Dynamics of classical and quantum spin systems with random fields

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We study the effect of quenched time-independent random fields coupled linearly to the order parameter on the dynamical critical behavior of spin systems. We assume that the dynamics is described by a Langevin equation without conservation of the order parameter. It is shown that the dominant fluctuations are those induced by the random fields and, therefore, thermal fluctuations are irrelevant. This allows us to establish a relation between this model and a quantum spin system in the presence of a quenched random field. Moreover, we find that *only* static exponents in D dimensions are the same as those of the pure $(D-2)$-dimensional theory, but the dynamical exponent z does not satisfy this relation. The quantum system in D dimensions is studied through its $(D+1)$-dimensional equivalent model where the quenched random fields are totally correlated in the additional imaginary-time (τ) direction. The system is anisotropic, and there is a new exponent z_A associated with the scaling behavior in the τ direction. We find the relation $z=2z_A$ to all orders in perturbation theory. For the zero-temperature quantum model we find that the static (zero-frequency) exponents are the same as those of the $(D-3)$-dimensional pure quantum model. At finite temperature, when the classical system is finite in the τ direction, we predict a crossover to D-dimensional classical behavior in nonstatic response and correlation functions, with crossover exponent $(z_A\nu_{(D-2)})^{-1}$, where $\nu_{(D-2)}$ is the exponent ν for the $(D-2)$-dimensional pure classical system. The static correlation functions do not have this crossover behavior and are the same as those of the $(D-3)$-dimensional pure quantum model. The dimensional shift in static quantities for both quantum and classical dynamical systems is a consequence of a supersymmetry in the underlying field theory. The exponent $z_A=1+\eta+O(1/N^2)$ in the large-N limit or $z_A=1+c\eta+O(\epsilon^4)$ in the ϵ expansion where $c=1-\frac{3}{4}[(N+2)/(N+8)^2]\epsilon$ and η is the same as in the $(D-2)$-dimensional pure classical system. We also study the above classical random-field Ising problem using the interface approach, but are unable to draw any definite conclusion about the dynamics at the lower critical dimension $D=3$.

I. INTRODUCTION

The study of the static critical behavior of spin systems with quenched random fields coupled linearly to the order parameter has been the subject of recent work.[1–4] It has been recognized that the leading contribution to the critical behavior arises from diagrams which are treelike before averaging over the random field, and that order by order in perturbation theory these diagrams in D dimensions are identically equal to the corresponding diagrams for the pure case in $D-2$ dimensions.[3–5]

This dimensionality shift by 2 has been shown to be a consequence of a hidden supersymmetry in the Lagrangian field theory.[6] More recently, this result was extended beyond perturbation theory using the supersymmetric model.[7]

The tree diagrams mentioned above are generated by the classical equations of motion, and the loop expansion arises after averaging over the random-field probability distribution. This means that the relevant fluctuations are due to the random fields and thermal disorder is irrelevant.

In this paper we try to understand whether these features of the static critical behavior in the presence of quenched random fields hold also for the case of the dynamical critical phenomena of spin systems coupled to time-independent quenched random fields. We analyze the simplest dynamical model without conservation of the order parameter, described by a Langevin equation of motion.[8,9] Several interesting questions arise in this case. In the first place we note that the Langevin equation is first order in the time derivatives and that this will certainly make it difficult to find a supersymmetry transformation in the Lagrangian, for which second-order derivatives are required. In dynamical critical phenomena, correlation and response functions are related by the fluctuation-dissipation theorem, and this may be modified if again thermal disorder is shown to be irrelevant, and only fluctuations of the (time-independent) random fields are responsible for the critical behavior.

We will show that this problem of Langevin dynamics is closely related to a model recently proposed by Aharony, Gefen, and Shapir[10] (hereafter AGS) to study the critical properties of a D-dimensional zero-temperature quantum spin system in the presence of random fields.

In AGS the equivalent $(D+1)$-dimensional classical system, with the random field infinitely correlated in the new τ (imaginary-time) direction and uncorrelated in D dimensions, is studied. Again it is argued that the most infrared divergent diagrams are treelike before averaging over the random-field configurations. They calculated the lowest-order contribution to the spin-spin correlation function and concluded that there is a dimensional shift $D \to D-3$. Their analysis led them to conclude that at finite temperature when the classical $(D+1)$-dimensional system is finite in the (imaginary-) time direction $0 \leq \tau \leq \beta = 1/kT$, there is no quantum-to-classical crossover. In Sec. II we investigate this further and conclude that zero-frequency (static) correlation functions like $\int d\tau \langle S(x,\tau) S(x',0) \rangle$ do not have crossover behavior, and their critical properties are the same as the $(D-3)$-dimensional quantum system. Therefore the dimensional shift $D \to D-3$ holds for static correlation functions. However, nonstatic response or correlation functions like the equal-time correlation function $\langle S(x,\tau) S(0,\tau) \rangle$ do have a crossover behavior to that of a classical D-dimensional system. We find that the fact the random field is totally correlated in the τ direction makes the system anisotropic, space and time scale in a different way, and there is an exponent z_A that characterizes this different scaling behavior.[11] There are different correlation lengths in space and time, and for the system with finite thickness β in the τ direction, we expect two different behaviors if the correlation length in the τ direction ξ_τ is smaller or larger than β. When $\xi_\tau \ll \beta$ we expect the system to behave as if it were infinite in τ, namely at $T=0$, therefore pure quantum behavior. For $\xi_\tau \gg \beta$ the classical system will behave as if it were D dimensional ($T \to \infty$), and we expect classical D-dimensional behavior. This crossover will take place when $\xi_\tau \approx \beta$, and we find the crossover exponent is $(z_A \nu_{(D-2)})^{-1}$; $\nu_{(D-2)}$ is the exponent ν for the $(D-2)$-dimensional classical pure system.

As it was pointed out before, the most divergent contributions arise from tree diagrams before averaging over the random field. These diagrams are generated by the classical equation of motion which is second order in the time derivatives. At this point, one is tempted to draw a similarity between this equation of motion and the Langevin equation. However, there are substantial differences: The latter involves a Gaussian noise and first-order derivatives in time and, therefore, the perturbative expansion involves retarded propagators and also averages over the random (Gaussian) noise.

In Sec. III we analyze the Langevin dynamics of a spin system without conservation of the order parameter and linearly coupled to a time-independent quenched random field. We show that the infrared behavior is dominated by diagrams that only involve the average over random fields and, therefore, the noise term can be neglected. Since the random field is time independent the perturbative expansion is in terms of nonretarded propagators. We find a dimensional shift $D \to D-2$ for static exponents, and the dynamical exponent z is related to z_A of the AGS model by $z = 2z_A$. In Sec. IV we calculate the exponent z_A and show that it is related to the static exponent η up to order ϵ^3 in the ϵ expansion, and up to order $1/N$ in the large-N limit. We find $z_A = 1 + c\eta + O(\epsilon^4)$, where

$$c = \left[1 - \frac{3}{4} \frac{N+2}{(N+8)^2} \epsilon \right]$$

and $z_A = 1 + \eta + O(1/N^2)$, where η is the exponent of the $(D-2)$-dimensional pure system. The relation between z_A and η is a consequence of a symmetry of the Feynman diagrams to that order.

Section V is devoted to a discussion of the physically interesting case of the dynamics for Ising-type systems with random fields for $D=3$. The statics are described by the interface model of Kogon and Wallace,[12] and we propose a Langevin equation to study the dynamics. However, the theory is highly nonlocal in time and the interactions are nonpolynomial. It is not clear to us if the model is renormalizable and we are unable to draw any conclusions about the dynamical exponent.

In the Appendix we discuss technical details of the supersymmetry involved, and the Fourier transform in superspace is introduced. The dimensional shift by 2 is proved using the superpropagator.

II. QUANTUM SYSTEM

The correspondence between a D-dimensional quantum spin system at zero temperature with uncorrelated random longitudinal field, and a $(D+1)$-dimensional classical system with a random field uncorrelated in D dimensions but infinitely correlated in the new (imaginary-) time direction has been studied by AGS.[10] The classical system is described by a Ginzburg-Landau free-energy functional

$$F=\int d^Dx\,[\tfrac{1}{2}(\vec{\nabla}\varphi)^2+V(\varphi)+h\varphi]\ .$$

Before studying the $(D+1)$-dimensional problem with time-independent fields, we will survey the main features of the "static" D-dimensional classical system with uncorrelated random fields. Looking at the perturbative series it has been argued that the diagrams that contribute most to the critical properties are those that are treelike before averaging over the fields. Upon averaging, the loop expansion is generated and two "branches" of the tree diagram are joined together giving rise to a squared propagator. The tree diagrams are generated by the classical equation of motion,

$$-\nabla^2\varphi+V'(\varphi)+h=0\ , \tag{2.1}$$

with

$$\langle\langle h(x)\rangle\rangle=0,\quad \langle\langle h(x)h(x')\rangle\rangle \propto \delta(x-x')\ , \tag{2.2}$$

where enclosure by the double angular brackets stands for the quenched average generated by the probability distribution,

$$P[h]\sim\exp\left[-\tfrac{1}{2}\int h^2(x)d^dx\right]\ . \tag{2.3}$$

Using standard techniques,[13,14] we can write

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle\sim\int\mathscr{D}h\,\mathscr{D}\varphi\,\varphi(x_1)\cdots\varphi(x_n)\delta(-\nabla^2\varphi+V'(\varphi)+h)\times\det[-\nabla^2+V''(\varphi)]\exp\left[-\tfrac{1}{2}\int h^2(y)d^Dy\right]\ .$$

The δ function can be replaced using a response field $\hat{\varphi}$, and the determinant can be written using anticommuting scalar (ghost) fields $\bar{\psi}$ and ψ. After integrating over the random field, one is led to

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle\sim\int\mathscr{D}\hat{\varphi}\,\mathscr{D}\varphi\,\mathscr{D}\psi\exp\left[-\int d^Dy\,\mathscr{L}[\hat{\varphi},\varphi,\psi]\right]\varphi(x_1)\cdots\varphi(x_n)\ , \tag{2.4}$$

with

$$\mathscr{L}=-\tfrac{1}{2}\hat{\varphi}^2+\hat{\varphi}[-\nabla^2\varphi+V'(\varphi)]+\bar{\psi}[-\nabla^2+V''(\varphi)]\psi\ . \tag{2.5}$$

The above derivation assumes the uniqueness of the solution of (2.1). While this may not be justified in general, we are concerned only with demonstrating a perturbative result. Within perturbation theory, the solution is unique. In fact, the Feynman diagrams corresponding to (2.5) reproduce, order by order, the most-infrared-singular diagrams of the original problem. It has been noted by Parisi and Sourlas[6] that this Lagrangian is invariant under the unexpected supersymmetric transformations,

$$\begin{aligned}&\delta\varphi=-\bar{a}\epsilon_\mu x_\mu\psi,\quad \delta\hat{\varphi}=2\bar{a}\epsilon_\mu\partial_\mu\psi\ ,\\ &\delta\psi=0,\quad \delta\bar{\psi}=\bar{a}(\epsilon_\mu x_\mu\hat{\varphi}+2\epsilon_\mu\partial_\mu\varphi)\ ,\end{aligned} \tag{2.6}$$

where $\bar{a}$ is an infinitesimal anticommuting number and ϵ_μ an arbitrary vector.

These authors then introduce a superspace characterized by D commuting coordinates x_μ and two anticommuting coordinates $\theta,\bar{\theta}$ with the property

$$\theta^2=\bar{\theta}^2=\theta\bar{\theta}+\bar{\theta}\theta=0 \tag{2.7}$$

and the superfield

$$\Phi(x,\theta,\bar{\theta})=\varphi(x)+\bar{\theta}\psi(x)+\bar{\psi}(x)\theta+\theta\bar{\theta}\hat{\varphi}(x)\ . \tag{2.8}$$

The change in the superfield under the transformations (2.6) can be written as[15]

$$\begin{aligned}&\delta\Phi(x,\theta,\bar{\theta})=\bar{a}\epsilon_\mu Q_\mu\Phi(x,\theta,\bar{\theta})\ ,\\ &Q_\mu=\left[\theta\frac{\partial}{\partial x_\mu}-x_\mu\frac{\partial}{\partial\bar{\theta}}\right]\ .\end{aligned} \tag{2.9}$$

Clearly Q_μ is the generator of superrotations in the $\{x,\theta,\bar{\theta}\}$ superspace. In terms of the superfield, the action in (2.4) can be written as $\int d\bar{\theta}\,d\theta\,\mathscr{L}_{\rm SS}[\Phi]$, where

$$\begin{aligned}&\mathscr{L}_{\rm SS}[\Phi]=-\tfrac{1}{2}\Phi\Delta_{\rm SS}\Phi+V(\Phi)\ ,\\ &\Delta_{\rm SS}=\nabla^2+\frac{\partial^2}{\partial\bar{\theta}\,\partial\theta}\ .\end{aligned} \tag{2.10}$$

The invariance of (2.10) under the transformations (2.9) allow Parisi and Sourlas[6] to prove the dimensional shift $D\to D-2$ to all orders. In the Appendix we introduce the Fourier transform in superspace and prove the dimensional reduction in momentum space.

Let us define anticommuting variables $\bar{\alpha},\alpha$ conjugate to $\theta,\bar{\theta}$ with the property,[16]

$$\int d\bar{\alpha}=0=\int d\alpha,\quad \int d\alpha\,\alpha=\int d\bar{\alpha}\,\bar{\alpha}=1\,,$$
$$\alpha^2=\bar{\alpha}^2=\bar{\alpha}\alpha+\alpha\bar{\alpha}=0\,. \tag{2.11}$$

The superpropagator in momentum superspace is (see Appendix)

$$G_{SS}(k,\alpha,\bar{\alpha})=\frac{1}{k^2+\bar{\alpha}\alpha}=G(k)+\bar{\alpha}\alpha\frac{\partial G(k)}{\partial k^2}\,, \tag{2.12}$$

where $G(k)=1/k^2$ is the propagator for the classical pure system. The rules for perturbation theory in momentum space are (a) conserve $k,\alpha,\bar{\alpha}$ at each vertex and (b) to each loop associate $\int d^Dk\,d\bar{\alpha}\,d\alpha$ with the rules (2.11). In the Appendix we show that to every order in perturbation theory, the contribution of each graph is the same as that of the same graph in the pure $(D-2)$-dimensional system. Let us study the AGS model in this context. The $(D+1)$-dimensional classical theory is again described by a classical equation of motion with a random field,

$$-\frac{\partial^2}{\partial\tau^2}\varphi-\nabla^2\varphi+V'(\varphi)+h=0\,, \tag{2.13}$$

with

$$\langle\langle h(x)\rangle\rangle=0,\quad \langle\langle h(x)h(x')\rangle\rangle\propto\delta(x-x')\,.$$

In this case the field h is time independent and therefore infinitely correlated in the τ direction. We can repeat the steps leading to Eqs. (2.4) and (2.5) and end up with the Lagrangian density

$$\mathcal{L}=-\tfrac{1}{2}\int d\tau'\left[\hat{\varphi}(\tau')\hat{\varphi}(\tau)+\hat{\varphi}\left[-\frac{\partial^2\varphi}{\partial\tau^2}-\nabla^2\varphi+V'(\varphi)\right]+\bar{\psi}\left[-\frac{\partial^2}{\partial\tau^2}-\nabla^2+V''(\varphi)\right]\psi\right]. \tag{2.14}$$

We then see that the first term is nonlocal in time as a consequence of the time independence of the random field. This anisotropy in the theory does not allow us to write a symmetry transformation of the type given by Eq. (2.6). However, we can still formally write the action in terms of the superfield (2.8), and we find

$$\int d^dx\,d\tau\,d\theta\,d\bar{\theta}\,\mathcal{L}[\Phi]\,,$$

where

$$\mathcal{L}[\Phi]=-\tfrac{1}{2}\int d\tau'\Phi(\tau)\frac{\partial^2}{\partial\bar{\theta}\,\partial\theta}\Phi(\tau')-\tfrac{1}{2}\Phi(\partial_\tau^2+\nabla^2)\Phi+V(\Phi)\,. \tag{2.15}$$

It is easy to see that the free propagator for this theory is given by

$$G(x-x',t-t',\theta-\theta')=\int\exp[ik(x-x')+i\omega(t-t')]\left[\frac{\delta(\omega)}{(\omega^2+k^2)^2}+(\theta-\theta')(\bar{\theta}-\bar{\theta}')\frac{1}{(\omega^2+k^2)}\right]dk\,d\omega\,. \tag{2.16}$$

Performing the transform defined in the Appendix, we find, in momentum superspace,

$$G(\omega,k,\alpha)=\frac{1}{\omega^2+k^2+\bar{\alpha}\alpha\delta(\omega)}=-\frac{\bar{\alpha}\alpha\delta(\omega)}{(\omega^2+k^2)^2}+\frac{1}{(\omega^2+k^2)}\,. \tag{2.17}$$

Since the propagator is anisotropic we expect correlation functions to scale differently in the time direction. We then introduce an anisotropy parameter γ_0 (Ref. 17) which will be nontrivially renormalized:

$$G(\omega,k,\alpha)=\frac{1}{\gamma_0\omega^2+k^2+\bar{\alpha}\alpha\delta(\omega)}\,. \tag{2.18}$$

The rules for the perturbative expansion are analogous to aforementioned (a) and (b), except that $\int d^dk\,d\omega\,d\bar{\alpha}\,d\alpha$ now corresponds to each loop. It can be seen that every internal $\int d\bar{\alpha}\,d\alpha$ brings a factor $\delta(\omega)$ that cancels the internal $\int d\omega$, so that all internal frequencies vanish and only the external frequencies flow through the diagram. It is then immediately realized that when the external frequencies vanish (static limit) the correlation functions can be constructed in perturbation theory using the effective propagator

$$G_{\text{eff}}(k,\alpha)=\frac{1}{k^2+\bar{\alpha}\alpha}\,, \tag{2.19}$$

which is the same propagator as (2.12), and the arguments given in the Appendix for dimensional reduction as a consequence of supersymmetry hold in this case.

We conclude then that static correlation functions in $D+1$ dimensions are the same as the correlation functions of the $(D-2)$-dimensional pure classical system and this means a dimensional shift $D\to D-3$ for the quantum system. However, this result does not hold for time-dependent correlations.

In the models described by (2.5) and (2.14), the response functions are

$$G_{\hat{\varphi}\varphi}=\langle\hat{\varphi}(x,\tau)\varphi(x',\tau')\rangle ,$$

whose inverse is the one-particle-irreducible (1PI) $\Gamma_{\hat{\varphi}\varphi}$. It is easy to show by power counting[18,19] that near $D=6$ [for $V(\varphi)\sim\varphi^4$] there is one more quantity to be renormalized besides the wave function and coupling constant. This is the anisotropy parameter γ_0 which is related to $\partial\Gamma_{\hat{\varphi}\varphi}/\partial\omega^2$, whose divergences will not be canceled by wave-function renormalization because the full correlation function is anisotropic. For the scaling behavior of response functions we will need the quantity

$$\hat{\xi}^*_A=\left.\frac{\kappa\partial\ln\gamma_R}{\partial\kappa}\right|_{\mathrm{fp}} , \tag{2.20}$$

where fp stands for the infrared-state fixed point and κ is some momentum scale. γ_R is defined (in minimal subtraction)[18] by

$$\frac{\partial\Gamma_{R\hat{\varphi}\varphi}}{\partial\omega^2}=\gamma_R+\cdots , \quad \Gamma_{R\hat{\varphi}\varphi}=Z_\varphi\Gamma_{\hat{\varphi}\varphi} ,$$
$$\frac{\partial\Gamma_{R\hat{\varphi}\varphi}}{\partial k^2}=1+\cdots , \tag{2.21}$$

where the ellipses stand for unspecified finite terms. The response function obeys the renormalization-group equation (RGE) at the critical point

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta_g\frac{\partial}{\partial g_R}+\xi_A\frac{\partial}{\partial\gamma_R}-\gamma_\phi\right]\times\Gamma_{R\hat{\varphi}\varphi}(\omega,k,g_R,\gamma_R,\kappa)=0 , \tag{2.22}$$

$$\beta_g=\kappa\frac{\partial g_R}{\partial\kappa},\quad \gamma_\phi=\kappa\frac{\partial\ln Z_\varphi}{\partial\kappa} ,$$

where g_R is the renormalized coupling constant. The solution of (2.22) in the scaling regime is

$$\Gamma_{R\hat{\varphi}\varphi}(\omega,k)\sim k^{2-\eta}\Psi_{\hat{\varphi}\varphi}\left[\frac{\omega}{k^{z_A}}\right] , \tag{2.23}$$

where $\Psi_{\hat{\varphi}\varphi}$ is a universal dimensionless scaling function and $z_A=1-\hat{\xi}^*_A/2$. We can solve (2.22) away from the critical point and find[17]

$$\Gamma_{R\hat{\varphi}\varphi}(\omega,p)\sim t^\gamma\Phi(\omega\xi_\tau,p\xi) , \tag{2.24}$$

where t $[=(T-T_c)/T_c]$ is the reduced temperature and ξ is the correlation length $\xi=t^{-\nu}$, $\xi_\tau=\xi^{z_A}$, and γ and ν are the usual exponents for the classical $(D-2)$-dimensional pure system since these are static ($\omega=0$) exponents.

If we consider the original quantum system at finite temperature T, the equivalent classical system is finite in the imaginary time direction with thickness $\beta=1/kT$ ($k=$Boltzmann's constant). If the correlation length in the time direction $\xi_\tau=\xi^{z_A}$ is such that $\xi_\tau\ll\beta$, the system behaves as if it were infinite in this direction, indeed like a $T=0$ quantum system. In the other limit when $\xi_\tau\gg\beta$ the system behaves as a D-dimensional classical system ($T\to\infty$), so that we expect a crossover behavior for the finite-size system when $\xi_\tau\approx\beta$. In terms of the coupling U of the original quantum system [$t\simeq(U-U_c)/U_c$],

$$|U-U_c|\sim T^{1/z_A\nu_{(D-2)}} . \tag{2.25}$$

Following Ref. 17 we define the crossover exponent as $(Z_A\nu_{(D-2)})^{-1}$, where $\nu_{(D-2)}$ is the exponent ν for the $(D-2)$-dimensional pure system.

We can solve the RGE for other correlation functions and we observe that the scaling function will be of the form given in (2.24). We then find that the static response or correlation functions do not have crossover behavior, indeed for the response function

$$\Gamma_{R\hat{\varphi}\varphi}(0,p)\sim t^\gamma\Phi(p\xi) , \tag{2.26}$$

and this then implies the dimensional shift $D\to D-3$ for the quantum system and therefore no crossover, according to the arguments of AGS. However, equal-time response or correlation functions will have this crossover:

$$\int d\omega\,\Gamma_{R\hat{\varphi}\varphi}(\omega,p)\sim t^\gamma\xi_\tau^{-1}\Phi(p\xi) . \tag{2.27}$$

when $\xi_\tau\gg\beta$, Eq. (2.27) will no longer be valid, and in this case we expect D-dimensional behavior.

III. DYNAMICS OF SYSTEMS WITH QUENCHED RANDOM FIELDS

In this section we study a simple model to describe the dynamics of spin systems in the presence of a time-independent (quenched) random field. The simplest description of the kinetics is in terms of the Langevin equation without conservation of the order parameter[8,9,14]:

$$\tilde{\gamma}_0\frac{\partial\varphi}{\partial t}=-\frac{\delta\mathscr{H}}{\delta\varphi}+\xi(x,t) , \tag{3.1}$$

where $\tilde{\gamma}_0$ is the inverse of the diffusion constant (for the model without conservation), $\mathscr{H}$ is the free-energy functional, and $\xi(x,t)$ is a random Gaussian noise with correlations:

$$\langle \xi(x,t)\rangle = 0 \,, \qquad \langle \xi(x',t')\xi(x,t)\rangle = 2\tilde{\gamma}_0\delta(x-x')\delta(t-t') \,, \tag{3.2}$$

$$\mathscr{H} = \int d^dx[(\nabla\varphi)^2 + V(\varphi) + h\varphi] \,, \qquad \langle\langle h(x)h(x')\rangle\rangle \sim \delta(x-x') \,, \tag{3.3}$$

so that (3.1) becomes

$$\tilde{\gamma}_0\frac{\partial\varphi}{\partial t} = -[-\nabla^2\varphi + V'(\varphi) + h] + \xi(x,t) \,. \tag{3.4}$$

This stochastic equation can be written as a field theory with a Martin-Siggia-Rose Lagrangian.[13,20,21] However, we prefer to study it in a simpler way by means of the iterative perturbative expansion.[9] To carry out this expansion we need the free response and correlation functions, the free terms in (3.4) corresponding to $V'(\varphi)=0$,

$$(-i\tilde{\gamma}_0\omega + k^2)\varphi_0(k,\omega) = -h(k,\omega) + \xi(k,\omega) \,. \tag{3.4'}$$

We identify $G_0(k,\omega) = (-i\tilde{\gamma}_0\omega + k^2)^{-1}$ with the (retarded) free response function.

The free correlation function is given by

$$\langle\, \langle\langle \varphi_0(k,\omega)\varphi_0(-k,-\omega)\rangle\rangle \,\rangle = \frac{2\tilde{\gamma}_0 + \delta(\omega)}{|-i\omega\tilde{\gamma}_0 + k^2|^2} \,. \tag{3.5}$$

We note that the first term in (3.5) is a consequence of the fluctuation-dissipation theorem, but the second is not and is completely due to the fluctuations induced by the random field. For definiteness we study the case of $V(\varphi) = (\lambda/4!)\varphi^4$, but the following arguments are general and do not depend on the form of $V(\varphi)$. Equation (3.4) can be written as

$$\varphi(k,\omega) = \varphi_0(k,\omega) - \frac{\lambda}{3!}G_0(k,\omega)\int dk_1dk_2d\omega_1d\omega_2\varphi(k_1,\omega_1)\varphi(k_2,\omega_2)\varphi(k-k_1-k_2,\omega-\omega_1-\omega_2) \,. \tag{3.6}$$

Equation (3.6) is represented in Fig. 1. The iteration of this formula generates tree graphs where the branches are φ_0's. Averaging over noise and random field means that the branches must be contracted pairwise in all possible ways and each contraction is given by (3.5). The diagrams must be ordered in time because of the retarded nature of the response functions. However, we recognize from (3.5) that the most infrared divergent contributions arise only from the average over the random field. Indeed, when the contour integrals on the ω's of the internal loops are performed the noise term in (3.5) contributes with its residue at the (causal) poles $\sim 1/k^2$, whereas the term proportional to $\delta(\omega)$ yields a factor $\sim 1/(k^2)^2$. So the most infrared contributions come from the average over the random fields, and therefore we can neglect the noise term altogether. This also implies that the internal-loop frequencies vanish and the external frequencies flow freely through the graph. The intermediate states are nonretarded. This is clear since retardation was a

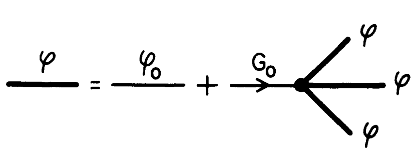

FIG. 1. Graphical representation of Eq. (3.6) in the text. Thin lines correspond to φ_0, thick lines to φ, and a line with an arrow to G_0. The vertex is $-\lambda/3!$.

consequence of the fluctuation-dissipation theorem in the intermediate states, and this was overridden when we dropped the noise terms. This is analogous, then, to the model described by (2.14), and indeed the reader can be convinced that 1PI diagrams for response and correlation functions can be generated with the following rules: (a) Draw the corresponding 1PI diagrams for the pure system; (b) to each internal line associate a propagator

$$G(k,\omega,\alpha) = [i\omega\tilde{\gamma}_0 + k^2 + \bar{\alpha}\alpha\delta(\omega)]^{-1} \,,$$

where the properties of $\bar{\alpha},\alpha$ are given by (2.11); (c) at each vertex conserve k,ω and $\bar{\alpha},\alpha$; (d) to each loop associate $\int d^dk\, d\omega\, d\bar{\alpha}\, d\alpha$. It is clear to see that with these rules, integration over $\bar{\alpha},\alpha$ in internal loops will generate all possible pairs of contracted branches[22] with a factor $\sim 1/(k^2)^2$ and a factor $\delta(\omega)$ that cancels $\int d\omega$, the response functions being calculated for external $\bar{\alpha}=\alpha=0$. Again in the static limit (when the external frequencies vanish) we effectively generate the perturbative contributions using the propagator (2.12). We recover the supersymmetry in the form given in the Appendix and, therefore, the dimensionality shift $D \to D-2$. However, nonstatic response or correlation functions do not bear this property. The theory is anisotropic in the time direction, and by the same power-counting arguments as before we expect one more quantity to be renormalized, namely the diffusion coefficient $\tilde{\gamma}_0$ associated with $\partial\Gamma_{\rm res}/\partial(-i\omega)$, where $\Gamma_{\rm res}$ is the 1PI response function kernel, with

$$\Gamma^R_{\rm res}=Z_\varphi\Gamma_{\rm res}\,,\quad \frac{\partial\Gamma^R_{\rm res}(k,\omega)}{\partial k^2}=\cdots\,,\quad \frac{\partial\Gamma^R_{\rm res}(k,\omega)}{\partial(-i\omega)}=\tilde\gamma_R+\cdots\,, \tag{3.7}$$

where the ellipses stand for unspecified finite terms. If we define in (2.21) and (3.7)

$$\gamma_R=\gamma_0 Z_\gamma,\ \ \tilde\gamma_R=\tilde\gamma_0 Z_{\tilde\gamma}\,, \tag{3.8}$$

we then recognize $Z_\gamma=Z_{\tilde\gamma}$.

Solving the RGE obeyed by the response kernel in the scaling regime, we find

$$\Gamma^R_{\rm res}(k,\omega)\sim k^{2-\eta}\chi\left\{\frac{\omega}{k^z}\right\}.\quad z=2-\left.\frac{\kappa\partial\ln\tilde\gamma_R}{\partial\kappa}\right|_{\rm fp}=2z_A\,, \tag{3.9}$$

where z_A is given by (2.22). The equality $z=2z_A$ is a consequence of $Z_\gamma=Z_{\tilde\gamma}$ in (3.8). We see then that the dynamical exponents of Langevin kinetics and the AGS model are indeed simply related, and that the static exponents in D dimensions are the same as those in $D-2$ dimensions for the pure system.

IV. CALCULATION OF z_A

In the previous sections we have shown the relation between the dynamical exponents of two different models. The calculation of these exponents would involve the setup of the perturbative expansion and renormalization-group apparatus. However, in this section we show that we can calculate the dynamical exponents easily up to order ϵ^3 in the ϵ expansion or order $1/N$ in the large-N limit, with the knowledge of the static exponent η up to this order. This is a consequence of an unexpected symmetry in a certain set of Feynman diagrams.

Even though we do not know of any physical realization of an N-component spin system coupled to quenched random fields,[23] the large-N expansion is interesting in its own right. It provides an interpolation between the upper and lower critical dimensionality of the system and complements the ϵ expansion. In our case it will provide us with results which can be translated to the ϵ expansion up to order ϵ^3. We will begin by looking at the diagrams in the large-N limit. The order parameter now is an N-component vector $\vec\varphi$. Equations (2.1) and (2.2) now read

$$-\nabla^2\phi_i-V_{i'}(\varphi)+h_i=0\,, \tag{2.1$'$}$$

$$\langle\langle h_i(x)\rangle\rangle=0\,,\quad \langle\langle h_i(x)h_j(x')\rangle\rangle\propto\delta_{ij}\delta(x-x')\,, \tag{2.2$'$}$$

where the potential $V(\varphi)$ is $O(N)$-invariant, $V_i'(\varphi)=\delta V(\varphi)/\delta\varphi_i$, and $\vec{\rm h}$ is an N-component vector. Hereafter we denote by a crossed propagator the quantity $\partial G(k)/\partial k^2$ $[G(k)=1/k^2]$. If we apply the rules (a) and (b) given in Secs. II and III, we see that every diagram has a distribution of crossed propagators such that when they are split open, the diagram is treelike (Fig. 2). In the large-N limit, the first nontrivial contribution to $\Gamma_{\hat\varphi\varphi}$ is of order $1/N$ and is given by a series of "string-bubble" diagrams (Fig. 3), where for m bubbles we have $m+1$ crossed propagators. We can either have one cross in each bubble and one cross in the lower line, or one bubble with two crosses and none in the lower line. Let us analyze for a diagram with m bubbles the contribution when one particular bubble has one and two crosses [Figs. 4(a)–4(c)] and the distribution of crosses in the rest of the diagram is fixed.

For each particular diagram, we redefine the loop momenta in such a way that the external momentum p flows along noncrossed lines, denoting a circled propagator by $\partial G(k_i,p)/\partial p^2$ and using

$$\frac{\partial}{\partial p^2}G(p-k_1-k_2)=G'(p-k_1-k_2)\left[\frac{(p-k_1-k_2)^2+p^2-(k_1+k_2)^2}{2p^2}\right]. \tag{4.1}$$

To calculate the exponent η we need the wave-function renormalization constant Z_φ which (in minimal subtraction) is obtained imposing $\partial\Gamma_R(p)/\partial p^2=1$ plus finite terms, as $\epsilon\to0$, where Γ is the response function. Now take the partial derivative with respect to p^2 of the diagrams in Fig. 4, to be evaluated at $\omega=0$ and look at the contribution where circled propagators are distributed as in Fig. 5. After some change of variables we find that the sum of diagrams in Fig. 5 amounts to (at external $\omega=0$)

$$G'(k_4)G'(k_1+k_2-k_4)G'(p-k_1-k_2)F(k_1+k_2)\,,$$

where $G'(k)=\partial G(k)/\partial k^2$ and F is a function that describes the distribution of crosses in the rest of the diagram. In order to compute z_A for the AGS

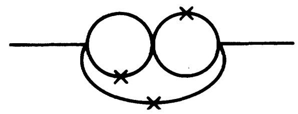

FIG. 2. Third-order contribution to $\Gamma_{\hat{\varphi}\varphi}$. Lines with crosses stand for $\partial G(k)/\partial k^2$.

model, we need $\partial\Gamma/\partial\omega^2$. Looking at the structure of the propagator we see that

$$\frac{\partial G(k,\omega)}{\partial\omega^2}=\frac{\partial G}{\partial k^2}(k,\omega)\ ,$$

so that taking $\partial/\partial\omega^2$ means inserting another cross. Note that the external frequencies ω flow along the noncrossed lines. Again looking at diagrams in Fig. 4, when we take $[\partial\Gamma/\partial\omega^2]_{\omega=0}$, circles are replaced by crosses in Fig. 5, and the sum of these three terms gives

$$3G'(k_4)G'(k_1+k_2-k_4)G'(p-k_1-k_2)F(k_1+k_2)\gamma_0\ .$$

This analysis can be carried out for every bubble in the diagram, and for any string-bubble diagram. We then conclude

$$\left.\frac{\partial\Gamma_l}{\partial\omega^2}\right|_{\omega=0}=3\gamma_0\left.\frac{\partial\Gamma_l}{\partial p^2}\right|_{\omega=0}\ , \tag{4.2}$$

where Γ_l stands for the loop corrections to the response kernel [clearly (4.2) does not hold for zeroth order]. In the minimal subtraction scheme we can write (g equals the coupling constant)

$$\begin{aligned}\frac{\partial\Gamma}{\partial\omega^2}&=\gamma_0\left[1+\sum_n a_n(\epsilon)g^n\right]+\cdots\ ,\\ \frac{\partial\Gamma}{\partial p^2}&=\left[1+\sum_n z_n(\epsilon)g^n\right]+\cdots\ ,\end{aligned} \tag{4.3}$$

where the ellipses represent unspecified finite terms. From (4.2), $a_n(\epsilon)=3z_n(\epsilon)$. If we define

$$\begin{aligned}\gamma_R&=\gamma_0 Z_\varphi\left[1+\sum_n a_n(\epsilon)g^n\right]\ ,\\ Z_\varphi^{-1}&=\left[1+\sum_n z_n(\epsilon)g^n\right]\ ,\end{aligned} \tag{4.4}$$

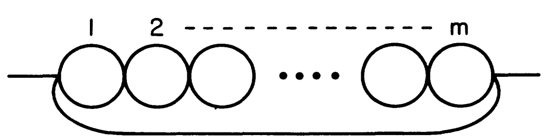

FIG. 3. String-bubble diagram with m bubbles.

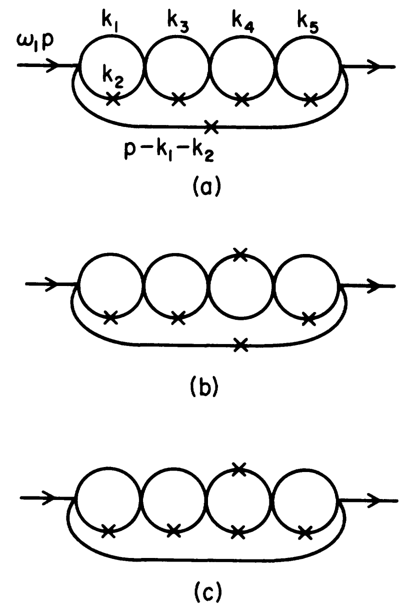

FIG. 4. (a) and (b) are contributions with one cross per bubble; (c) is one bubble with two crosses.

we get

$$\gamma_R=\gamma_0(3-2Z_\varphi)\ . \tag{4.5}$$

From the definition (2.20) we then find,

$$\hat{\epsilon}_A^*=-\frac{2\eta}{3Z_\varphi^{-1*}-2}\ , \tag{4.6}$$

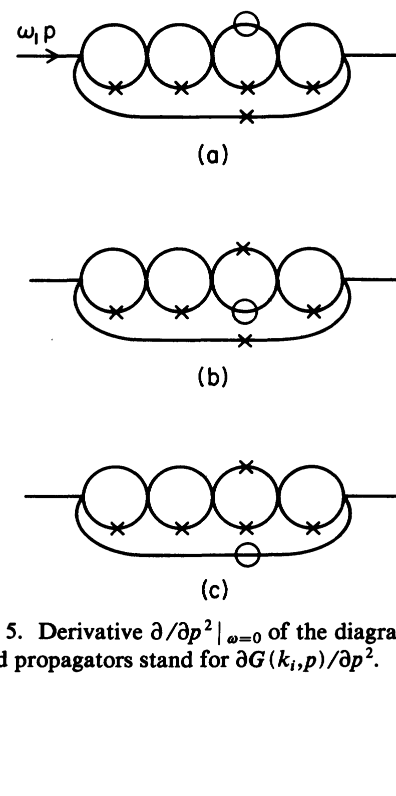

FIG. 5. Derivative $\partial/\partial p^2|_{\omega=0}$ of the diagrams of Fig. 4 circled propagators stand for $\partial G(k_i,p)/\partial p^2$.

where Z_φ^{-1*} is the value of Z_φ^{-1} at the infrared-stable fixed point. But in the large-N limit, $Z_\varphi^{-1*} \simeq 1+O(1/N)$, and since $\eta \approx O(1/N)$,

$$\xi_A^* = -2\eta + O(1/N^2), \quad z_A = 1+\eta+O(1/N^2) . \tag{4.7}$$

We should not keep $O(1/N^2)$ terms because relation (4.2) only holds for string-bubble diagrams.

In the ϵ expansion the first two corrections are of the form of string-bubble diagrams (Fig. 6) and relation (4.2) holds for these graphs, so we can use (4.6) to order ϵ^3. However, one might worry about the universality of Z_φ^{-1*}. Indeed, this function depends upon the renormalization scheme, but to $O(\epsilon)$, this function *is* universal as it is the first nontrivial order in the renormalization-group β function. To order ϵ^3 we then find

$$\xi_A^* = -2c\eta + O(\epsilon^4), \quad z_A = 1+c\eta+O(\epsilon^4) , \qquad c = \left[1 - \frac{3}{4}\frac{N+2}{(N+8)^2}\epsilon \right] . \tag{4.8}$$

Again since (4.2) holds only for diagrams contributing up to $O(\epsilon^3)$, and since η starts at $O(\epsilon^2)$, we only need Z_φ up to $O(\epsilon)$. Since η is a static exponent, it is the same as in the pure $(D-2)$-dimensional classical model. We predict, for the $(N=1)$ Ising model, that $z_A = 1+(0.0185)\epsilon^2+(0.0182)\epsilon^3+O(\epsilon^4)$. A Padé extrapolation for $\epsilon=3$ yields $z_A \approx 0.92$ for the $D=3$ Ising model.

V. CRITICAL DYNAMICS FOR ISING MODEL WITH RANDOM FIELDS IN THREE DIMENSIONS

The static critical properties of Ising-type systems have been studied by Wallace and Zia[24] in the context of the interface model. These authors argued that deviations from a planar sharp interface (capillary waves) are analogous to the spin waves for a continuous-spin system; a field-theoretical analysis of these Goldstone modes indicated that the lower critical dimensionality for Ising-type systems is 1. The Hamiltonian that describes the interface model is invariant under Euclidean rotations which amount to a rigid rotation of the surface (and also under translations of the interface). This invariance is responsible for the renormalizability of the model.

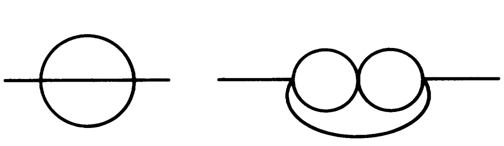

FIG. 6. String-bubble diagrams that contribute up to third order to $\Gamma_{\hat{\varphi}\varphi}$.

Indeed Ward identities ensure that the field does not acquire an anomalous dimension, and therefore only renormalization of the temperature variable is needed. The model has been studied away from the critical dimension in the ϵ expansion[24,25] $(\epsilon=D-1)$.

More recently, Bausch *et al.* have studied the dynamics of the interface model.[26] They recognized that the Langevin equation must be modified to take into account the Euclidean invariance properties mentioned above. Their model is therefore renormalizable and they computed the dynamical exponent z in the ϵ expansion.

The question of the lower critical dimensionality of the Ising model in the presence of a quenched random field has raised much controversy. On the basis of the supersymmetry argument of Parisi and Sourlas this dimension is expected to be $D_c=3$, but Imry and Ma[2] argued that $D_c=2$. They found that below $D=2$ there is an instability in domain formation, even at $T=0$. With the use of the replica method Pytte *et al.*[27] showed that $D_c=3$. Subsequently, Kogon and Wallace[12] studied the supersymmetric generalization of the interface, and they concluded that $D_c=3$, and that the critical behavior in $D=3+\epsilon$ is the same as that of the $D=1+\epsilon$ pure interface as a consequence of this supersymmetry. More recently, an argument for $D_c=3$ has been given by Cardy,[7] where in the context of the supersymmetric model he found topological configurations akin to the domain wall for $D=1$, whose effect is to disorder the system at any finite randomness.

Our aim here is to study the kinetics of the Ising model with a random time-independent field at the lower critical dimensionality $D_c=3$ using the interface approach. Following Wallace and Zia we start from a Ginzburg-Landau free-energy functional,

$$F = \int d^D\vec{x}[(\vec{\nabla}\varphi)^2 + \mu^2(\varphi^2-1)^2] . \tag{5.1}$$

In the low-temperature limit $\mu^2 \to \infty$, the saddle-point contribution corresponds to the solution to the classical equation of motion. In order to take into account the fluctuations of the interface, the field configuration is

$$\Phi_f = \phi_c \left[\frac{z-f(x)}{[1+(\vec{\nabla}f)^2]^{1/2}} \right] , \tag{5.2}$$

where the field $f(x)$ represents the deviation from planar interface and x stands for $(D-1)$-dimensional components perpendicular to the interface. A random field in (5.1) corresponds, after substituting (5.2) to a term,

$$H_f=\int d^{D-1}x\,dz\,h(x,z)\phi_c\left[\frac{z-f(x)}{[1+(\vec{\nabla}f)^2]^{1/2}}\right] \quad (5.3)$$

in the Hamiltonian. Therefore the effective Hamiltonian describing the long-wavelength phenomena is

$$H=\frac{1}{\tilde{T}}\int d^{D-1}x[1+(\vec{\nabla}f)^2]^{1/2}+H_f \quad (5.4)$$

with

$$\langle\langle h(x,z)\rangle\rangle=0\,,$$
$$\langle\langle h(x,z)h(x',z')\rangle\rangle\propto\delta(x-x')\delta(z-z')\,. \quad (5.5)$$

The kinetics of this system with a nonconserved order parameter is described by the Langevin equation proposed by Bausch *et al.*[26]:

$$\frac{\partial f}{\partial t}=-\lambda\sqrt{g}\frac{\delta H}{\delta f}+\xi(x,t)\,,$$
$$g(x,t)=1+[\vec{\nabla}f(x,t)]^2\,, \quad (5.6)$$
$$\langle\xi(x,t)\xi(x',t')\rangle=2\lambda\sqrt{g}\,\delta(x-x')\delta(t-t')\,.$$

If we want to study the system at the lower critical dimension $D_c=3$, one might argue that we should expect Gaussian exponents and therefore $z=2$. However, we should bear in mind that in order to be able to say so one has to prove the renormalizability of the theory.

In what follows we will show that the theory defined by (5.3)–(5.6) is by no means easy to analyze, since highly nonlocal and nonpolynomial interactions arise due to the fact that the random field is totally correlated in time. Following Martin-Siggia-Rose theory we write (5.6) in terms of a dynamic generating functional. Introducing a response field $\tilde{f}$, and averaging over the noise leads to

$$\langle f(x,t)f(x',t')\rangle=\int\mathscr{D}f\,\mathscr{D}\tilde{f}\,\mathscr{D}h\exp\left\{-\int d^{D-1}x\,dt\left[-\lambda\sqrt{g}\tilde{f}^2+\tilde{f}\left[\frac{\partial f}{\partial t}+\lambda\sqrt{g}\frac{\delta H}{\delta f}\right]\right]-\int d^{D-1}x\,dz\,h^2(x,z)\right\}f(x,t)f(x',t')\,. \quad (5.7)$$

Performing the integration over h in (5.7) we get the following term in the exponential:

$$\tilde{F}(x,t)=\int dt'\sqrt{g(x,t)}\sqrt{g(x,t')}\tilde{f}(x,t)\tilde{f}(x,t')\times\int dz\frac{\delta}{\delta f(x,t)}\varphi_c\left[\frac{z-f(x,t)}{\sqrt{g(x,t)}}\right]\frac{\delta\varphi_c}{\delta f(x,t')}\left[\frac{z-f(x,t')}{\sqrt{g(x,t')}}\right]. \quad (5.8)$$

We can go a step further and argue that, at $\tilde{T}\approx0$, the interface is nearly sharp and we can approximate $\varphi_c'(z)$ by $\delta(z)$ plus terms with higher powers of $\tilde{T}$; then (5.8) reduces to

$$\tilde{F}(x,t)\approx\int dt'\sqrt{g(x,t)}\tilde{f}(x,t)\tilde{f}(x,t')\delta\left[\frac{f(x,t)-f(x,t')}{\sqrt{g(x,t')}}\right]+\cdots\,. \quad (5.9)$$

Either (5.8) or (5.9) are highly nonlocal in time, and the time-dependent interaction is nonpolynomial. A perturbative analysis does not seem possible, and we do not see a way of proving the renormalizability, or even to understand whether the zero-frequency (static) limit of the theory corresponds to the well-known results. These complications do not allow us then to affirm that we should expect $z=2$ at $D_c=3$. We believe this point deserves to be investigated in greater detail and further work should be aimed in this direction.

VI. CONCLUSIONS

In this paper we have studied the effect of quenched random fields on quantum spin systems and on critical dynamics of classical spin systems. The former has been considered through the equivalent classical spin model in one more dimension, imaginary time, with the random field totally correlated in the time direction. We proved by using supersymmetry arguments that time-independent correlation functions in D dimensions are the same as those of the $(D-3)$-dimensional pure quantum problem to all orders in perturbation theory. However, time-dependent correlation functions behave differently, and do not bear this property. Moreover, the anisotropy of the system forces us to introduce an anisotropy parameter that is nontrivially renormalized, giving rise to a new exponent z_A, which describes the scaling properties of nonstatic response and correlation functions.

Our conclusions with respect to the static behavior agree with those first obtained by AGS.[10] However, in that paper no reference was made to the possibility of anisotropic scaling or to the failure of dimensionality reduction for nonstatic quantities.

The correlation length in the time direction is $\xi_\tau = t^{-z_A \nu_{(D-2)}}$, where $\nu_{(D-2)}$ is the exponent ν for the $(D-2)$-dimensional classical pure theory. At finite temperature, when the classical system is finite in the time direction with thickness $\beta = 1/kT$, we expect a finite-size crossover when $\xi_\tau \sim \beta$ from quantum behavior for $\xi_\tau \ll \beta$ to a D-dimensional classical behavior for $\xi_\tau \ll \beta$. The crossover exponent is $(z_A \nu_{(D-2)})^{-1}$. We predict that this crossover can be observed in nonstatic correlation functions like equal-time spin-spin correlation, but static (zero-frequency) quantities do not have crossover behavior.

The dynamics of the spin system in D dimensions with a quenched (time-independent) random field without conservation of the order parameter is described by a Langevin equation. We pointed out that the most infrared divergent contributions arise from fluctuations in the random field and that the noise term is irrelevant. This allowed us to prove that the static exponents are the same as those of the $(D-2)$-dimensional pure system to all orders in perturbation theory, again as a consequence of the supersymmetry recovered in the static (zero-frequency) limit. It was also recognized that the perturbation expansion for the time-dependent response functions was related to all orders to those of the quantum system mentioned above, and this allowed us to find the dynamical exponent $z = 2z_A$.

We also calculated the exponent z_A, which up to order ϵ^3 in the ϵ expansion ($\epsilon = 6 - D$), or up to order $1/N$ in the large-N limit, is related to the exponent η. We find, for $N = 1$, that $z_A = 1 + (0.0185)\epsilon^2 + (0.0182)\epsilon^3 + O(\epsilon^4)$. A Padé extrapolation yields $z_A \approx 0.92$ for $\epsilon = 3$.

Finally, we looked at the dynamics of an Ising-type system coupled to a time-independent quenched random field, using the interface model at the lower critical dimensionality $D_c = 3$. One would naively expect a Gaussian dynamical exponent $z = 2$, consistent with the estimate given above. However, we show that the underlying field theory is far from being trivial and the infinite correlation of the random field in the time direction brings highly nonlocal and nonpolynomial interactions along this direction. A perturbative analysis was not feasible and we were not able to draw any conclusions about the renormalizability of the theory, and therefore the value of the dynamical exponent z.

ACKNOWLEDGMENT

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APPENDIX: FOURIER TRANSFORM IN SUPERSPACE AND DIMENSIONAL SHIFT

In this Appendix we develop the perturbative rules in momentum superspace. If we introduce the variables α, $\bar{\alpha}$ conjugate to $\theta,\bar{\theta}$ with the Grassmann algebra

$$\begin{gathered} \alpha^2 = \bar{\alpha}^2 = \alpha\bar{\alpha} + \bar{\alpha}\alpha = 0 , \\ \int d\alpha = \int d\bar{\alpha} = 0 , \\ \int d\alpha\, \alpha = \int d\bar{\alpha}\, \bar{\alpha} = 1 , \end{gathered} \tag{A1}$$

and define the Fourier transform in anticommuting variables as

$$\Phi_F(\alpha,\bar{\alpha}) = \int d\bar{\theta}\, d\theta \exp[i(\bar{\alpha}\theta + \bar{\theta}\alpha)]\Phi(\theta,\bar{\theta}) , \tag{A2}$$

the following property follows:

$$\int d\bar{\theta}\, d\theta \exp[i(\bar{\alpha}\theta + \bar{\theta}\alpha)] = \alpha\bar{\alpha} = \delta(\alpha)\delta(\bar{\alpha}) .$$

The superspace Fourier transform is

$$\Phi(x,\theta,\bar{\theta}) = \int \exp(-ikx)\exp[-i(\bar{\alpha}\theta + \theta\alpha)]\Phi_F(k,\alpha) . \tag{A3}$$

The quadratic part of the Parisi and Sourlas[6] superaction is

$$A_F = \int dx\, d\bar{\theta}\, d\theta\, \Phi(x,\theta) \left[-\Delta - \frac{\partial^2}{\partial\bar{\theta}\,\partial\theta} \right] \Phi(x,\theta) . \tag{A4}$$

Using (A1)–(A3) we find

$$A_F = \int dk\, d\bar{\alpha}\, d\alpha\, \Phi_F(k,\alpha)(k^2 + \bar{\alpha}\alpha)\Phi_F(-k,-\alpha) . \tag{A5}$$

The superpropagator can be read off

$$G_{ss}(k,\alpha) = \frac{1}{k^2 + \bar{\alpha}\alpha} . \tag{A6}$$

This propagator can also be found performing the transforms (A2) and (A3) in the Green function $\langle \Phi(x,\theta)\Phi(x',\theta') \rangle$ that can be computed from the field components

$$\langle \Phi(x,\theta)\Phi(x',\theta)\rangle = \int dk\,\exp[-ik(x-x')]\left[\frac{1}{(k^2)^2}+(\theta-\theta')(\bar{\theta}-\bar{\theta}')\frac{1}{k^2}\right], \tag{A7}$$

$$G_{ss}(k,\alpha)=\int d(x-x')d(\bar{\theta}-\bar{\theta}')d(\theta-\theta')$$

$$\times \exp[ik(x-x')\exp\{i[\bar{\alpha}(\theta-\theta')+(\bar{\theta}-\bar{\theta}')\alpha]\}\langle\Phi(x,\theta)\Phi(x',\theta')\rangle \tag{A8}$$

we find (A6).

We can compute the transform (A3) for the superfield given in the text, and we find

$$\langle \Phi_F(k,\alpha)\Phi_F(k',\alpha')\rangle = G_{ss}(k,\alpha)\delta(k+k')\delta(\alpha+\alpha')\delta(\bar{\alpha}+\bar{\alpha}') . \tag{A9}$$

To all orders in perturbation theory we can read off the field components Green functions by looking at the coefficients of α,α'. Indeed the supersymmetry

$$k_\mu \to k_\mu - \tfrac{1}{2}\epsilon_\mu(\bar{\alpha}a+\bar{a}\alpha) ,$$
$$\alpha \to \alpha + \epsilon_\mu k_\mu a , \tag{A10}$$
$$\bar{\alpha}\to\bar{\alpha}+\epsilon_\mu k_\mu \bar{a} ,$$

with $a,\bar{a}$ anticommuting variables, ensures that the full Green function is of the form

$$G(k^2+\bar{\alpha}\alpha)\delta(k+k')\delta(\alpha+\alpha')\delta(\bar{\alpha}+\bar{\alpha}') ,$$

where k,α,α' are external variables, so that

$$\langle\varphi\hat{\varphi}\rangle \sim \bar{\alpha}\alpha+\bar{\alpha}'\alpha' ,$$
$$\langle\psi\bar{\psi}\rangle \sim \bar{\alpha}\alpha'+\bar{\alpha}'\alpha , \tag{A11}$$
$$\langle\varphi\varphi\rangle \sim \bar{\alpha}\alpha\bar{\alpha}'\alpha' .$$

Here the relational operator "$\sim$" stands for "coefficient of"; so we see that the momentum dependence of $\langle\varphi\hat{\varphi}\rangle$ and $\langle\psi\bar{\psi}\rangle$ are the same and the response function $\langle\varphi\hat{\varphi}\rangle$ is found setting the external α variables to zero.

To prove the dimensional shift by 2 we follow the steps of Ref. 22, and we write the momentum $k=(\vec{k},q)$, where $\vec{k}$ is a $(D-2)$-dimensional vector and q is a two-dimensional vector. To every order we find an expression of the form

$$I(\vec{p})=\int \prod_{i=1}^{L} dk_i d\bar{\alpha}_i d\alpha_i F(\vec{p},\vec{k}_i,q_i,\alpha_i,\bar{\alpha}_i) , \tag{A12}$$

where $\vec{p}$ are a set of external momenta and chosen to be defined in a $(D-2)$-dimensional space, and F is a product of superpropagators (A6), and L is the number of loops in the diagram considered. The supertransformation (A10) can be cast in terms of the q_i's choosing ϵ_μ to be an arbitrary two-dimensional vector.

Consider

$$J(\vec{p},\vec{k}_i,q_1,\alpha_1,\bar{\alpha}_1)=\int \prod_{i=2}^{L} d^2q_i d\bar{\alpha}_i d\alpha_i \times F(\vec{p},k_i,q_i,\alpha_i,\bar{\alpha}_i) . \tag{A13}$$

This quantity is a supersymmetric invariant therefore it is of the form $J(\vec{p},\vec{k}_i,q_1^2+\bar{\alpha}_1\alpha_1)$ and

$$I(\vec{p})=\int \prod_{i=1}^{L} d\vec{k}_i \int d^2q_1 d\bar{\alpha}_1 d\alpha_1 J(\vec{p},\vec{k}_i,q_1^2+\bar{\alpha}_1\alpha_1)$$
$$=\int \prod_{i=1}^{L} d\vec{k}_i \int d^2q_1 \frac{\partial J(\vec{p},\vec{k}_i,q_1^2)}{\partial q_1^2}$$
$$=\int d\vec{k}_i J(\vec{p},\vec{k}_i,0) . \tag{A14}$$

We can integrate over all q_i's in the same way and we find

$$I(\vec{p})=\int \prod_{i=1}^{L} d\vec{k}_i F(\vec{p},\vec{k}_i) . \tag{A15}$$

This is the expression we get for the pure system in $D-2$ dimensions; indeed, $F(\vec{p},\vec{k}_i)$ is a product of propagators $G(k)$.

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