Localization in a disordered elastic medium near two dimensions

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We describe the transition from extended to localized modes in a disordered elastic medium in $2+\epsilon$ dimensions as a phase transition in an appropriate nonlinear σ model. The latter is derived by considering fluctuations about the mean-field approximation to the replica field Lagrangian for the system. Within this framework, we calculate the averaged one- and two-particle phonon Green's functions obtaining the phonon density of states and frequency-dependent, zero-temperature thermal diffusivity, respectively. Momentum-shell integration of the nonlinear σ model reveals how this diffusivity renormalizes at longer length scales and hence the nature of normal modes at a given frequency. We demonstrate that all finite-frequency phonons in one and two dimensions are localized with low-frequency localization lengths diverging as $1/\omega^2$ and e^{1/ω^2}, respectively, and that a mobility edge, ω_*, separating low-frequency extended states from high-frequency localized states exists above $d=2$. The phonon localization length at this mobility edge is shown to diverge as $|\omega-\omega_*|^{-1/\epsilon}$.

I. INTRODUCTION

The transition from extended to localized states of an electron in a random potential is a problem of considerable interest in solid-state physics. Recent studies have shown that it is possible to describe this phenomenon as a phase transition in an appropriate field theory.[1,2] Within the framework of such a field theory, the fundamental nature of the "Anderson transition"[3] becomes manifest. It also leads to a derivation from first principles of the scaling theory of localization of Abrahams *et al.*[4] Here we exploit the field-theoretical formulation to study the transition from extended to localized states in a disordered elastic medium supporting longitudinal and transverse oscillations. Such a model, as we shall show, lends insight into the nature of phonon eigenstates, sound propagation, and thermal transport in a disordered solid at low temperatures. We demonstrate that all finite-frequency states are localized in one and two dimensions, and that a mobility edge exists above two dimensions. Localization lengths are calculated in all cases. These results are contrasted with those of an electron propagating in a random potential. In particular, we find that above two dimensions all phonon states sufficiently close to zero frequency remain extended. Phonon localization in one dimension has been studied extensively (see Ishii[5] for a review) and recently the effect of phonon localization on thermal conductivity has been discussed by Jackle.[6] In two dimensions the effects of phonon localization on the Kosterlitz-Thouless transition have been discussed by Miyake and Ito.[7]

II. THE MODEL

As in usual theories of critical phenomena, we postulate that the nature of the localization transition is independent of the details of lattice structure and microscopic interactions. This universality hypothesis forms the basis of our description of phonons in a continuum field-theory limit. Specifically, we consider a disordered elastic medium in d dimensions characterized by a vector displacement field $\phi_i(x,t)$ where the index i runs over the d spatial dimensions. For small deviations from equilibrium, the dynamical properties of the system are determined by the Lagrangian[8]

$$L=\int d^dx\left[\tfrac{1}{2}m(x)\dot{\phi}_i\dot{\phi}_i-\tfrac{1}{2}\lambda(\vec{\nabla}\cdot\vec{\phi})^2-\mu u_{ik}u_{ik}\right], \tag{2.1a}$$

where λ and μ are the usual Lamé coefficients measuring stiffness to bulk and shear deformation, and the strain tensor takes the form

$$u_{ik}=\frac{1}{2}\left[\frac{\partial\phi_i}{\partial x_k}+\frac{\partial\phi_k}{\partial x_i}\right]. \tag{2.1b}$$

We have introduced the randomness into the kinetic energy density by means of a random mass density $m(x)$ of the form

$$m(x)=m_0+m'(x) . \tag{2.2a}$$

m_0 is the average mass density and we characterize the fluctuating mass density by the correlation function

$$\langle m'(x)m'(y)\rangle=\gamma^2\delta^d(x-y) , \tag{2.2b}$$

where γ is a parameter measuring the amount of disorder.

Although in the present paper, we consider in detail only mass disorder, we assert that at the mean-field level of approximation which we make, introduction of randomness into the Lamé coefficients merely adds mathematical complexity but essentially no new physics. A model supporting this claim is discussed in Sec. VIII. As in the analogous problem of an electron propagating in a Gaussian random white-noise potential considered by McKane and Stone,[2] the disorder involves fluctuations of all wavelengths with equal probability as indicated by the δ-function nature of the correlation function (2.2b).

The nature of the normal modes of (2.1) is studied by introducing a point-localized excitation or impulse, and examining its evolution in time on average as it scatters from random fluctuations in its environment. If the initial excitation, when considered in a normal-mode expansion, contains states which are extended, the random scattering gives rise to diffusive behavior. Localized states suppress this diffusive behavior on sufficiently long length scales in a manner revealed by a renormalization-group transformation. This is what we now proceed to describe.

III. THE AVERAGE DIFFUSIVITY

The dynamical properties of the elastic medium described by Eq. (2.1) are governed by the associated Euler-Lagrange equation

$$m(x)\ddot{\vec{\phi}}_i=(\lambda+\mu)\partial_i(\vec{\nabla}\cdot\vec{\phi})+\mu\nabla^2\vec{\phi}_i, \quad i=1,\ldots,d \tag{3.1}$$

where $\partial_i\equiv\partial/\partial x_i$.

In the absence of disorder, i.e., $m(x)=m_0$, the solutions of (3.1) decouple into purely longitudinal ($\vec{\nabla}\times\vec{\phi}=0$) and purely transverse ($\vec{\nabla}\cdot\vec{\phi}=0$) waves. However, in the presence of a spatially varying mass, which gives rise to random scattering, all normal modes will contain, in general, a mixture of both polarizations.[8]

We consider the response of the ith component of the displacement field at position x to a velocity impulse in $\phi_j(y)$ as defined by the retarded Green's functions:

$$m(x)\ddot{G}_{ij}^{+}(x,y,t)=(\lambda+\mu)\partial_i\partial_k G_{kj}^{+}+\mu\nabla^2 G_{ij}^{+} , \tag{3.2a}$$

$$G_{ij}^{+}(x,y,t)=0, \quad t\le 0 \tag{3.2b}$$

$$\dot{G}_{ij}^{+}(x,y,0)=\delta_{ij}\delta^d(x-y) . \tag{3.2c}$$

The time evolution of the deformation is then given in terms of the initial velocity by

$$\phi_i(x,t)=\int d^dy\, G_{ij}^{+}(x,y,t)\dot{\phi}_j(y,0) . \tag{3.2d}$$

For the purpose of studying the subsequent diffusive behavior of an initial velocity impulse that is confined to a small region of space, for example near $y=0$, this expression can be approximated by evaluating the Green's function at $y=0$ and taking it out of the integration. Furthermore, since averaging restores isotropy to the medium, it is sufficient to consider the initial excitation of only one component of the vector field ϕ_i, such as $i=1$. The random scattering ensures that after a short time, waves of all possible polarizations will be activated. Under these conditions we may rewrite (3.2d) as

$$\phi_i(x,t)\simeq cG_{i1}^{+}(x,0,t) , \tag{3.3a}$$

where

$$c\equiv\int d^dy\,\dot{\phi}_1(y,0) . \tag{3.3b}$$

The initial conditions (3.2b) and (3.2c) can be incorporated into the equation of motion by Fourier transformation,

$$G_{ij}(x,y,\omega_+)=\int_{-\infty}^{\infty}dt\, e^{i\omega_+ t}G_{ij}^{+}(x,y,t) . \tag{3.4}$$

We use the notation $\omega_\pm=\omega\pm i\eta$ for the retarded and advanced Green's functions, respectively. Since the Lagrangian (3.1) has no explicit time dependence, the energy density

$$E(x,t)=\tfrac{1}{2}m(x)\dot{\vec{\phi}}_i\dot{\vec{\phi}}_i+\tfrac{1}{2}\lambda(\vec{\nabla}\cdot\vec{\phi})^2+\mu u_{ik}u_{ik} \tag{3.5a}$$

is a locally conserved quantity and satisfies a continuity equation of the form

$$\frac{\partial E(x,t)}{\partial t}=-\vec{\nabla}\cdot\vec{J} . \tag{3.5b}$$

In formulating a theory for the propagation of an initially injected disturbance, and in particular its subsequent diffusive behavior, it is such a conserved density that is of interest. We define a diffusion coefficient for our disordered elastic medium to be

$$D \equiv \lim_{t\to\infty} \frac{t^{-1}\int d^d x\, x^2 E(x,t)}{\int d^d x E(x,t)} . \tag{3.6}$$

Clearly, if all states are localized, the mean-square spatial extent of our initially injected energy density will remain localized as $t\to\infty$ so that the diffusion coefficient (3.6) vanishes. In the opposite extreme of a perfectly ordered medium possessing propagating waves, the mean-square distance traversed by the energy density from its origin would be proportional to the square of the time elapsed, leading to an infinite diffusion coefficient. We will show that the intermediate regime of extended waves undergoing random scattering gives rise to a finite diffusivity on average.

The long-time behavior of the kinetic-energy density of the medium in response to an initial impulse of the form described by (3.3) can be expressed in terms of the retarded and advanced Green's functions as

$$\lim_{t\to\infty} \frac{1}{t^\alpha}\frac{m(x)}{2}\dot{\phi}_i\dot{\phi}_i = c^2\frac{m(x)}{2}\lim_{\eta\to 0}\left[(2\eta)^{1+\alpha}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2 G_{k1}(x,0,\omega_+) \times G_{k1}(x,0,\omega_-)\right] . \tag{3.7}$$

It is a straightforward exercise to show that the potential and kinetic energies contribute equally to the total energy so that the denominator of (3.6) may be simply expressed as

$$\lim_{t\to\infty}\int d^d x\, E(x,t) = \lim_{\eta\to 0}\frac{\eta c^2}{\pi}\int_{-\infty}^{\infty} d\omega \int d^d x\, m(x)\omega^2 G_{k1}(x,0,\omega_+) \times G_{k1}(x,0,\omega_-) . \tag{3.8}$$

A similar analysis for the numerator in (3.6) yields[9]

$$\lim_{t\to\infty}\frac{1}{t}\int d^d x\, x^2 E(x,t) = \lim_{\eta\to 0}\frac{2\eta^2 c^2}{\pi}\int_{-\infty}^{\infty} d\omega \int d^d x\, x^2 m(x)\omega^2 G_{k1}(x,0,\omega_+)G_{k1}(x,0,\omega_-) . \tag{3.9}$$

Combining and symmetrizing over all possible initial polarizations we obtain the fundamental result for the average diffusivity in the limit of weak disorder:

$$D \simeq \lim_{\eta\to 0} 2\eta \frac{\int_{-\infty}^{\infty} d\omega\,\omega^2 \int d^d x\, x^2 \langle G_{ki}(x,0,\omega_+)G_{ki}(x,0,\omega_-)\rangle_{\text{ensemble}}}{\int_{-\infty}^{\infty} d\omega\,\omega^2 \int d^d x \langle G_{ki}(x,0,\omega_+)G_{ki}(x,0,\omega_-)\rangle_{\text{ensemble}}} . \tag{3.10}$$

We stipulate here that the expression (3.10) is valid only in the limit $\gamma \ll m_0$ of weak disorder since we have replaced the $m(x)$ appearing explicitly in (3.8) and (3.9) by its average value m_0. The averaging over all realizations of the random mass field is denoted here by an enclosure by angular brackets. The numerator and denominator in (3.10) can be averaged separately because the denominator is simply the total energy of the medium, which is the same for all realizations of the mass field.

The frequency sum appearing in Eq. (3.9) suggests a spectral decomposition of the average diffusivity:

$$D \equiv \frac{\int_{-\infty}^{\infty} d\omega\, E(\omega) D_0(\omega)}{\int_{-\infty}^{\infty} d\omega\, E(\omega)} , \tag{3.11}$$

where $E(\omega)$ is an energy density associated with modes of frequency ω. This can be regarded as a defining equation for a frequency-dependent diffusivity $D_0(\omega)$. It is this quantity which we now proceed to evaluate in a mean-field approximation.

IV. METHOD OF CALCULATION

We now describe the method of calculation of the averaged one- and two-particle Green's functions relevant to the phonon density of states and diffusivity, respectively. In order to facilitate the process of averaging over the random mass density, we utilize a replica field representation of the Green's functions. The one-particle Green's function may be written as

$$G_{ij}(x,y,\omega_+) = -\lim_{n_+\to 0}\int D\phi\, \phi_i^1(x)\phi_j^1(y) e^{-L^+} , \tag{4.1a}$$

where

$$L^{\pm} = \tfrac{1}{2}\sum_{\alpha=1}^{n_\pm}\int d^d x\, \phi_i^\alpha(x) A_{ij}(\omega_\pm)\phi_j^\alpha(x) \tag{4.1b}$$

and

$$A_{ij}(\omega_\pm) = [\omega_\pm^2 m(x) + \mu\nabla^2]\delta_{ij} + (\lambda+\mu)\partial_i\partial_j . \tag{4.1c}$$

Here, we use the abbreviated symbol $D\phi$ to denote

functional integration over all fields $\{\phi_i^\alpha\}$. For the two-particle Green's function we introduce two sets of replica indices denoted with + and − signs:

$$|G_{ij}(x,y,\omega_+)|^2 = \lim_{\substack{n_+\to 0\\ n_-\to 0}} \int D\phi\, \phi_i^{1+}(x)\phi_j^{1+}(y)\phi_i^{1-}(x)\phi_j^{1-}(y) \times e^{-(L^+ + L^-)} . \tag{4.2}$$

Averaging over all realizations of the random mass field we obtain

$$\langle |G_{ij}(x,y,\omega_+)|^2 \rangle_{\text{ensemble}} = \lim_{\substack{n_+\to 0\\ n_-\to 0}} \int D\phi\, \phi_i^{1+}(x)\phi_j^{1+}(y)\phi_i^{1-}(x)\phi_j^{1-}(y) \times \exp[-(L_0^+ + L_0^- + L_{\text{int}})] \tag{4.3a}$$

where L_0^+ and L_0^- are obtained from (4.1b) by replacing $A_{ij}(\omega_+)$ by $A_{ij}^0(\omega_+)$, that is, $m(x)$ by m_0 in (4.1c), and

$$L_{\text{int}} \equiv -\frac{\gamma^2\omega^4}{8} \int d^dx\, \phi_i^\mu \phi_i^\mu \phi_j^\nu \phi_j^\nu . \tag{4.3b}$$

The replica indices μ and ν here run freely from 1 to $n_+ + n_-$. In performing the average, we have used a probability distribution for the fluctuating part of the mass field $m'(x)$ of the form

$$P[m'(x)] \propto \exp\left[-\frac{1}{2\gamma^2}\int d^dx [m'(x)]^2\right] . \tag{4.4}$$

A Gaussian distribution of this form gives rise to a quartic coupling in the interaction part of the Lagrangian (4.3b). Strictly speaking, such a distribution includes a tail of physically inadmissible negative masses

$$m(x) = m_0 + m'(x) < 0 . \tag{4.5}$$

In principle, this unphysical tail could be removed by introducing a modified probability distribution which would produce couplings of a higher order. However, as we will show, interactions of a higher order than quartic are irrelevant to the critical behavior, and so we retain the form of the Lagrangian given by (4.3b).

Following standard methods, we introduce a conjugate field $Q_{\mu\nu}^{ij}$, where i and j are spatial indices and μ and ν are replica indices. The interacting part of the field theory (4.3b) can be represented in terms of Q as

$$e^{-L_{\text{int}}} = \text{const} \times \int DQ \exp\left[-\tfrac{1}{2}\int d^dx (Q_{\mu\nu}^{ij} Q_{\mu\nu}^{ij} + \gamma\omega^2 \phi_i^\mu \phi_j^\nu Q_{\mu\nu}^{ij})\right] . \tag{4.6}$$

This substitution clearly reduces the field theory to a Q-dependent quadratic form on the ϕ fields. Having introduced this conjugate field, all relevant Green's functions may be obtained from the generating function

$$Z(J) \equiv \lim_{\substack{n_+\to 0\\ n_-\to 0}} \int D\phi DQ \exp\left[-\tfrac{1}{2}\langle \phi, A(Q)\phi\rangle - \tfrac{1}{2}\int d^dx (Q_{\mu\nu}^{ij} Q_{\mu\nu}^{ij} - 2J_{\mu\nu}^{ij} Q_{\mu\nu}^{ij})\right] , \tag{4.7}$$

where the quadratic form involves a d-dimensional spatial integration and sums over the spatial and replica indices associated with the fields ϕ and Q:

$$\langle \underline{\phi}, A(Q)\underline{\phi}\rangle \equiv \int d^dx (\underline{\phi}_i^+(x), \underline{\phi}_i^-(x)) \begin{bmatrix} \underline{A}_{ij}^0(\omega_+) + \gamma\omega^2 \underline{Q}_{++}^{ij}(x) & \gamma\omega^2 \underline{Q}_{+-}^{ij}(x) \\ \gamma\omega^2 \underline{Q}_{-+}^{ij}(x) & \underline{A}_{ij}^0(\omega_-) + \gamma\omega^2 \underline{Q}_{--}^{ij}(x) \end{bmatrix} \begin{bmatrix} \underline{\phi}_j^+(x) \\ \underline{\phi}_j^-(x) \end{bmatrix} . \tag{4.8}$$

The superscripts + and − on the ϕ fields denote n_+- and n_-- component vectors, respectively, in replica space and likewise for each i and j, $\underline{Q}_{++}^{ij}(x)$ is an $n_+ \times n_-$ matrix. $\underline{A}_{ij}^0$ denotes A_{ij}^0 multiplied by the identity matrix in replica space. $Z(J)$ is essentially the functional integral (4.3a) with the addition of a souce $J_{\mu\nu}^{ij}$. By evaluating derivatives of $Z(J)$ with respect to J at $J=0$ before and after integration over ϕ in (4.7), we may express the averaged one- and two-particle Green's functions in terms of correlation functions involving only Q. Namely,

$$\langle Q_{++}^{ij}\rangle = \frac{\gamma\omega^2}{2}\langle G_{ij}(x,x,\omega_+)\rangle_{\text{ensemble}} , \tag{4.9a}$$

and similarly with two differentiations

$$\langle Q^{ij}_{+-}(x)Q^{kl}_{-+}(0)\rangle = \frac{\gamma^2\omega^4}{4}\langle G_{il}(x,0,\omega_+)G_{jk}(x,0,\omega_-)\rangle_{\text{ensemble}} . \tag{4.9b}$$

Here, Q^{ij}_{++} denotes the (1,1) element of the matrix $\underline{Q}^{ij}_{++}$. This latter correlation function is precisely that which appears in the average diffusivity (3.10). These are the quantities which we now proceed to evaluate in a mean-field approximation.

V. THE AVERAGED ONE-PARTICLE GREEN'S FUNCTION

By integrating over the ϕ fields in (4.7) we obtain a field theory involving only the Q field:

$$Z(0)=\lim_{\substack{n_+\to 0\\ n_-\to 0}}\int DQ\, e^{-L[Q]} , \tag{5.1a}$$

where

$$L[Q]\equiv \tfrac{1}{2}\ln\det\underline{A}(Q)+\tfrac{1}{2}\int d^dx\, Q^{ij}_{\mu\nu}Q^{ij}_{\mu\nu} . \tag{5.1b}$$

Mean-field theory corresponds to evaluating (5.1a) in a saddle-point approximation in the vicinity of a stationary point of $L[Q]$. A saddle point Q_0 of the Q field is a point such that $L[Q]$ contains no terms linear in a small displacement $\hat{Q}\equiv Q-Q_0$. Furthermore, if we expand L about Q_0, keeping only quadratic fluctuations in $\hat{Q}$, then the saddle-point value of Q determines the averaged one-particle Green's function of (4.9a):

$$Q_0=\langle Q\rangle . \tag{5.2}$$

We may formally perform such an expansion by writing the matrix $\underline{A}(Q)$ of Eq. (4.8) as

$$\underline{A}(Q)=\underline{A}(Q_0)+\gamma\omega^2\underline{\hat{Q}} \tag{5.3}$$

and then making use of the identity

$$\ln\det\underline{A}(Q)=\ln\det\underline{A}(Q_0)+\gamma\omega^2\mathrm{Tr}\underline{A}^{-1}(Q_0)\underline{\hat{Q}} -\tfrac{1}{2}\gamma^2\omega^4\mathrm{Tr}\underline{A}^{-1}(Q_0)\underline{\hat{Q}}\underline{A}^{-1}(Q_0)\underline{\hat{Q}} +\cdots . \tag{5.4}$$

We restrict our attention for the time being to those stationary points of $L[Q]$ that are relevant to the evaluation of the averaged one-particle Green's function, namely those which are diagonal in the replica indices μ and ν. Nondiagonal saddle points need only be taken into account at the stage of renormalization as will be explained later. In the present case, we may write

$$\underline{A}^{-1}_{ij}(Q_0)=\begin{bmatrix} G_{ij}(x,y,\omega_+)\underline{I} & \underline{0}\\ \underline{0} & G_{ij}(x,y,\omega_-)\underline{I}\end{bmatrix} , \tag{5.5a}$$

where

$$G_{ij}(x,y,\omega_\pm)\equiv\{[\underline{A}^0(\omega_\pm)+\gamma\omega^2\underline{Q}^0_{++}\underline{I}]^{-1}_{ij}\}_{xy} .$$

Using the expansion (5.4) and (5.1b), the condition for a diagonal stationary point is simply

$$Q^0_{++}=-\frac{\gamma\omega^2}{2}G(x,x,\omega_+) , \tag{5.6a}$$

$$Q^0_{--}=-\frac{\gamma\omega^2}{2}G(x,x,\omega_-) , \tag{5.6b}$$

$$Q^0_{+-}=Q^0_{-+}=0 . \tag{5.6c}$$

Here we have made use of the symmetry

$$\langle Q^{ij}_{++}(x)\rangle=\delta_{ij}Q^0_{++} . \tag{5.7a}$$

An analogous relation, of course, holds for $\underline{G}$:

$$G(x,x,\omega_+)\equiv\frac{1}{d}G_{ii}(x,x,\omega_+) . \tag{5.7b}$$

Physically, $\langle Q^{ij}_{++}(x)\rangle$ measures the average response of the ith component of the displacement field to an impulse in the jth component, both at the same point x. Since averaging restores isotropy to the medium, there is no preferred direction for the jth component to respond to and so the symmetry relation (5.7) is self-evident.

Equations (5.6) constitute a "coherent-potential approximation" (CPA) to the one-particle Green's function. The physical nature of this approximation becomes more apparent upon evaluating the on-site element of the CPA Green's function using the momentum representation of the operator

$$A^0_{ij}(\omega_+)=(m_0\omega_+^2+\gamma Q^0_{++}\omega_+^2-\mu k^2)\delta_{ij} -(\lambda+\mu)k_ik_j . \tag{5.8}$$

The matrix k_ik_j has one eigenvalue k^2 and the remaining $(d-1)$ eigenvalues are zero, allowing the trace appearing in (5.7b) to be expressed in terms of the eigenvalues of the matrix G_{ij},

$$G_L^+(k)\equiv[(m_0+\gamma Q^0_{++})\omega_+^2-(\lambda+2\mu)k^2]^{-1} \tag{5.9a}$$

and

$$G_T^+(k)\equiv[(m_0+\gamma Q^0_{++})\omega_+^2-\mu k^2]^{-1} . \tag{5.9b}$$

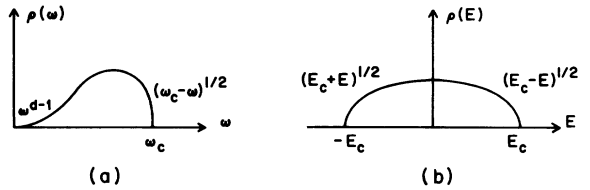

FIG. 1. Comparison of the phonon density of states (a) and the electron density of states (b) in the coherent-potential approximation.

Therefore,

$$Q^0_{++}=-\frac{\gamma\omega^2}{2d}\int d^dk[G_L^+(k)+(d-1)G_T^+(k)] . \tag{5.10}$$

G_L and G_T are longitudinal and transverse propagators, respectively. The form of Eqs. (5.9) reveals that the coherent-potential approximation consists of replacing the disordered elastic medium by an effective medium with a renormalized mass $m_0+\gamma Q^0_{++}$ which is complex whenever $\mathrm{Im}Q^0_{++}\neq 0$. The complex mass simply expresses, of course, the fact that k is not a good "quantum number" for the disordered medium and that states of a given k decay as a result of scattering.

It is shown in Appendix A that the CPA density of states

$$\rho(\omega)\propto\frac{1}{\omega}\mathrm{Im}Q^0_{++} \tag{5.11}$$

has ω^{d-1} behavior near $\omega=0$ and $(\omega_c-\omega)^{1/2}$ behavior at the upper band edge ω_c. Figure 1 shows a comparison of the phonon and electron density of states in the CPA.

VI. MEAN-FIELD THEORY FOR DIFFUSIVITY

By means of Eq. (4.9b), we may reexpress the diffusion coefficient (3.10) entirely in terms of the Q fields:

$$D=\lim_{\eta\to 0}2\eta\frac{\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}\int d^dx\,x^2\langle\hat{Q}^{kk}_{+-}(x)\hat{Q}^{ii}_{-+}(0)\rangle}{\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}\int d^dx\langle\hat{Q}^{kk}_{+-}(x)\hat{Q}^{ii}_{-+}(0)\rangle} . \tag{6.1}$$

We have used here the fact that $Q^0_{+-}=0$.

In evaluating D, we again retain only quadratic fluctuations in $\hat{Q}$ for the expansion of $L[Q]$ about its saddle point:

$$L[\hat{Q}]=\tfrac{1}{2}\int d^dx\,\hat{Q}^{ij}_{\mu\nu}\hat{Q}^{ij}_{\mu\nu}-\tfrac{1}{4}\gamma^2\omega^4\mathrm{Tr}\underline{A}^{-1}(Q_0)\underline{\hat{Q}}\underline{A}^{-1}(Q_0)\underline{\hat{Q}}+\cdots . \tag{6.2}$$

The terms of this expansion relevant to the evaluation of D may be written as

$$\tfrac{1}{4}\int d^dx\,d^dy\,\hat{Q}^{ij}_{+-}(x)C_{ijkl}(x-y)\hat{Q}^{kl}_{-+}(y) . \tag{6.3a}$$

With the use of the explicit form of matrix $A^{-1}(Q_0)$ given in (5.5a) it is straightforward to show that

$$C_{ijkl}(x-y)=2\delta^d(x-y)\delta_{ik}\delta_{jl}-\gamma^2\omega^4G_{li}(y,x,\omega_+)G_{jk}(x,y,\omega_-) . \tag{6.3b}$$

The correlation function appearing in (6.1) can now be formally expressed as

$$\langle\hat{Q}^{kk}_{+-}(x)\hat{Q}^{ii}_{-+}(0)\rangle=\langle x\mid C^{-1}_{kkii}\mid 0\rangle , \tag{6.4a}$$

where $\underline{C}^{-1}$ is defined by

$$\int d^dz\,C_{ijmn}(x-z)C^{-1}_{mnkl}(z-y)=\delta_{ik}\delta_{jl}\delta^d(x-y) . \tag{6.4b}$$

$\underline{C}$ is a function of the difference of its two coordinate space arguments so that by introducing a Fourier transform

$$C_{ijkl}(p)=2\delta_{ik}\delta_{jl}-\gamma^2\omega^4\int d^dx\,e^{ip\cdot x}\times G_{li}(0,x,\omega_+)G_{jk}(x,0,\omega_-) , \tag{6.5}$$

the diffusion coefficient can be expressed in terms of (6.5) as

$$D=-\lim_{\eta\to 0}2\eta\frac{\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_i}C^{-1}_{jjkk}(p)\mid_{p=0}}{\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}C^{-1}_{jjkk}(0)} . \tag{6.6}$$

From (6.5) it can also be seen that $\underline{C}$ is a Hermitian operator and thus can be diagonalized in its Cartesian indices by a unitary transformation

$$C_{ijkl}(p)=\sum_{m=1}^{d}\sum_{n=1}^{d}e^{mn}(p)v^{*mn}_{ij}(p)v^{mn}_{kl}(p) . \tag{6.7a}$$

Here, $e^{mn}(p)$ are the associated eigenvalues and the eigenvectors v^{mn} satisfy the orthonormality condition

$$v_{ij}^{*mn}(p)v_{ij}^{m'n'}(p)=\delta_{mm'}\delta_{nn'} . \tag{6.7b}$$

Examination of (6.6) reveals that $D=0$ unless at least one of the eigenvalues $e^{mn}(p)$ vanishes in the limit as $\eta\to 0$ for $p=0$. We show in Appendix B that there is in fact only one such eigenvalue which we denote $e^{11}(p)$, and for which the corresponding eigenvector $v_{kl}^{11}(p)=\delta_{kl}d^{-1/2}$. Since this is the only contributing eigenvalue in (6.6) as $\eta\to 0$, we may replace $C_{jjkk}^{-1}(p)$ in the same equation by the projection

$$\frac{1}{e^{11}(p)}v_{ii}^{*11}(p)v_{jj}^{11}(p)=\frac{d}{e_0^{11}+e_{rs}^{11}p_rp_s+\cdots} . \tag{6.8}$$

In terms of the low-momentum expansion coefficients of $e^{11}(p)$, namely e_0^{11} and e_{rs}^{11}, the diffusion coefficient becomes

$$D=\lim_{\eta\to 0}2\eta\frac{\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}\frac{C_2(\omega)}{C_0^2(\omega)}}{\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}\frac{1}{C_0(\omega)}} , \tag{6.9a}$$

where

$$C_0(\omega)\equiv e_0^{11}(\omega) \tag{6.9b}$$

and

$$C_2(\omega)\equiv\sum_i e_{ii}^{11}(\omega) . \tag{6.9c}$$

We infer, using (3.11), that the energy density and frequency-dependent diffusivity are given by

$$E(\omega)\propto\frac{1}{\omega^2 C_0(\omega)} \tag{6.10a}$$

and

$$D_0(\omega)\propto\frac{C_2(\omega)}{C_0(\omega)} . \tag{6.10b}$$

We leave the explicit evaluation of the functions $C_0(\omega)$ and $C_2(\omega)$ to Appendix B and simply state the results here. From (6.5) we see that

$$C_{ijkl}(p)=C_{ijkl}^0+C_{ijkl}^{mn}p_mp_n+\cdots , \tag{6.11a}$$

where

$$C_{ijkl}^0\equiv 2\delta_{ik}\delta_{jl}-\gamma^2\omega^4\int d^dp\, G_{li}^{+}(p)G_{jk}^{-}(p) \tag{6.11b}$$

and

$$C_{ijkl}^{mn}\equiv\gamma^2\omega^4\int d^dp\,\frac{\partial}{\partial p_m}G_{li}^{+}(p)\frac{\partial}{\partial p_n}G_{jk}^{-}(p) . \tag{6.11c}$$

$C_0(\omega)$ and $C_2(\omega)$ are the eigenvalues corresponding to the eigenvector δ_{kl} of C_{ijkl}^0 and C_{ijkl}^{mm} respectively:

$$C_0(\omega)=\frac{4\eta m_0 d}{\gamma\omega\,\mathrm{Im}Q_{++}^0}\sim\eta\gamma^{-2}\omega^{-(d+1)} \tag{6.12a}$$

near $\omega=0$, and

$$C_2(\omega)=\frac{\gamma\omega^4}{d}\int d^dp\frac{\partial}{\partial p_m}G_{li}^{+}(p)\frac{\partial}{\partial p_m}G_{il}^{-}(p)$$

$$\sim\gamma^{-4}\omega^{-(2d+2)} . \tag{6.12b}$$

Using (6.10b) we arrive at the result for the band-edge behavior of the diffusivity:

$$D_0(\omega)\sim\gamma^{-2}\omega^{-(d+1)} . \tag{6.13}$$

In $d=3$, this gives an ω^4 low-frequency phonon scattering cross section, reminiscent of Rayleigh scattering. Physically, such behavior arises because of the extreme short-range nature of the scattering centers as expressed by the δ-function correlation of the random mass disorder.

VII. RENORMALIZATION

We wish to determine the nature of the normal modes of frequency ω by considering how $D_0(\omega)$ renormalizes as we integrate over short-wavelength fluctuations of the field theory and rescale to longer lengths. Following Schäfer and Wegner,[1] we define an effective Lagrangian based on (6.2) and (6.3), retaining only second-order fluctuations in $\hat{Q}$ and only second-order terms in the low-momentum expansion (6.8) of the operator $\underline{C}$. In coordinate space this becomes

$$L^{\mathrm{eff}}[\hat{Q}]=\tfrac{1}{4}C_{ijkl}^{mn}(\omega)\int d^dx\,\partial_m\hat{Q}_{\mu\nu}^{ij}(x)\partial_n\hat{Q}_{\nu\mu}^{kl}(x) . \tag{7.1}$$

We neglect here the remaining nonzero eigenvalues of the matrix C_{ijkl}^0, and all other finite-mass terms in obtaining this massless field theory in $\hat{Q}$. Also, since Q_0 is independent of x as a result of translational symmetry, (7.1) is invariant under the substitution $\hat{Q}\to Q$.

In the spirit of the previous mean-field approximation, we would like to restrict Q to the vicinity of the previously determined saddle point Q_0. However, in doing so, we note that the exact Lagrangian $L[Q]$ described by Eq. (5.1) has an internal global symmetry. Namely, $L[Q]$ is invariant if an orthogonal transformation $\underline{U}\in O(n_++n_-)$ on the replica indices is simultaneously applied to each of the matrices $\underline{Q}^{ij}$:

$$\underline{Q}^{ij}\rightarrow\underline{U}^T\underline{Q}^{ij}\underline{U}\ . \tag{7.2}$$

This is a direct consequence of the fact that when $\eta\rightarrow 0$, all of the replica fields ϕ_i^μ of (4.3) are equivalent and can be freely rotated among themselves. Therefore, for every orthogonal transformation $\underline{U}$ that can be applied to the original diagonal saddle point $\underline{Q}_0$, there is a new saddle point $\underline{U}^T\underline{Q}_0\underline{U}$. Instead of restricting $\underline{Q}$ in (7.1) to a neighborhood of the original saddle point $\underline{Q}_0$, we should extend it to a neighborhood that encompasses the entire manifold of saddle points generated by the transformations (7.2). The simplest nontrivial neighborhood of this type is given by

$$\begin{aligned}\underline{Q}(x)&\equiv\underline{U}^T(x)\underline{Q}_0\underline{U}(x)\\&=\mathrm{Re}(Q^0_{++})\begin{bmatrix}\underline{I}&\underline{0}\\\underline{0}&\underline{I}\end{bmatrix}\\&\quad+i\,\mathrm{Im}(Q^0_{++})\underline{U}^T(x)\begin{bmatrix}\underline{I}&\underline{0}\\\underline{0}&-\underline{I}\end{bmatrix}\underline{U}(x)\ .\end{aligned} \tag{7.3}$$

Here $\underline{U}$ is allowed to fluctuate from point to point in space. These fluctuations correspond to the fluctuations away from the saddle-point manifold that we will allow in the effective Lagrangian (7.1). We have used the symmetry condition (5.7a) to eliminate the spatial indices on $\underline{Q}(x)$. With the restriction (7.3), the effective Lagrangian simplifies to

$$L^{\mathrm{eff}}[Q]=\tfrac{1}{4}C^{mn}_{iijj}(\omega)\int d^dx\,\partial_m Q_{\mu\nu}(x)\partial_n Q_{\nu\mu}(x)\ . \tag{7.4}$$

But $C^{mn}_{iijj}=C_2(\omega)\delta_{nm}$, as can be verified by comparing (6.11c) with (6.12b). Thus by rescaling $Q(x)$ in (7.4) and dropping the real part which is independent of x, we arrive at the final form of the effective Lagrangian:

$$L^{\mathrm{eff}}[Q]=\frac{1}{g_0^2}\int d^dx\,\partial_m Q_{\mu\nu}(x)\partial_m Q_{\nu\mu}(x)\ , \tag{7.5a}$$

where

$$\frac{1}{g_0^2}=\frac{C_2(\omega)}{4}(\mathrm{Im}\langle Q_{++}\rangle)^2\sim\frac{1}{\omega^2} \tag{7.5b}$$

and

$$\underline{Q}(x)=i\underline{U}^T(x)\begin{bmatrix}\underline{I}&\underline{0}\\\underline{0}&-\underline{I}\end{bmatrix}\underline{U}(x)\ . \tag{7.5c}$$

Combining (6.10b) and (6.12a) we see that the coupling constant here is the frequency-dependent bare conductance

$$\frac{1}{g_0^2}\propto D_0(\omega)\rho(\omega)\ . \tag{7.6}$$

The physical picture that emerges from this generalized nonlinear σ model is that of a phase transition in a fixed-length spin system. Straightforward momentum shell integration and rescaling, or otherwise,[10–12] yields the following differential recursion relation for the resistance g^2 in $d=2+\epsilon$ dimensions:

$$\frac{dg^2(L)}{d\ln L}=-\epsilon g^2-(n_+ +n_- -2)(g^2)^2\ . \tag{7.7}$$

In the limit $n_+ +n_-\rightarrow 0$, we obtain the fundamental scaling relation for the zero-temperature thermal resistance

$$\frac{dg^2(L)}{d\ln L}=-\epsilon g^2+2(g^2)^2\ . \tag{7.8}$$

This is identical in form to the scaling relation for the resistance of an electron in a random potential.[1,2] In two dimensions and below, there is a single unstable fixed point at $g^2=0$. Flow is toward infinite resistance on longer length scales so that all states are localized. In both one and two dimensions we obtain localization lengths that diverge as we approach the single extended state at $\omega=0$. For $d=1$ (7.8) can be approximated by

$$\frac{dg^2}{d\ln L}=g^2\ , \tag{7.9a}$$

which has the solution

$$\frac{g^2}{g_0^2}\simeq L/L_0\ . \tag{7.9b}$$

The length scale L, at which the resistance g^2 reaches some fixed value such as $g^2=1$, defines the localization length ξ:

$$\xi/L_0\sim\frac{1}{\omega^2}\quad(d=1) \tag{7.9c}$$

where we have made use of the asymptotic behavior (7.5b) of the bare conductance $1/g_0^2$. A similar analysis for $d=2$ yields an even more rapidly diverging localization length near zero frequency[7]:

$$\xi/L_0\sim e^{1/\omega^2}\quad(d=2)\ . \tag{7.10}$$

Above two dimensions ($\epsilon>0$), the $g^2=0$ fixed point exchanges stability with a new fixed point at $g^{*2}=\epsilon/2$. This new unstable fixed point represents a mobility edge between localized and extended phonon states. Solving (7.8) in general for $\epsilon\neq 0$ we obtain

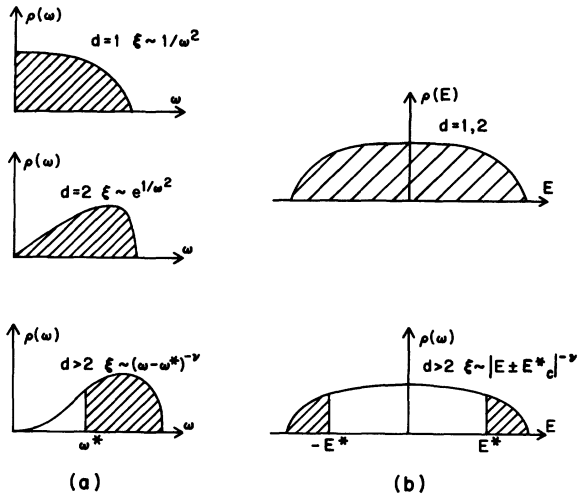

FIG. 2. For both phonons (a) and electrons (b) all states are localized (shaded) in two or less dimensions. Above two dimensions high-frequency phonon states may become localized whereas the electron band may show localization at both the upper and lower band edges.

$$L/L_0=\left[\frac{g_0^2}{g^2}\left[\frac{g^2-\epsilon/2}{g_0^2-\epsilon/2}\right]\right]^{1/\epsilon} . \tag{7.11}$$

Since $\epsilon/2$ corresponds to a phonon mobility edge ω^*, we may write

$$g_0^2-\epsilon/2\sim(\omega-\omega^*) . \tag{7.12}$$

Setting $g^2=1$, we see that the localization length diverges at the mobility edge as

$$\xi/L_0\sim\frac{1}{(\omega-\omega^*)^{1/\epsilon}} . \tag{7.13}$$

This is of course the same localization length exponent that occurs in the problem of electron localization in $2+\epsilon$ dimensions. Also, in the region of extended states the conductivity

$$\frac{1}{g^2}\sim(\omega^*-\omega)^t,\quad t=(d-2)\nu=1 . \tag{7.14}$$

The feature that distinguishes phonon localization from that of electrons is the asymptotic form of the bare conductance (7.5b). For sufficiently low frequencies the bare resistance $g_0^2\sim\omega^2$ will always lie below the nontrivial fixed point at $\epsilon/2$, and hence flow will be toward zero resistance. Therefore, low-frequency phonon states in $2+\epsilon$ dimensions are extended. This is to be contrasted with results for an electron in a random potential, where it can be shown that the bare conductance at either CPA band edge $\pm E_c$ is of the form

$$\frac{1}{g_0^2}\sim|E\pm E_c| . \tag{7.15}$$

Here, the bare resistance will lie above the fixed point $\epsilon/2$ and be renormalized to infinity, giving rise to localized states at the top and bottom of the band. These results are summarized in Fig. 2.

VIII. DISCUSSION

We have shown that the transition from extended to localized states of a disordered elastic medium, like the analogous phenomenon for an electron in a random potential, can be described as a phase transition in a generalized fixed-length spin system. The "spin" here is actually a complex matrix and the resistance g^2 plays the role of a temperature variable. The theory reveals a consistent preference for localization of the high-frequency modes by way of the diverging localization lengths as $\omega\to0$ in two dimensions and below, as well as the existence of a single mobility edge separating high-frequency localized states from low-frequency extended states in $2+\epsilon$ dimensions. In all cases the $\omega=0$ mode corresponding to uniform translation of the entire medium is extended. The roots of this preference can of course be traced to the problem of a single defect in a phonon system. For example, in a periodic one-dimensional system of masses connected by springs, it is easy to verify that the introduction of a single light mass results in the splitting of a single localized state from the top of the phonon band.

We have presented only the case of "site-diagonal" disorder. We have studied the effects of introducing disorder into the Lamé coefficients in (2.1a) in a simpler scalar theory described by a Lagrangian

$$L=\int d^dx[\tfrac{1}{2}m_0\dot{\phi}\dot{\phi}-\tfrac{1}{2}V(x)\partial_i\phi\partial_i\phi] , \tag{8.1}$$

where the fluctuating part of the spring constants $V(x)=V_0+V'(x)$ satisfies

$$\langle V'(x)V'(y)\rangle=\gamma^2\delta^d(x-y) .$$

The localization properties of this model are in fact the same as those summarized in Fig. 2.

The low-temperature thermal transport properties of the medium are determined by the thermal conductivity

$$K(T)\equiv\int d\omega\,\rho(\omega)C(\omega,T)D_0(\omega) , \tag{8.2a}$$

where $C(\omega,T)$ is the contribution of modes of frequency ω to the specific heat:

$$C(\omega,T)\equiv\hbar\omega\frac{\partial}{\partial T}(e^{\hbar\omega/k_BT}-1)^{-1} . \tag{8.2b}$$

From Eqs. (7.5b) and (7.6) we see that mean-field theory predicts a low-frequency ω^{-2} behavior for the quantity $\rho(\omega)D_0(\omega)$. As a result, the thermal conductivity $K(T)$ has an infrared divergence of the form

$$\int d\omega \frac{1}{\omega^2} , \tag{8.3}$$

or in other words, it diverges linearly with the length L of the sample. This is the usual result for a model that does not take into account inelastic scattering processes arising from lattice anharmonicity.[13] Phonon-phonon interactions, however, which in a real solid render the thermal conductivity finite, introduce into the problem a new length scale, the inelastic mean free path, which must be longer than the localization length ξ if localization of phonons in the harmonic approximation is to be important. The situation is further complicated by the fact that the localization length diverges in the low-frequency regime accessible by low-temperature experiments. The competing effects of localization and inelastic scattering on the thermal diffusivity have been discussed recently by Jackle.[6] He suggested that in certain types of solids which have a quasi-one-dimensional chainlike structure a temperature range may exist in which phonon localization is important. Another way to observe localization might be to study phonons injected into solids at sufficiently low temperatures where inelastic scattering effects are small.

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APPENDIX A: DENSITY OF STATES IN THE COHERENT-POTENTIAL APPROXIMATION

The density of normal modes in the range ω^2 to $\omega^2+d\omega^2$ is given by

$$N(\omega^2)\equiv \frac{1}{\pi}\mathrm{Im}G_{ii}(x,x,\omega_-) . \tag{A1}$$

The density of states $\rho(\omega)$ is obtained by multiplying (A1) by 2ω. Combining (A1) and (5.10) yields

$$\rho(\omega)=-\frac{2\omega}{\pi}\mathrm{Im}\int_0^\Lambda d^dk[G_L^+(k)+(d-1)G_T^+(k)] , \tag{A2}$$

where Λ is the ultraviolet cutoff of the field theory. The nature of the density of states near $\omega=0$ is determined by the fact that

$$\mathrm{Im}\int_0^\Lambda \frac{dk\,k^{d-1}}{k^2-z}\sim z^{(d-2)/2} \tag{A3}$$

as $z\to 0$. With the use of Eqs. (5.9), this yields

$$\rho(\omega)\sim\omega(\omega^2)^{(d-2)/2}\sim\omega^{d-1} . \tag{A4}$$

The upper band edge ω_c is determined by the cutoff Λ. We obtain the relevant exponent by rewriting Eq. (5.10) in terms of $G^+\equiv-(2/\gamma\omega^2)Q^0_{++}$ and expanding it about its value at ω_c, which we denote by G_c^+:

$$\delta G^+\equiv G^+-G_c^+ , \tag{A5a}$$

$$\delta\omega^2\equiv\omega^2-\omega_c^2\simeq 2\omega_c(\omega-\omega_c) . \tag{A5b}$$

It is straightforward to show that

$$\delta G^+=\sum_{n=1}^{\infty} c_n[a(\delta\omega)^2-b\,\delta G^+]^n , \tag{A6a}$$

where

$$a=m_0-\gamma\omega_c^2G_c^+ , \tag{A6b}$$

$$b=\frac{\gamma}{2}\omega_c^4 . \tag{A6c}$$

It follows from (5.10) that as $\omega\to\omega_c$,

$$\mathrm{Im}G^+=\frac{b}{d}\mathrm{Im}G^+\int_0^\Lambda d^dk\left[\frac{1}{[\mathrm{Re}G_L^{-1}(k)]^2}+\frac{(d-1)}{[\mathrm{Re}G_T^{-1}(k)]^2}\right] . \tag{A7}$$

Here, we have used the fact that $\mathrm{Im}G_L=\mathrm{Im}G_T=\mathrm{Im}G^+=0$ when $\omega=\omega_c$. Comparing (A7) with the expansion (A6) yields the fundamental result $-bc_1=1$. For small excursions from the band edge ω_c, (A6a) can then be simplified to

$$0=c_1a(\delta\omega)^2+c_2[a(\delta\omega)^2-b\,\delta G^+]^2 . \tag{A8}$$

Solving for δG^+ and taking the imaginary part, we obtained the desired result

$$\mathrm{Im}G^+\sim(\omega_c-\omega)^{1/2} . \tag{A9}$$

APPENDIX B: EVALUATION OF $C_0(\omega)$ AND $C_2(\omega)$

We demonstrate the existence of a Goldstone mode in the Q_{+-} channel by showing that $C_0(\omega)$ vanishes as $\eta\to 0$. The averaged one-particle

Green's function in (6.11) can be written in the form

$$G_{ij}^{+}(p)=G_L^{+}(p)\frac{p_i p_j}{p^2}+G_T^{+}(p)\left[\delta_{ij}-\frac{p_i p_j}{p^2}\right]. \tag{B1}$$

Clearly $G_{ij}^{+}(p)$ is odd in p_i and p_j for $i\neq j$. Therefore,

$$C_0(\omega)=2-\frac{\gamma^2\omega^4}{d}\int d^d p\, G_{ii}^{+}(p)G_{il}^{-}(p) \tag{B2a}$$

$$=2-\frac{\gamma^2\omega^4}{d}\int d^d p\, F(p)\,, \tag{B2b}$$

where

$$F(p)=G_L^{+}(p)G_L^{-}(p)+(d-1)G_T^{+}(p)G_T^{-}(p)\,. \tag{B2c}$$

This can be simplified by writing the imaginary part of Eq. (5.10) in terms of $G^{+}\equiv-(2/\gamma\omega_{+}^2)Q_{++}^0$:

$$\mathrm{Im}G^{+}=\frac{1}{d}\left[\frac{\gamma^2\omega^4}{2}\mathrm{Im}G^{+}+2\omega\eta(\gamma^2\omega^2\mathrm{Re}G^{+}-m_0)\right]\times\int d^d p\, F(p)\,. \tag{B3}$$

Dividing by $\mathrm{Im}G^{+}$ and rearranging we observe that

$$\tfrac{1}{2}C_0(\omega)=2\omega\eta\frac{\gamma^2\omega^2\mathrm{Re}G^{+}-m_0}{\mathrm{Im}G^{+}}\int d^d p\, F(p)\,. \tag{B4}$$

Taking the limit $\eta\to 0$ in (B3) we observe also that the integral appearing in (B4) is simply $2d/\gamma^2\omega^4$. In the limit of weak disorder $\gamma\ll m_0$, we obtain the result

$$C_0(\omega)=-\frac{8dm_0\eta}{\gamma^2\omega^3\mathrm{Im}G^{+}}\sim\eta\gamma^{-2}\omega^{-(d+1)}\,. \tag{B5}$$

The calculation of $C_2(\omega)$ given in (6.12b) is likewise facilitated by the decomposition (B1):

$$\frac{\partial}{\partial p_m}G_{ii}^{+}(p)\frac{\partial}{\partial p_m}G_{il}^{-}(p)=(d-1)\frac{\partial G_T^{+}}{\partial p_m}\frac{\partial G_T^{-}}{\partial p_m}+\frac{\partial G_L^{+}}{\partial p_m}\frac{\partial G_L^{-}}{\partial p_m}+\frac{2(d-1)}{p^2}(G_L^{+}-G_T^{+})(G_L^{-}-G_T^{-})\,. \tag{B6}$$

With the use of the above equation and the relations

$$\frac{\partial}{\partial p_m}G_T^{+}=2\mu p_m G_T^{+}G_T^{+} \tag{B7a}$$

and

$$\mathrm{Im}G_T^{+}=-\gamma\omega^2 Q_{++}^0 G_T^{+}G_T^{-}\,, \tag{B7b}$$

it is possible to express (6.12b) as

$$C_2(\omega)=\frac{8}{d(\mathrm{Im}Q_{++}^0)^2}\times\int d^d p\, p^2[2\mu^2(d-1)(\mathrm{Im}G_T^{+})^2+2(\lambda+2\mu)^2(\mathrm{Im}G_L^{+})^2+(\lambda+\mu)^2(d-1)\mathrm{Im}G_T^{+}\mathrm{Im}G_L^{+}]\,. \tag{B8}$$

Consider for instance the first term here, which can be rewritten using (5.9) as

$$I\equiv\int d^d p\, p^2(\mathrm{Im}G_T^{+})^2=f^2\int\frac{d^d p\, p^2}{[(h-\mu p^2)^2+f^2]^2}\,, \tag{B9a}$$

where

$$f(\omega)=\gamma\omega^2\mathrm{Im}Q_{++}^0\sim\omega^{d+2} \tag{B9b}$$

and

$$h(\omega)=\omega^2(m_0+\gamma\mathrm{Re}Q_{++}^0)\sim\omega^2\,. \tag{B9c}$$

Since f vanishes rapidly as $\omega\to 0$, the dominant contribution to the integral I comes from the region $\mu p^2\simeq h$. It is straightforward to show that for sufficiently small ω

$$I\sim h^{d/2}f^{-1}\sim\omega^d(\gamma\omega^2\mathrm{Im}Q_{++}^0)^{-1}\,. \tag{B10}$$

Using the fact that $\mathrm{Im}Q_{++}^0\sim\gamma\omega^d$ gives the result

$$C_2(\omega)\sim\gamma^{-4}\omega^{-(2d+2)}\,. \tag{B11}$$

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