^{-2}] \ln q$$

and

$$-\frac{\beta F_{\text{ferro}}}{N} = \tfrac{1}{2}\bar{K}_0 \ln q \ ,$$

where Q_s satisfies $\delta F_{SG}/\delta Q_s = 0$. In particular, at zero temperature,

$$(\bar{K}_0/\bar{K})_c = 8\sqrt{2\pi}e^{-2} = 2.714 \ .$$

These results are used to draw the phase diagram in Fig. 3.

Since we have explicit values of the free energy in paramagnetic and ferromagnetic phases, we can calculate the latent heat along the phase boundary. The values of internal energy in the two phases are

$$E_{\text{para}} = -\frac{1}{2\beta} N \bar{K}^2 \ln q$$

and (3.14)

$$E_{\text{ferro}} = -\frac{1}{2\beta} N \bar{K}_0 \ln q \ .$$

Thus on the boundary defined by $\bar{K}_0/\bar{K} = 2/\bar{K} + \bar{K}/2$ or $\bar{K}_0 = 2 + \bar{K}^2/2$,

$$\Delta E = E_{\text{para}} - E_{\text{ferro}} = \frac{N}{2\beta}(\ln q)\tfrac{1}{2}(4 - \bar{K}^2) \ , \quad (3.15)$$

which is positive if $\bar{K} < 2$ and negative when $\bar{K} > 2$. This change of sign in ΔE is consistent with the shape of the phase boundary; the paramagnetic state is at lower temperatures than the ferromagnetic state if $\bar{K} > 2$. The point $\bar{K} = 2$ and $\bar{K}_0 = 4$, where $\Delta E = 0$, is on the exactly solvable line $\bar{K}_0=\bar{K}^2$.

We have found that the spin-glass transition is of first order when q is very large while it is of second order if $q=3$. One is naturally tempted to ask what happens for intermediate q. The question is answered by expanding the free energy in powers of Q. We here assume that $m=0$ because we are interested only in the spin-glass transition. The expansion is

$$-\beta F/N=\ln q+(q-1)[\tfrac{1}{4}\tilde{K}^2+\tfrac{1}{4}\tilde{K}^2(1-\tilde{K}^2)Q^2 +\tfrac{1}{12}\tilde{K}^6(6-q)Q^3 +C_8\tilde{K}^8Q^4+\dots], \quad (3.16)$$

where c_8 is some positive coefficient. It is clearly seen that $\tilde{K}_c=1$ is the critical point of a second-order transition if $q\le 6$. However, when $q>6$, the negative coefficient of Q^3 would result in a first-order instability at $\tilde{K}<1$.

IV. BETHE LATTICE

If the randomness is the discrete one (2.4), the model can be solved exactly on the Bethe lattice. There is no frustration on the Bethe lattice because of the absence of loops. Nevertheless we show that the system undergoes a phase transition similar to the spin-glass transition on lattices with loops.

In the absence of external fields the free energy has no dependence on the randomness of (2.4). In fact the following gauge transformation readily eliminates the randomness from the free energy:

$$\lambda_j\to(\pi\tau)\lambda_j \quad (4.1)$$

where the product is over the bonds between an arbitrarily chosen site 0 and j, and τ represents the randomness: $J_{ij}^{(r)}=J\tau_{ij}^r$. For instance, in Fig. 4,

$$\lambda_0\to\lambda_0,\ \ \lambda_1\to\tau_1\lambda_1,\ \ \lambda_2\to\tau_1\tau_2\lambda_2,\dots .$$

The transformation (4.1) leaves the trace over λ invariant and changes the interaction (2.3) into a nonrandom one (2.2) (apart from a constant). Therefore the solution to the nonrandom Potts model on the

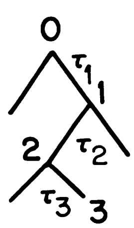

FIG. 4. There exists a unique path between two sites on the Bethe lattice. In (4.1) the spin variable is multiplied by all τ along the path between 0 and j.

Bethe lattice[6] applies and we find

$$-\beta F=(N-1)\ln(e^{\beta J}+q-1)+\ln q \quad (4.2)$$

for any configuration of the randomness. The internal energy and specific heat are readily derived from (4.2) and it is confirmed that the exact results (2.9) and (2.10) are satisfied when $K=K_p$.

The correlation function can be calculated in a similar way. By the gauge transformation (4.1), the correlation function is changed as

$$\langle\lambda_i^r\lambda_j^{q-r}\rangle\to(\pi\tau)\langle\lambda_i^r\lambda_j^{q-r}\rangle_{\text{pure}}\,, \quad (4.3)$$

where the product is over the bonds between i and j, and the last factor represents the correlation function of the nonrandom model[6] with the same $\beta J/q$. Since the randomness is now separated from spin variables in (4.3), the configurational average is easily carried out to yield

$$[\langle\lambda_i^r\lambda_j^{q-r}\rangle]_{\text{av}}=\left[\frac{(e^{\beta J}-1)(pq-1)}{(e^{\beta J}+q-1)(q-1)}\right]^{|i-j|}, \quad (4.4)$$

where $|i-j|$ denotes the distance between the two sites and the explicit expression of the nonrandom correlation function[6] has been used. An immediate conclusion from (4.4) is the absence of ferromagnetic long-range order because the correlation always decays exponentially. However, as in the nonrandom case,[6] the special structure of the Bethe lattice allows a divergence of the susceptibility in the absence of long-range order. We simply replace the correlation function of the nonrandom model[6] by (4.4), and find that the susceptibility, sum of (4.4) over i and j, diverges if $\bar{B}\mu^2(p)\ge 1$ where $\bar{B}+1$ is the coordination number and

$$\mu(p)=\frac{(e^{\beta J}-1)(pq-1)}{(e^{\beta J}+q-1)(q-1)}\,.$$

Thus $\bar{B}\mu^2=1$ defines the phase boundary between the finite-χ and infinite-χ phases. The spin-glass correlation function

$$\langle\lambda_i^r\lambda_j^{q-r}\rangle\langle\lambda_i^{q-r}\lambda_j^r\rangle \quad (4.5)$$

can also be calculated. According to (4.3), (4.5) is transformed to the product of two nonrandom correlation functions. Hence the configurational average of (4.5) simply gives

$$[\langle\lambda_i^r\lambda_j^{q-r}\rangle\langle\lambda_i^{q-r}\lambda_j^r\rangle]_{\text{av}}=[\mu(p=1)]^{2|i-j|}\,. \quad (4.6)$$

The corresponding susceptibility χ_2, sum of (4.6) over i and j, diverges if $\bar{B}\mu^4(1)\ge 1$. The boundary

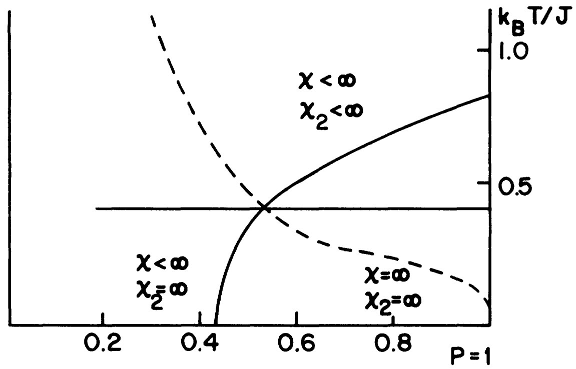

FIG. 5. Phase diagram of the model with $q=5$, $\bar{B}=3$ on the Bethe lattice. Solid lines indicate where the susceptibilities χ and χ_2 begin to diverge. The broken line represents the exactly solvable subspace.

between the finite-χ_2 and infinite-χ_2 phases is thus $\bar{B}\mu^4(1)=1$.

It is possible to eliminate the surface effects following Wang and Wu.[6] Again the conditions for divergence of χ and χ_2 are obtained from the formulas of Wang and Wu simply by replacing the nonrandom correlation functions by the corresponding random ones (4.4) and (4.6). χ turns out to be divergent when $\bar{B}\mu(p)\geq 1$ and χ_2 diverges if $\bar{B}\mu^2(1)\geq 1$. The resulting phase diagram is illustrated in Fig. 5 for $q=5$ and $\bar{B}=3$.

In the nonrandom model, the critical condition of χ with surface effects eliminated exactly agrees with the Bethe-Peierls transition temperature.[6] In our present random case, if $q=2$, the critical conditions $\bar{B}\mu(p)=1$ and $\bar{B}\mu^2(1)=1$ for the divergence of χ and χ_2, respectively, agree with the expression of the corresponding phase boundaries derived from the Bethe-Peierls method.[11] It should also be noticed that the tricritical point, where both χ and χ_2 begin to diverge, is on the exactly solvable line $K=K_p$. For instance, if we take the surface effects into account, the tricritical point should satisfy $\mu(p)=\mu^2(1)$ because

$$\bar{B}\mu^2(p)=\bar{B}\mu^4(1)=1.$$

But $\mu(p)=\mu^2(1)$ is nothing but $K=K_p$ from the definition of K_p (see Sec. II). The same relation $\mu(p)=\mu^2(1)$ should be satisfied by the tricritical point if we eliminate the surface effects, which leads us to the same conclusion.

V. COMMENT

We proposed a random Potts model which obeys a gauge symmetry characteristic of random systems. We could derive exact results and solve the model in the infinite-ranged case and on the Bethe lattice. In the infinite-ranged model, the spin-glass transition was shown to be of first order if q exceeds 6. This may be an artifact of the infinite-ranged model. Investigation of this problem by other methods, such as the renormalization group, Monte Carlo simulation, etc., will provide us with the answer for real dimensional systems. The structure of the Bethe lattice is very special and the Potts model on it does not undergo a first-order transition even in the nonrandom case.[6] Therefore the solution of the random model on the Bethe lattice does not give us any clue to this problem of critical q.

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APPENDIX: FREE ENERGY IN THE LARGE-q LIMIT.

In this Appendix we evaluate the integral expression in the free energy (3.1) for large q. Let us call the integral I. With a dummy variable y, I can be written as

$$I=(2\pi)^{-q/2}\int\left[\prod_r dz_r\right]\exp\left[-\tfrac{1}{2}\sum_r z_r z_{q-r}\right]\int_{-\infty}^{\infty}dy\exp\left[-\frac{y^2}{2}\right]\ln \operatorname{Tr}_{(\lambda)}\exp\left[\sum_r \Omega(z_r)\lambda^{-r}+\tilde{K}Q^{1/2}y\right].$$

A change of variables

$$\sum_r z_r\omega^{k(q-r)}+y=\sqrt{q}\,x_k,\quad (k=0,1,\ldots,q-1)$$

and explicit evaluation of trace over λ yield

$$I=(2\pi)^{-q/2}\int_{-\infty}^{\infty}\left[\prod_k dx_k\right]\exp\left[-\tfrac{1}{2}\sum_k x_k^2\right]\ln[\exp(q\tilde{K}_0 m+\tilde{K}\sqrt{qQ}x_0)+\exp(\tilde{K}\sqrt{qQ}x_1)+\cdots+\exp(\tilde{K}\sqrt{qQ}x_{q-1})]-\tilde{K}_0 m$$

$$\equiv J-\tilde{K}_0 m\ . \tag{A1}$$

If we regard the integral as an average of a function of stochastic variables $\{x_k\}_{k=1,\ldots,q-1}$, we may apply the law of large numbers[12] when $q\to\infty$. In fact, the sum of exponential functions of x_k

$$S_q\equiv\sum_{k=1}^{q-1}f_k\equiv\sum_{k=1}^{q-1}\exp(\tilde{K}\sqrt{qQ}x_k)$$

satisfies the Chebyshev inequality[12]

$$P[\,|S_q/(q-1)-\mu\,|>\epsilon]\le\epsilon^{-2}(q-1)^{-1}\sigma^2 \tag{A2}$$

for any $\epsilon>0$, where μ and σ^2 are the mean and variance of a single f_k:

$$\mu=\exp(\tilde{K}^2qQ/2)$$

and

$$\sigma^2=\exp(2\tilde{K}^2qQ)-\exp(\tilde{K}^2qQ)\ .$$

It is easy to see that the rhs of (A2) vanishes as $q\to\infty$ if

$$\tilde{K}\sqrt{qQ}/\sqrt{\ln q}=\bar{K}\sqrt{Q}<1/\sqrt{2}\ .$$

Therefore in this limit S_q in (A1) assumes its mean value $(q-1)\mu\approx q\mu$ with probability one,

$$J=(2\pi)^{-1/2}\int dx_0\exp(-\tfrac{1}{2}x_0^2)\times\ln(e^{(q\tilde{K}_0m+\tilde{K}\sqrt{qQ}x_0)}+q\,e^{(\tilde{K}^2qQ/2)})\ . \tag{A3}$$

We now rescale the constants $\tilde{K}$ and $\tilde{K}_0$ as in (3.6) and take the larger term in the logarithm in (A3) for each x_0, assuming $q\gg1$:

$$J=(2\pi)^{-1/2}\int_A^{\infty}dx_0\exp(-\tfrac{1}{2}x_0^2)\times(\bar{K}_0m+\bar{K}\sqrt{Q}x_0)\ln q+(2\pi)^{-1/2}\int_{-\infty}^{A}dx_0\exp(-\tfrac{1}{2}x_0^2)\times(\tfrac{1}{2}\bar{K}^2Q+1)\ln q\ ,$$

where

$$A=(\tfrac{1}{2}\bar{K}\sqrt{Q}+1/\bar{K}\sqrt{Q}-\bar{K}_0m/\bar{K}\sqrt{Q})\sqrt{\ln q}\ .$$

Hence, if $\bar{K}^2Q/2+1>\bar{K}_0m$, $A>0$ and

$$J=(\tfrac{1}{2}\bar{K}^2Q+1)\ln q\ , \tag{A4}$$

while

$$J=\bar{K}_0m\ln q \tag{A5}$$

if $\bar{K}^2Q/2+1<\bar{K}_0m$. The difference of I and J, $\tilde{K}_0m$ as in (A1), is small compared to I and J on the present scale and is neglected. Thus (A4) and (A5), inserted in (3.1), readily give the relations (3.7), (3.9), and (3.11).

The integral in the large $\bar{K}$ region ($\bar{K}\sqrt{Q}>\sqrt{2}$) is evaluated as follows. When $m=0$ and $q\gg1$, we may take one of the terms in the log of (A1) and neglect all the others:

$$I\approx J=\bar{K}\sqrt{Q\ln q}\,(2\pi)^{-q/2}\times\int_{-\infty}^{\infty}\left[\prod_k dx_k\right]\exp\left[-\tfrac{1}{2}\sum_k x_k^2\right]\times\max(x_0,\ldots,x_{q-1})\ , \tag{A6}$$

where the scaling (3.6) has already been taken into account. (A6) can further be rewritten as

$$I=\bar{K}\sqrt{Q\ln q}\,(2\pi)^{-q/2}q\times\int_{-\infty}^{\infty}dx_0\exp(-\tfrac{1}{2}x_0^2)\times x_0\left[\int_{-\infty}^{x_0}dx\,\exp(-\tfrac{1}{2}x^2)\right]^{q-1}\ ,$$

which may be evaluated by steepest descents to yield

$$I=\bar{K}\sqrt{Q}\,4\sqrt{2\pi}e^{-2}\ln q \tag{A7}$$

as in (3.8). The limit of validity of (A7) is obtained from the estimation of correction terms which were neglected in going from (A1) to (A6). The first-order correction is

$$I_1=q^2(2\pi)^{-q/2}\int_{-\infty}^{\infty}dx_0\int_{-\infty}^{x_0}dx_1\cdots dx_{q-1}\exp\left[-\tfrac{1}{2}\sum_k x_k^2+\bar{K}\sqrt{Q\ln q}\,(x_1-x_0)\right]\ .$$

By steepest descents, we find in the $q \to \infty$ limit,

$$I_1 \approx q^{(1-\bar{K}\sqrt{Q}/\sqrt{2})^2}\sqrt{\ln q} \gg I$$

if $\bar{K}\sqrt{Q} < \sqrt{2}$, and

$$I_1 \approx O(1) \ll I$$

if $\bar{K}\sqrt{Q} > \sqrt{2}$. Therefore the formula (A7) is valid only when $\bar{K}\sqrt{Q} > \sqrt{2}$.

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