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High-temperature—series expansion for random Ising magnets

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We use high-temperature—series expansion to study the magnetic properties of d-dimensional hypercubic Ising spin systems with a random distribution of exchange interaction. Our series is valid for an arbitrary distribution of exchange. We examine the case of a concentration of p ferromagnetic bonds and $1-p$ antiferromagnetic bonds of equal magnitude. We find regions of a spin-glass phase in the concentration-temperature phase diagram sandwiched between regions of ferromagnetic and antiferromagnetic order.

INTRODUCTION

There has been a great deal of interest in the properties of magnetic systems in the presence of disorder. Well-studied examples of such systems are spin glasses,[1] in which the magnetic spins interact via exchange interactions which are random in strength and in sign. It is found that at low temperatures such systems freeze into a ground state, in which the spin density does not show any well-defined long-range ordering. Two distinct types of theoretical models have been proposed to describe these systems. One model, due to Sherrington and Kirkpatrick,[2] consists of long-ranged interactions occurring between every spin in the system, which simulates the long-ranged Ruderman-Kittel-Kasuya-Yosida (RKKY) interactions occurring in systems such as *Cu*Mn spin-glasses. Another model, more appropriate to $Eu_xSr_{1-x}S$,[3] is based on the competition of ferromagnetic and antiferromagnetic short-ranged interactions. In this latter model,[4] it is unclear what the low-temperature phase diagram looks like. It is clear that such a system will order ferromagnetically when the concentration of antiferromagnetic bonds is sufficiently small. However, as the fraction of antiferromagnetic bonds is increased, does the low-temperature phase change to a spin-glass or paramagnetic phase?

In one dimension,[5] the situation is perfectly clear, the system is either ferromagnetic or antiferromagnetic at $T=0$ when all the bonds are of one type. For all other intermediate types of distributions, the system exhibits a $T=0$ phase transition to a spin-glass phase. For higher dimensions, real-space renormalization schemes indicate that there is no spin-glass phase,[6] contrary to the results of Monte Carlo calculations.[7] For d greater or equal to four, high-temperature—series expansions have been performed[8] for symmetric distributions of positive and negative exchange interactions $P(J)=P(-J)$, and it has been found that there exists a paramagnetic to spin-glass phase transition. These studies have not provided information about the phase diagram for $d<4$. We shall extend the high-temperature—series expansions to apply to the case of nonsymmetric distributions of exchange interactions in order to ascertain the phase diagram over a wider range of distributions.

It is with these aims in mind that we shall perform high-temperature—series expansion for random Ising ferromagnets for d-dimensional hypercubic lattices. In Sec. II we shall derive the free-energy and magnetic susceptibility series. In Sec. III we derive the susceptibility series corresponding to the Edwards-Anderson order parameter. In Sec. IV we shall analyze these series and present the results. Since in order to obtain phase diagrams we have to use the high-temperature series down to $T=0$, in which case there is no definite expansion parameter, we shall check our predictions by examining the case corresponding to diluted ferromagnets. Estimates of the points where T_c falls to zero can then be compared to the well-known percolation concentration. This can be used to give a test of the reasonableness and reliability of our results.

HIGH-TEMPERATURE EXPANSION

The thermodynamic and static correlation functions for a particular realization of a random sys-

tem can be obtained from the free energy by taking the appropriate derivatives with respect to the thermodynamic fields. However, the pertinent quantities to study are not those corresponding to a particular realization of a random system, but rather one should study the distribution of the properties pertaining to a whole ensemble of realizations of the random system. We shall calculate the average values of the thermodynamic functions from the average free energy by taking the appropriate derivatives. This is permissible since the derivative is a linear operator and it commutes with the average over the random ensemble. The free energy is defined in terms of the logarithm of the partition function

$$-\beta F = \ln \mathrm{Tr}_{\{S_i\}} \exp(-\beta H) , \tag{1}$$

where H is the Hamiltonian of the system.

We shall consider a set of Ising spins positioned at the lattice sites of a d-dimensional hypercubic lattice. The spins are assumed to interact via a nearest-neighbor exchange interaction, in which the exchange parameter J is distributed randomly. The Hamiltonian H is given as the sum of the exchange energy and a Zeeman energy due to a spatially varying magnetic field

$$H = -\sum_{ij} J_{ij} S_i S_j - \sum_i H_i S_i , \tag{2}$$

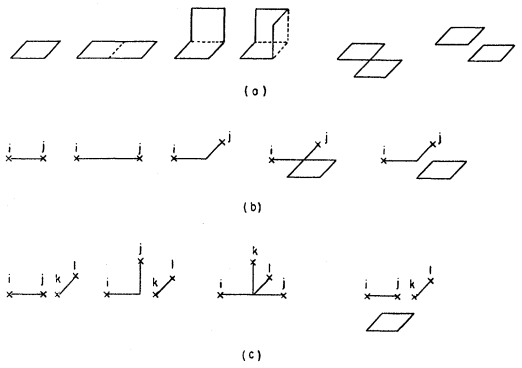

FIG. 1. Typical terms in the high-temperature expansion of the partition function. (a) Typical zeroth-order term in $\tanh\beta H$; (b) typical second-order term in $\tanh\beta H$; (c) typical fourth-order term in $\tanh\beta H$.

where H_i is the magnetic field at site i, and S_i the spin at site i can take on the values $+1$ or -1. We shall calculate the free energy by the primitive method of series expansion,[9] i.e., by expanding in powers of $\tanh\beta J_{ij}$. We shall utilize the idempotency relations for S_i, giving the identities

$$\exp(\beta J_{ij} S_i S_j) = \cosh\beta J_{ij}(1 + S_i S_j \tanh\beta J_{ij}) ,$$

$$\exp(\beta H_i S_i) = \cosh\beta H_i(1 + S_i \tanh\beta H_i) .$$

On substituting these relations in the free energy we obtain

$$-\beta F = \sum_{ij} \ln(\cosh\beta J_{ij}) + \sum_i \ln(\cosh\beta H_i) + \ln\left[\mathrm{Tr}_{(S_i)} \prod_{ij} (1 + S_i S_j \tanh\beta J_{ij}) \prod_i (1 + S_i \tanh\beta H_i)\right] .$$

The high-temperature–series expansion is obtained by expanding the multiple products in powers of $\tanh\beta J_{ij}$ and $\tanh\beta H_i$. In this manner we find

$$\begin{aligned} -\beta F = &-\sum_{ij} \tfrac{1}{2}\ln(1-\tanh^2\beta J_{ij}) - \sum_i \tfrac{1}{2}\ln(1-\tanh^2\beta H_i) \\ &+ \ln\left[\mathrm{Tr}_{(S_i)} \left[1 + \sum_{ij} S_i S_j \tanh\beta J_{ij} + \sum_{ij>kl} S_i S_j S_k S_l \tanh\beta J_{ij} \tanh\beta J_{kl} + \cdots\right]\right. \\ &\left. \times \left[1 + \sum_m S_m \tanh\beta H_m + \sum_{m>n} S_m S_n \tanh\beta H_m \tanh\beta H_n + \cdots\right]\right] . \end{aligned} \tag{3}$$

On performing the trace over the spins $\{S_i\}$, we note that only the terms which contain even powers of the spins at any site are nonvanishing. For each of these nonvanishing terms, the trace yields a common factor of 2^N, where N is the total number of spins in the system.

We shall classify the terms in powers of $\tanh\beta H$. The terms of zeroth order can be put in correspondence with a set of closed polygons on the d-dimensional hypercubic lattice, each side representing an exchange interaction. Since each term in the expansion can only contain a particular bond one or zero times, the polygons may not share a common side. Typical terms are depicted diagrammatically in Fig. 1(a). The terms of first order in $\tanh\beta H$ are identically zero, as are all terms of odd order in $\tanh\beta H$.

The next nonzero terms involve $\tanh\beta H_i \tanh\beta H_j$, with i not equal to j. The external-field vertices (sites i and j) have an odd number of bonds emanating from them. Therefore, each term corresponds to one closed path connecting site i to j, and a set of closed polygons. Terms of this nature are depicted diagrammatically in Fig. 1(b).

The remaining terms that we shall consider are those of fourth order in $\tanh\beta H$. These terms contain four distinct vertices (i,j,k,l). An odd number of bonds emanate from each vertex. This results in two paths connecting the vertices in pairs, and a set of closed polygons shown in Fig. 1(c).

Since the partition function is expected to vary exponentially with N, the number of spins, and since the number of diagrams consisting of n separated structures has a leading N dependence of N^n, we expect that on taking the logarithm the effect of the separated structures will cancel, yielding a free energy that should consist of linked clusters. For a rigorous proof of the linked-cluster theorem, the interested reader is referred to the excellent review article by Domb.[10] The linked-cluster expansion results in terms which may have multiply repeated bonds or external-field vertices. On averaging over the distribution of exchange interactions, the terms with multiply repeated bonds separate from the rest. The free energy can then be written as a functional of the distribution function $P(J)$ for the exchange interactions

$$-\beta F\{P(J)\} = N\ln 2 + A\{P(J)\} + \sum_{ij} B_{ij}\{P(J)\}\tanh\beta H_i \tanh\beta H_j + \sum_{ijkl} C_{ijkl}\{P(J)\}\tanh\beta H_i \tanh\beta H_j \tanh\beta H_k \tanh\beta H_l + \cdots . \tag{4}$$

The coefficient $A\{P(J)\}$ is given as a series in powers of $\tanh\beta J$

$$A\{P(J)\} = \sum_n \sum_{m_1,\ldots,m_n} A^{(n)}_{m_1,m_2,\ldots,m_n} \langle \tanh\beta J^1\rangle^{m_1} \langle \tanh^2\beta J\rangle^{m_2} \cdots \langle \tanh^n \beta J\rangle^{m_n}$$

in which $\sum_{i=1}^{n} m_i i = n$. The coefficients $A^{(n)}_{m_1,m_2,\ldots,m_n}$ are given in Table I. They are broken down according to the number of dimensions i which a particular term spans, according to

$$A^{(n)}_{m_1,m_2,\ldots,m_n} = N \frac{\prod_{i=1}^{n-1}\{\delta_{m_i 0}\}}{2n}\delta_{m_n,1}\begin{bmatrix} d \\ 1 \end{bmatrix} + N\sum_{i=2}^{N} A^{(n,i)}_{m_1,m_2,\ldots,m_n}\begin{bmatrix} d \\ i \end{bmatrix}, \quad \text{where } \begin{bmatrix} d \\ i \end{bmatrix} = \frac{d!}{i!(d-i)!} . \tag{5}$$

TABLE I. The coefficients $A^{(n,i)}_{m_1,\ldots,m_n}$ in the free-energy expansion [Eq. (5)].

	$i=2$	$i=3$	$i=4$	$i=5$
$n=4$	$m_1=4$ 1			
$n=6$	$m_1=6$ 2	$m_1=6$ 16		
$n=8$	$m_1=8$ 7	$m_1=8$ 186	$m_1=8$ 648	
	$m_1=6,\ m_2=1$ -2	$m_1=6,\ m_2=1$ -12		
	$m_2=4$ $-1/2$			
$n=10$	$m_1=10$ 28	$m_1=10$ 2304	$m_1=10$ 23 072	$m_1=10$ 47 616
	$m_1=8,\ m_2=1$ -12	$m_1=8,\ m_2=1$ -288	$m_1=8,\ m_2=1$ -768	
	$m_1=6,\ m_2=2$ 0	$m_1=6,\ m_2=2$ -48		
	$m_1=4,\ m_2=3$ -4	$m_1=4,\ m_2=3$ -24		

TABLE II. The coefficients $B^{(n,i)}_{m_1,\ldots,m_n}$ in the free-energy expansion [Eq. (6)].

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
$n=1$	$m_1=1$						
	1						
$n=2$	$m_1=2$	$m_1=2$					
	1	4					
$n=3$	$m_1=3$	$m_1=3$	$m_1=3$				
	1	16	24				
$n=4$	$m_1=4$	$m_1=4$	$m_1=4$	$m_1=4$			
	1	48	216	192			
$n=5$	$m_1=5$	$m_1=5$	$m_1=5$	$m_1=5$	$m_1=5$		
	1	140	1344	3072	1920		
	$m_1=3,\ m_2=1$	$m_1=3,\ m_2=1$					
	0	−4					
$n=6$	$m_1=6$	$m_1=6$	$m_1=6$	$m_1=6$	$m_1=6$	$m_1=6$	
	1	388	7296	32 064	48 000	23 040	
	$m_1=4,\ m_2=1$	$m_1=4,\ m_2=1$	$m_1=4,\ m_2=1$				
	0	−16	−48				
	$m_1=2,\ m_2=2$	$m_1=2,\ m_2=2$					
	0	−4					
$n=7$	$m_1=7$	$m_1=7$	$m_1=7$	$m_1=7$	$m_1=7$	$m_1=7$	$m_1=7$
	1	1080	37 416	279 360	750 720	829 440	322 560
	$m_1=5,\ m_2=1$	$m_1=5,\ m_2=1$	$m_1=5,\ m_2=1$	$m_1=5,\ m_2=1$			
	0	−76	−576	−576			
	$m_1=3,\ m_2=2$	$m_1=3,\ m_2=2$	$m_1=3,\ m_2=2$				
	0	−16	−48				
	$m_1=1,\ m_2=3$	$m_1=1,\ m_2=3$					
	0	−4					
$n=8$	$m_1=8$	$m_1=8$	$m_1=8$	$m_1=8$	$m_1=8$	$m_1=8$	$m_1=8$
	1	2948	184 872	2 214 528	9 498 240	18 109 440	15 805 440
	$m_1=6,\ m_2=1$	$m_1=6,\ m_2=1$	$m_1=6,\ m_2=1$	$m_1=6,\ m_2=1$	$m_1=6,\ m_2=1$		
	0	−272	−4416	−11 904	−7680		
	$m_1=4,\ m_2=2$	$m_1=4,\ m_2=2$	$m_1=4,\ m_2=2$	$m_1=4,\ m_2=2$			
	0	−76	−576	−576			
	$m_1=2,\ m_2=3$	$m_1=2,\ m_2=3$	$m_1=2,\ m_2=3$				
	0	−16	−48				

The first term represents the direct interaction between the Nd pairs of spins. The second term represents the indirect interactions, and as such, always involves dimensions greater than 1. Since A represents the free energy in the absence of a magnetic field, the series is not expected to exhibit drastic anomalies in the vicinity of the critical temperature.

The next term in the expansion in powers of $\tanh\beta H$ is of second order. The coefficient B_{ij} is the zero-field spin-spin correlation function $\langle S_i S_j \rangle$. For the case of a uniform-applied magnetic field, $H_i = H$ for all i, the coefficient is simply related to the zero-field ferromagnetic susceptibility. In this case the coefficient is independent of i and j, and the sum can be written as

$$\sum_{ij} B_{ij}\{P(J)\} = \frac{N}{2} + N \sum_{m_1,m_2,\ldots,m_n} \sum_{n} \sum_{i} \begin{bmatrix} d \\ i \end{bmatrix} B^{(n,i)}_{m_1,m_2,\ldots,m_n} \langle \tanh\beta J \rangle^{m_1} \langle \tanh\beta J^2 \rangle^{m_2} \cdots \langle \tanh\beta J^n \rangle^{m_n} , \tag{6}$$

where the sum over the m_i are restricted such that $\sum_{i=1}^{n} i m_i = n$. The coefficients $B^{(n,i)}_{m_1,m_2,\ldots,m_n}$ are given in Table II. This series is expected to exhibit a sharp singularity, at the temperature at which the systems order ferromagnetically. The critical exponent γ is expected to be much larger than that of the specific-heat anoma-

TABLE III. The coefficients $C^{(n,i)}_{m_1,\ldots,m_n}$ in the free-energy expansion [Eq. (7)].

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$n=2$	$m_1=2$	$m_1=2$			
	-1	-4			
	$m_1=0,\ m_2=1$				
	$-1/2$				
$n=3$	$m_1=3$	$m_1=3$	$m_1=3$		
	-2	-40	-64		
	$m_1=1,\ m_2=1$	$m_1=1,\ m_2=1$			
	-2	-8			
$n=4$	$m_1=4$	$m_1=4$	$m_1=4$	$m_1=4$	
	-3	-224	-1056	-992	
	$m_1=2,\ m_2=1$	$m_1=2,\ m_2=1$	$m_1=2,\ m_2=1$		
	-4	-76	-120		
	$m_1=0,\ m_2=2$	$m_1=0,\ m_2=2$			
	$-1/2$	-2			
$n=5$	$m_1=5$	$m_1=5$	$m_1=5$	$m_1=5$	$m_1=5$
	-4	-1016	$-10\,608$	$-26\,240$	$-18\,640$
	$m_1=3,\ m_2=1$	$m_1=3,\ m_2=1$	$m_1=3,\ m_2=1$	$m_1=3,\ m_2=1$	
	-6	-368	-1680	-1920	
	$m_1=1,\ m_2=2$	$m_1=1,\ m_2=2$	$m_1=1,\ m_2=2$		
	-2	-32	-96		

ly α. This series will be used to search for the instability of the paramagnetic system to a ferromagnetically ordered phase.

The corresponding transition from the paramagnetic to an antiferromagnetic phase, can be found from the response function which relates the antiferromagnetic sublattice order parameter to a staggered magnetic field. For a simple hypercubic lattice, this response function is simply related to the series $\sum_{ij} B_{ij}\{P(J)\}$ by multiplying the nth-order term by $(-1)^n$. In this manner we can establish the position of both the ferromagnetic and antiferromagnetic instability.

The next term in the free energy is of fourth order in $\tanh\beta H$. The coefficient $C_{ijkl}\{P(J)\}$ is related to four-spin correlation $\langle S_i S_j S_k S_l \rangle$. For a uniform magnetic field, the coefficients can be summed over yielding

$$\sum_{ijkl} C_{ijkl}\{P(J)\} = N/4 + N \sum_{m_1,m_2,\ldots,m_n} \sum_{n} \sum_{i} \begin{bmatrix} d \\ i \end{bmatrix} C^{(n,i)}_{m_1,m_2,\ldots,m_n} \langle \tanh\beta J \rangle^{m_1} \langle \tanh\beta J^2 \rangle^{m_2} \cdots \langle \tanh\beta J^n \rangle^{m_n} , \quad (7)$$

where the coefficients $C^{(n,i)}_{m_1,m_2,\ldots,m_n}$ are given in Table III. In compiling Tables I—III, extensive use was made of the paper by Fisher and Gaunt[11] which contains the high-temperature—serics expansion for a system of Ising spins on a d-dimensional hypercubic lattice with a uniform nearest-neighbor exchange interaction. Therefore, it is quite natural that our results reduce to those of Fisher and Gaunt in the case of a single δ-function distribution of exchange interactions

$$P(J_{ij}) = \delta(J_{ij} - J) .$$

For a more general distribution of bonds $P(J)$, the above series allows us to calculate the average values of the thermodynamic quantities in the paramagnetic phase. A divergence of the series can be related to thermodynamic instabilities of the system. However, a phase transition need not be directly related to the average thermodynamic quantities of a random ensemble, but could be related to the spread of the distributions of the quantities. One example of such a measure is the Edwards-Anderson order parameter and its corresponding susceptibility. In the next section we shall derive the high-temperature—series expansion for such measures.

THE EDWARDS-ANDERSON ORDER PARAMETER AND SUSCEPTIBILITY

In the high-temperature paramagnetic phase the thermal expectation value of $\langle S_i \rangle$ at any site is

zero. The spins are equally likely to be pointing up or down. At low temperatures, the paramagnetic state may be unstable to a magnetically ordered state in which the symmetry of the Hamiltonian is broken. For a regular ferromagnet or antiferromagnet, this state corresponds to one in which the expectation value $\langle S_i \rangle$ is nonzero. Each spin has a definite direction associated with the ground state into which the system has frozen. For regular magnetic systems the corresponding order parameter is $M_q = \sum_i \langle S_i \rangle e^{iqR_i}$, which is nonvanishing for some vector q. Edwards and Anderson[12] realized that for some random systems M_q will be zero for all values of q but that $\langle S_i \rangle$ could be finite. Edwards and Anderson utilized this concept of a finite $\langle S_i \rangle$ in the ground state, to define an order parameter Q defined by

$$Q = \sum_i \langle S_i \rangle^2 . \tag{8}$$

This order parameter will be zero in the paramagnetic state, and finite in the low-temperature "frozen" phase. Corresponding to this order parameter Q, one may define a response function via the relationship

$$Q = \sum_{jk} R_{jk} H_j H_k , \tag{9}$$

where H_j is the magnetic field at site j. For a uniform magnetic field, we find

$$Q = \chi^{\mathrm{EA}} H^2 .$$

For an idealized spin glass in which $P(J)$ is symmetric, the configurational average over the distribution of bonds results in

$$\chi^{\mathrm{EA}} = \beta^2 \sum_{jk} \sum_i \langle S_i S_j \rangle \langle S_i S_k \rangle \delta_{jk} / N, \qquad P(-J) = P(J) \tag{10}$$

since the configurational average is only nonzero if $j = k$. This is the Edwards and Anderson susceptibility, studied by Cherry and Domb[8(b)] and Fisch and Harris.[8(a)] We are interested in more general distributions, and therefore shall study the more general form

$$\chi^{\mathrm{EA}} = \beta^2 \sum_{ijk} \langle S_i S_j \rangle \langle S_i S_k \rangle / N .$$

We shall expand χ^{EA} in a power series in $\tanh\beta J$. The configurational average separates out terms involving repeated bonds. Therefore, we utilize the form

$$\chi^{\mathrm{EA}}\{P(J)\} = \sum_{\{m_i\},n} \begin{bmatrix} d \\ i \end{bmatrix} D^{(n,i)}_{m_1,\ldots,m_n} \langle \tanh\beta J \rangle^{m_1} \langle \tanh\beta J^2 \rangle^{m_2} \cdots \langle \tanh\beta J^n \rangle^{m_n} \tag{11}$$

where, as before $\sum_i m_i i = n$. The coefficients $D^{(n,i)}_{m_1,\ldots,m_n}$ are tabulated in Table IV. In the next section we shall analyze the series for χ^{EA} and the ordinary magnetic susceptibility and obtain a phase diagram corresponding to the random ferromagnet with various kinds of disorder.

ASYMPTOTIC ANALYSIS OF COEFFICIENTS—DETERMINATION OF THE CRITICAL TEMPERATURE

Once a series expansion is available for the ordinary susceptibility or Edwards-Anderson susceptibility, we would like to know the transition temperature suggested by it. There are several techniques available for analyzing the high-temperature series of the form

$$\chi = \sum_n A_n \omega^n, \quad \omega = \tanh\beta J . \tag{12}$$

They all rely on the assumption that near the critical point ω_c the susceptibility has the form

$$\chi(\omega) \approx (\omega_c - \omega)^{-\gamma} . \tag{13}$$

The value of ω_c and γ are best determined by Padé approximant of $d \ln\chi(\omega)/d\omega$, since

$$F(\omega) = d \ln\chi(\omega)/d\omega \approx +\gamma/(\omega_c - \omega) .$$

We can determine ω_c and Γ from Padé approximants of $F(\omega)$. Thus we take

$$F(\omega) = [L,M] = P_L(\omega)/Q_M(\omega) = \frac{p_0 + p_1\omega + \cdots + p_L\omega^L}{q_1 + q_1\omega + \cdots + q_M\omega^M} , \tag{14}$$

and the coefficients $p_0, \ldots, p_L$ and $q_1, \ldots, q_M$ are chosen to fit the first $L + M + 1$ terms in $F(\omega)$ expansion. The roots of $Q_M(\omega)$ provide the possible values of ω_c and the residues of the Padé approxi-

TABLE IV. The coefficient $D^{(n,i)}_{m_1,\ldots,m_n}$ in the Edwards-Anderson susceptibility [Eq. (11)].

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$n=1$	$m_1=1$ 4					
$n=2$	$m_1=2$ 6	$m_1=2$ 24				
	$m_2=1$ 2					
$n=3$	$m_1=3$ 8	$m_1=3$ 128	$m_1=3$ 192			
	$m_1=1,\ m_2=1$ 4	$m_1=1,\ m_2=1$ 16				
$n=4$	$m_1=4$ 10	$m_1=4$ 504	$m_1=4$ 2160	$m_1=4$ 1920		
	$m_1=2,\ m_2=1$ 4	$m_1=2,\ m_2=1$ 88	$m_1=2,\ m_2=1$ 144			
	$m_1=0,\ m_2=2$ 2	$m_1=0,\ m_2=2$ 8				
$n=5$	$m_1=5$ 12	$m_1=5$ 1976	$m_1=5$ 16 416	$m_1=5$ 36 864	$m_1=5$ 23 040	
	$m_1=3,\ m_2=1$ 4	$m_1=3,\ m_2=1$ 336	$m_1=3,\ m_2=1$ 1632	$m_1=3,\ m_2=1$ 1536		
	$m_1=1,\ m_2=2$ 4	$m_1=1,\ m_2=2$ 64	$m_1=1,\ m_2=2$ 96			
$n=6$	$m_1=6$ 14	$m_1=6$ 5888	$m_1=6$ 105 840	$m_1=6$ 452 352	$m_1=6$ 672 000	$m_1=6$ 332 560
	$m_1=4,\ m_2=1$ 4	$m_1=4,\ m_2=1$ 1192	$m_1=4,\ m_2=1$ 12 384	$m_1=4,\ m_2=1$ 29 568	$m_1=4,\ m_2=1$ 19 200	
	$m_1=2,\ m_2=2$ 4	$m_1=2,\ m_2=2$ 240	$m_1=2,\ m_2=2$ 1248	$m_1=2,\ m_2=2$ 1152		
	$m_2=3$ 2	$m_2=3$ 32	$m_2=3$ 48			
	$m_1=3,\ m_3=1$ 0	$m_1=3,\ m_3=1$ -16				

mant provide the exponent γ. The smallest real value of ω_c with $\gamma>0$ is the required critical value of ω_c.

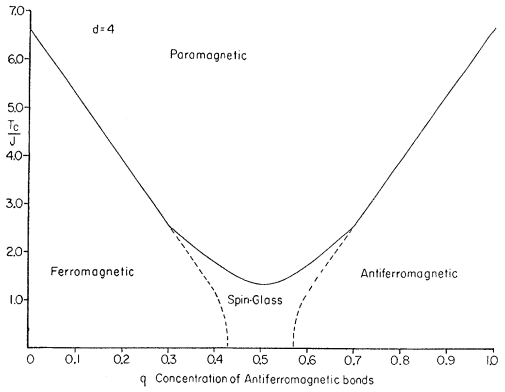

FIG. 2. Phase diagram of four-dimensional hypercubic Ising spin system with random distribution of $1-q$ ferromagnetic bonds and q antiferromagnetic bonds.

Since we have only the first 7 to 8 terms in the expansion for χ and χ^{EA}, the accuracy of the estimate is not expected to be very high. However, experience with Padé approximant show that 7 to 8 terms are adequate to estimate the critical temperature within a few percent.

In Figs. 2–4, we present the results of the D log Padé analysis for a distribution of ferromagnetic and antiferromagnetic bonds

$$P(J)=p\delta(J-J_0)+q\delta(J+J_0)\ ,$$

where $p+q=1$. When $p=1$, all the bonds are ferromagnetic and the D log Padé analysis yields the critical temperature and critical exponents pertinent to the paramagnetic-ferromagnetic phase transition. Analysis of the series for the magnetic susceptibility of $d=4$ yields a T_c of $T_c/J=6.664$ compared to the best estimate of $T_c/J=6.672$ given by Fisher and Gaunt.[11] The estimate of the critical exponent γ is

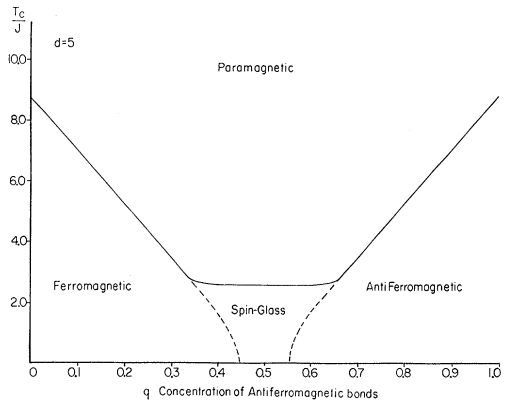

FIG. 3. Phase diagram of five-dimensional hypercubic Ising spin system with random distribution of $1-q$ ferromagnetic bonds and q antiferromagnetic bonds.

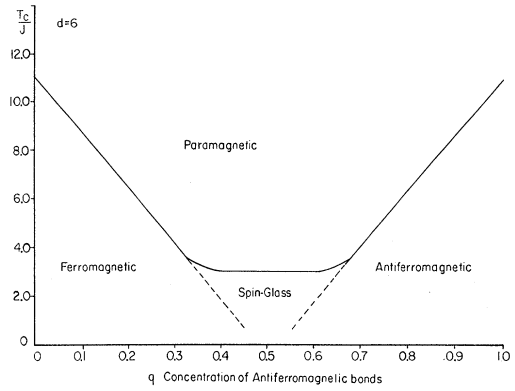

FIG. 4. Phase diagram of six-dimensional hypercubic Ising spin system with random distribution of $1-q$ ferromagnetic bonds and q antiferromagnetic bonds.

much more sensitive, we find $\Gamma=1.108$ compared with the value given by Fisher and Gaunt ($\gamma=1.094$). As the concentration of antiferromagnetic bonds q increases, the transition temperature T_c drops, in an almost linear manner. This linear decrease persists until a value of q such that

$$(1-2q)(2d-1)\sim 1 .$$

For this regime of concentrations T_c drops rapidly to zero, in logarithmic manner.

Analysis of the Edwards-Anderson susceptibility χ^{EA} series yields a similar behavior for the region of small q. For $d=4$, 5, and 6 the transition temperatures coincide with those obtained from the series for the magnetic susceptibility for q in the range $0\leq q\leq 0.3$. As q is increased beyond this range, the transition temperature taken from the Edwards-Anderson susceptibility is higher than the temperature at which the paramagnetic-ferromagnetic instability would occur. We take this as being indicative that a paramagnetic—spin-glass phase transition occurs, which preempts a ferromagnetic phase transition. The high-temperature series cannot be used to find further instabilities which occur in the low-temperature ordered phase. Thus the dashed line separating the ferromagnetic and spin-glass phase has no real significance. For large d, we find that our results continue on to those obtained, for the symmetric case $p=q$, by Fisch and Harris.[8(a)] For the case where $d=3$, the series for Edwards-Anderson susceptibility is poorly convergent, even in the ferromagnetic case $p=1$, and our estimate of T_c from this series is approximately 10% higher than that obtained from the magnetic susceptibility. We therefore cannot draw any concrete conclusions about the existence or nonexistence of a spin-glass phase for three-dimensional random magnets. This result is consistent with the earlier results which indicate that the lower critical-dimensional dimensionality as determined from high-temperature series is 4.[13] We notice that for large d the convergence of the series becomes more regular, and estimates of T_c more precise. While we have no direct way of determining higher critical dimensionality, our results show that the critical exponents are $\gamma=2$ at $p=1.0$ and $\gamma=1$ at $p=0.5$ for $d=6$. Since these are the mean-field values, it indicates that $d=6$ is the upper critical dimensionality.[14]

CRITICAL TEMPERATURE IN THE LIMIT OF LARGE d

For large d, one can express the nth-order terms in powers of $(2d-1)^{-1}$. Following Fisher and Gaunt,[11] we shall separate the coefficient a_n of the nth-order term in the susceptibility in terms of the number of n-step self-avoiding walks c_n and correction terms d_n. Thus

$$a_n=c_n+d_n ,$$

where for large d, c_n can be represented as a polynomial

$$c_n=2^n n!\begin{bmatrix} d \\ n \end{bmatrix}+2^{n-1}(n-1)!(n^2-2n+1)\begin{bmatrix} d \\ n-1 \end{bmatrix}$$

$$+2^{n-2}(n-2)!(3n^4-20n^3+48n^2-55n+36)\begin{bmatrix} d \\ n-2 \end{bmatrix}\Big/2!$$

$$+2^{n-3}(n-3)!(n^6-14n^5+79n^4-235n^3+412n^2-459n+330)\begin{bmatrix} d \\ n-3 \end{bmatrix}\Big/3!+\cdots$$

and similarly

$$d_n=-2^{n-3}(n-3)!(n-4)\begin{bmatrix} d \\ n-3 \end{bmatrix}\frac{\langle\tanh^2\beta J\rangle}{\langle\tanh\beta J\rangle^2}-2^{n-4}(n-4)!(n-5)\begin{bmatrix} d \\ n-4 \end{bmatrix}\frac{\langle\tanh^2\beta J\rangle^2}{\langle\tanh\beta J\rangle^4}$$

$$-2^{n-4}(n-4)!(n^3-12n^2+49n-74)\frac{\langle\tanh^2\beta J\rangle}{\langle\tanh\beta J\rangle^2}\begin{bmatrix} d \\ n-4 \end{bmatrix}$$

$$-2^{n-4}(n-4)!(n-6)\begin{bmatrix} d \\ n-4 \end{bmatrix}+\cdots .$$

We shall expand a_n in powers of $(2d-1)^{-1}$, by utilizing the series

$$\begin{bmatrix} d \\ r \end{bmatrix}=2d(2d-1)^{r-1}\left[1-\frac{(r-1)^2}{2d-1}+\frac{(r-1)(3r^3-13r^2+17r-6)}{6(2d-1)^2}\right.$$

$$\left.-\frac{(r-1)(r^5-9r^4+30r^3-45r^2+29r-6)}{6(2d-1)^3}+\cdots\right].$$

On substituting in the expression for a_n, we obtain

$$a_n=2d(2d-1)^{n-1}\left[-\frac{n-3}{(2d-1)^2}-\frac{(2n-13)+(n-4)\frac{\langle\tanh^2\beta J\rangle}{\langle\tanh\beta J\rangle^2}}{(2d-1)^3}\right.$$

$$\left.+\frac{\frac{n^2}{2}-\frac{15n}{2}+113-(n-10)\frac{\langle\tanh^2\beta J\rangle}{\langle\tanh\beta J\rangle^2}-(n-5)\frac{\langle\tanh^2\beta J\rangle^2}{\langle\tanh\beta J\rangle^4}}{(2d-1)^3}-\cdots\right].$$

On taking the logarithm of $a_n/2d$

$$\ln(a_n/2d)=(n-1)\ln(2d-1)-(n-3)/(2d-1)^2-\left[(2n-13)+(n-4)\frac{\langle\tanh^2\beta J\rangle}{\langle\tanh\beta J\rangle^2}\right]\Big/(2d-1)^3$$

$$-\left[\left[\frac{25n}{2}-\frac{217}{2}\right]+(n-10)\frac{\langle\tanh^2\beta J\rangle}{\langle\tanh\beta J\rangle^2}+(n-5)\frac{\langle\tanh^2\beta J\rangle^2}{\langle\tanh\beta J\rangle^4}\right]\Big/(2d-1)^4 .$$

Since, as noted by Fisher and Gaunt,[11] this series only contains terms of order n and lower it is consistent with an asymptotic form

$$a_n=Cn^{\gamma-1}\omega^n \text{ as } n\to\infty ,$$

where ω is given by

$$\omega=(2d-1)\left[1-\frac{1}{(2d-1)^2}-(2d-1)^{-3}\left[2+\frac{\langle\tanh^2\beta J\rangle}{\langle\tanh\beta J\rangle^2}\right]\right.$$

$$\left.-(2d-1)^{-4}\left[12+\frac{\langle\tanh^2\beta J\rangle}{\langle\tanh\beta J\rangle^2}+\frac{\langle\tanh^2\beta J\rangle^2}{\langle\tanh\beta J\rangle^4}\right]+\cdots\right]$$

and γ by its mean-field value 1. This is in accordance with expectations since the expansion is valid only for d greater than the upper critical dimensionality d^*. The critical temperature is given by the solution of the equation

$$\langle \tanh\beta J\rangle=\omega^{-1}=(2d-1)^{-1}\left[1+(2d-1)^{-2}+(2d-1)^{-3}\left[2+\frac{\langle\tanh^2\beta J\rangle}{\langle\tanh\beta J\rangle^2}\right]\right.$$

$$\left.+(2d-1)^{-4}\left[13+\frac{\langle\tanh^2\beta J\rangle}{\langle\tanh\beta J\rangle^2}+\frac{\langle\tanh\beta J^2\rangle^2}{\langle\tanh\beta J\rangle^4}\right]+\cdots\right]. \tag{15}$$

Although this series is probably only asymptotic, it does give remarkably good agreement with D log Padé analysis. As pointed out by Fisher and Gaunt,[11] this series yields estimates for the critical temperature of a regular, nondisordered ferromagnet to within 0.1% for $d=3$ and 0.3% for $d=4$. The worst error occurs for $d=2$, which is 5.6%. For the case of the random systems, one can see that as the disorder increases so does the error in the critical temperature. (See Table V). However, one can still make estimates that agree with the results of the D log Padé analysis to within 0.5% for $d=3$ and to 2% for $d=4$, providing that $\langle\tanh\beta J\rangle$ does not have a value close to the percolation probability p_c, which is given by

$$p_c=(2d-1)^{-1}[1+5(2d-1)^{-2}/2+\tfrac{15}{2}(2d-1)^{-3}+57(2d-1)^{-4}+\cdots]$$

(to within 2%), as found by Gaunt and Ruskin.[15]

As one can see, by inspection of the series for ω^{-1}, there should be small difference between the critical temperature of the quenched random system and the annealed system, when the disorder is small. These differences rapidly increase until the critical temperatures drop rapidly to zero near the percolation concentration. This behavior has been reported previously by Rappaport[16] for the case of bond dilution where

$$P(J)=p\delta(J-J_0)+(1-p)\delta(J)\ .$$

The behavior of T_c is shown in Fig. 5. However, for a spin-glass disorder, in which there are ferromagnetic and antiferromagnetic interactions, the difference between the quenched and annealed systems is much larger.

We shall now discuss the large d behavior of the Edwards-Anderson order parameter, in the case of a symmetric distribution of exchange interactions $P(J)=P(-J)$. Again we shall separate out the coefficient into a self-avoiding random walk part c_n and correction terms b_n. Thus

$$a_n=c_n-b_n\ ,$$

where

$$b_n=2^{n-2}(n-2)!\begin{bmatrix}d\\n-2\end{bmatrix}6(n-3)$$

$$+2^{n-3}(n-3)!\begin{bmatrix}d\\n-3\end{bmatrix}\left[(6n^3-54n^2+184n-296)-3(n-4)\frac{\langle\tanh^4\beta J\rangle}{\langle\tanh^2\beta J\rangle^2}\right]$$

$$+2^{n-4}(n-4)!\begin{bmatrix}d\\n-4\end{bmatrix}\left[(3n^5-53n^4+394n^3-1631n^2+4004n-5304)\right.$$

$$\left.-3(n^3-12n^2+49n-74)\frac{\langle\tanh^4\beta J\rangle}{\langle\tanh^2\beta J\rangle^2}+3(n-5)\frac{\langle\tanh^4\beta J\rangle^2}{\langle\tanh^2\beta J\rangle^4}\right]+\cdots\ .$$

On combining these terms we obtain the expression

$$a_n = 2d(2d-1)^{n-1}\left[1-\frac{7(n-3)}{(2d-1)^2}-\frac{(24n-147)-3(n-4)\dfrac{\langle\tanh^4\beta J\rangle}{\langle\tanh^2\beta J\rangle^2}}{(2d-1)^3}\right.$$
$$\left.+\frac{(49n^2/2-695n/2+1719)+3(n-10)\dfrac{\langle\tanh^4\beta J\rangle}{\langle\tanh^2\beta J\rangle^2}+3(n-5)\dfrac{\langle\tanh^4\beta J\rangle^2}{\langle\tanh^2\beta J\rangle^4}}{(2d-1)^4}+\cdots\right].$$

This is consistent with the form $a_n \sim n^{\gamma-1}\omega^n$ for large n. We can identify ω by taking the logarithm of $a_n/2d$,

$$\ln\frac{a_n}{2d} = (n-1)\ln(2d-1)-\frac{7(n-3)}{(2d-1)^2}-\frac{(24n-147)-3(n-4)\dfrac{\langle\tanh\beta J^4\rangle}{\langle\tanh\beta J^2\rangle^2}}{(2d-1)^3}$$
$$-\frac{\dfrac{401n}{2}-\dfrac{2997}{2}+3(n-10)\dfrac{\langle\tanh\beta J^4\rangle}{\langle\tanh\beta J^2\rangle^2}+3(n-5)\dfrac{\langle\tanh\beta J^4\rangle^2}{\langle\tanh\beta J^2\rangle^4}}{(2d-1)^4}+\cdots,$$

which gives ω as

$$\omega=(2d-1)\left[1-\frac{7}{(2d-1)^2}-\frac{24-\dfrac{3\langle\tanh\beta J^4\rangle}{\langle\tanh\beta J^2\rangle^2}}{(2d-1)^3}-\frac{176+\dfrac{3\langle\tanh^4\beta J\rangle}{\langle\tanh^2\beta J\rangle^2}+\dfrac{3\langle\tanh^4\beta J\rangle^2}{\langle\tanh\beta J^2\rangle^4}}{(2d-1)^4}+\cdots\right]$$

and $\gamma=1$. Thus the asymptotic, large d behavior of the Edwards-Anderson susceptibility yields a mean-field transition, and a transition temperature given by

$$\langle\tanh\beta J^2\rangle=(2d-1)^{-1}\left[1+\frac{7}{(2d-1)^2}+\frac{24-\dfrac{3\langle\tanh^4\beta J\rangle}{\langle\tanh^2\beta J\rangle^2}}{(2d-1)^3}\right.$$
$$\left.+\frac{225+\dfrac{3\langle\tanh^4\beta J\rangle}{\langle\tanh^2\beta J\rangle^2}+\dfrac{3\langle\tanh^4\beta J\rangle^2}{\langle\tanh^2\beta J\rangle^4}}{(2d-1)^4}+\cdots\right].$$

Comparison with the results of Fisch and Harris for the distribution

$$P(J)=\tfrac{1}{2}\delta(J-J_0)+\tfrac{1}{2}\delta(J+J_0)$$

yields agreement to within 2% between $d=6$ and

TABLE V. The percolation point is defined as the solution of Eq. (15) in which $T=0$ and $\langle\tanh^n\beta J\rangle=p_c$. The "exact" results are obtained by various methods and are taken from the article by Gaunt and Ruskin (Ref. 15).

p_c	$d=2$	$d=3$	$d=4$
Exact	0.5	0.246	0.160
Eq. (15)	0.498	0.230	0.153

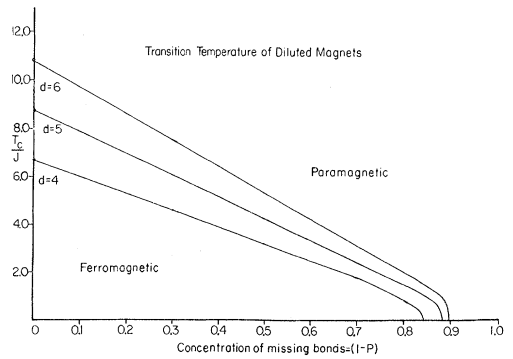

FIG. 5 Phase diagram of four-, five-, and six-dimensional hypercubic Ising spin system with random distribution of $1-p$ ferromagnetic bonds and p missing bonds.

$d=5$. However, at $d=4$, where Fisch and Harris[8(a)] find that the critical exponent $\gamma\to\infty$, the above series in $2d-1$ yields an estimate for $\langle\tanh^2\beta J\rangle$ which is 50% lower, and the series does not appear to be convergent.

SUMMARY

To summarize, we have shown that the high-temperature—series expansion predicts instabilities of the high-temperature paramagnetic phase to low-temperature ordered phases. For dimensions $d\geq 4$ we find that as the concentration of antiferromagnetic bonds is increased the ferromagnetic transition temperature decreases linearly. We find that, by further increasing the concentration of antiferromagnetic bonds, the low-temperature ferromagnetic instability is replaced by an instability to a spin-glass phase. As the distribution of bonds approaches the symmetric case $P(J)=P(-J)$, the transition temperature approaches that calculated by Fisch and Harris.[8(a)] Unfortunately, for $d=3$, we cannot draw any conclusions about the existence of a spin-glass phase. This is consistent with the picture of Fisch and Harris,[8(a)], who found that the critical exponent for the Edwards-Anderson susceptibility diverges as d approached 4, and the critical value of $\tanh\beta J$ became complex for $d\simeq 4$. Our results are in agreement with the idea that $d=4$ in the lower critical dimensionality and $d=6$ is the upper critical dimensionality.

In order to check the reliability of the high-temperature—series expansions, as applied to random systems, we have also examined the effect of random bond dilution. This has provided estimates of the percolation concentration which are in reasonable agreement with the estimates given by other authors, which suggests that it is reasonable to use the high-temperature—series expansion to predict a phase transition at $T=0$.

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