Quantized Hall conductance and edge states: Two-dimensional strips with a periodic potential

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We have investigated the energy levels of an electron in a periodic potential and a magnetic field, confined to a two-dimensional strip of infinite length but finite width, for two sets of boundary conditions. The bulk formula giving the quantized Hall conductance is explained in terms of a special gauge-invariance property of the edge states. This Communication clarifies the role of the geometry in the derivations of the quantized Hall effect.

To this day, the quantization of the Hall conductance σ_H of a two-dimensional electron gas in a magnetic field, when the Fermi level lies in an energy gap of extended states, has been explained with fair success along two different lines of argument. The analysis of Laughlin[1] and Halperin[2] makes use of an annular geometry and of considerations of gauge invariance to arrive at the formula

$$\sigma_H = ec\frac{\Delta N}{\Delta \Phi} \ , \tag{1}$$

where Φ is a solenoid magnetic flux confined to the interior of the hole in the annulus and ΔN is the number of electrons transferred from one edge to the other when Φ varies by increments of $\Phi_0 = hc/e$, the flux quantum. The specific integer value for σ_H (in units of e^2/h) within a given gap is not predicted by this formula.

The other line of argument, developed in particular by the Japanese school,[3] Středa,[4] and Thouless *et al.*,[5] is a bulk analysis starting from Kubo linear-response theory. Thus, in the presence of a periodic potential, the spectrum of Landau levels displays an amazing complexity,[6] but the value of the quantized Hall conductance obeys the simple formula

$$\sigma_H = ec\frac{\partial n}{\partial H} \ , \tag{2}$$

where n is the electronic density and H the applied magnetic field. Equation (2) applies, in particular, to an infinite periodic system, without impurities, when the Fermi level lies within a gap in the energy spectrum.

The purpose of this Communication is to clarify the role of the geometry (ring, strip, or bulk) and to reveal the gauge-invariance principle hidden in formula (2), by an analysis of the properties of edge states. For this purpose, we study the spectrum of Landau levels in two-dimensional regular strips which are infinite in extent in one direction and finite in the other, and we discuss the modifications which occur when the width is increased. For simplicity, no disordered potential or electron interactions are introduced in the analysis.

In a tight-binding formulation, our strips can be viewed as two-dimensional regular arrays of sites with N sites per row, parallel to the x axis, and unlimited in the y direction. For symmetry reasons, the origin of the x axis is taken at the middle of the strip and the vector potential is chosen as $A_y = Hx$, corresponding to a uniform perpendicular field H. The eigenvalue equation[6] for the tight-binding wave functions ψ reads

$$\epsilon\psi(x,y) + \psi(x+a,y) + \psi(x-a,y) + e^{-i\Gamma(x)}\psi(x,y+a) + e^{+i\Gamma(x)}\psi(x,y-a) = 0 \ , \tag{3}$$

at a bulk site (not located on the edges); a is the lattice spacing and $\Gamma(x) = (2\pi/\Phi_0)Hax$ is related to the circulation of the vector potential along a lattice bond pointing in the y direction; the energy eigenvalue ϵ is taken dimensionless. For a site located on the right (left) edge, an equation similar to (3) holds with the second (third) term missing. This defines the natural boundary conditions in the tight-binding problem.

In a different context,[7–9] our strips can be regarded as networks of superconducting wires with nodes at the lattice sites. To obtain the critical superconducting field, one has to solve the linearized Ginzburg-Landau equations which, in the bulk, are identical to (3) if one interprets ψ as the superconducting order parameter and if ϵ is defined as $-4\cos(a/\xi)$, where ξ is the coherence length. On the right (left) edge, not only is the second (third) term missing in (3) but, in contrast with the preceding formulation, ϵ must simultaneously be transformed into $3\epsilon/4$, because an edge node has three neighbors instead of four. This is the boundary condition coming naturally from the Kirchhoff continuity equations at the nodes, in the superconducting network problem.

Thus both systems have the same thermodynamic limit, when $N \to \infty$. The comparison of these two different kinds of boundary conditions, which turn out to be "repulsive" in the tight-binding problem and "attractive" in the network problem, provides useful hints to distinguish the universal from the peculiar properties of the edge states.

Before presenting our results, we make some introductory remarks. Because of the translation periodicity in the y direction, the eigenvalues and eigenfunctions can be indexed by a momentum k, $-\pi \leq k \leq +\pi$:

$$\psi_{\alpha,k}(x,y) = e^{ik(y/a)}\psi_{\alpha,k}(x,0) \quad . \tag{4}$$

Since x takes N values (N sites per row), one is left with the diagonalization of a $N \times N$ matrix which has N eigenvalues $\epsilon_\alpha(k,\phi)$ where $\alpha = 0,1,\ldots,(N-1)$ and $\phi = Ha^2/\Phi_0$ is the reduced magnetic flux, i.e., the flux per lattice cell divided by the flux quantum. It is easily checked that

$$\text{(i)} \quad \epsilon_\alpha(-k,\phi) = \epsilon_\alpha(+k,\phi) \quad ,$$

$$\text{(ii)} \quad \epsilon_\alpha(k,\phi+1) = \epsilon_\alpha(k,\phi) \quad ,$$

$$\epsilon_\alpha(k,\tfrac{1}{2}+\phi) = \epsilon_\alpha(k,\tfrac{1}{2}-\phi), \quad \text{for } N \text{ odd} \quad ,$$

$$\text{(iii)} \quad \epsilon_\alpha(k,\phi+2) = \epsilon_\alpha(k,\phi) \quad ,$$

$$\epsilon_\alpha(k,1+\phi) = \epsilon_\alpha(k,1-\phi), \quad \text{for } N \text{ even} \quad .$$

These symmetry properties allow one to restrict attention to the domain $0 \leq k \leq \pi$, $0 \leq \phi \leq 1$. In a (k,ϕ,ϵ) representation space, the spectrum consists of N energy surfaces which are well ordered, one above the other, beause they do not cross each other. As N increases, the number of energy sheets increases and each of them becomes more and more wrinkled according to some patterns that are analyzed below.

Two aspects of these energy sheets deserve special attention. One is the number of crossing points with a line (ϵ,ϕ) parallel to the k axis and the corresponding values of the tangent slope $\partial k/\partial\phi|_\epsilon$ at these points, because this governs the Hall conductance, as given by Eq. (6) below. The second is the projection of these energy sheets along the k axis, because this gives the spectrum as a function of magnetic field, to be compared with the bulk spectrum.[6] This comparison has a double interest: It provides a genesis of the bulk discontinuous spectrum in a manner which preserves at all stages (width N) the continuity of the eigenvalues as a function of field, and it reveals the role and the position of the edge states.

Analytic computation of the eigenvalues and eigenfunctions becomes rapidly unmanageable when N increases ($N > 4$). We have made extensive numerical investigations from which we extract here a few illustrative pictures.

Figure 1 shows an overview of the highest eigenvalue sheet as a function of k and ϕ, for $N = 7$ and tight-binding boundary conditions. For the same width ($N = 7$) and wire network boundary conditions, Fig. 2 shows a section of the spectrum at $k = 0$. Figure 3 shows three curves given by

$$\epsilon + 4 = 8p\phi, \quad p = 2,4,6 \quad , \tag{5}$$

and successive foldings on the edges $\epsilon = \pm 4$ of the zero-field spectrum. Equation (5) is analogous in form to the variation of Landau levels in free space. The folded curves of Fig. 3 constitute a skeleton of Fig. 2 through a succession of anticrossings. (The

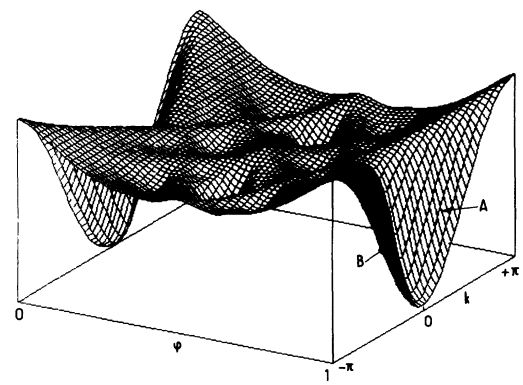

FIG. 1. Overview of the uppermost eigenvalue energy sheet, for strips of width $N = 7$ and tight-binding boundary conditions (see text). The energy ϵ is plotted as a function of the magnetic flux ϕ and of the momentum k. The two steep flat slopes, labeled A and B, joined by a fold centered at $k = 0$, correspond to edge states.

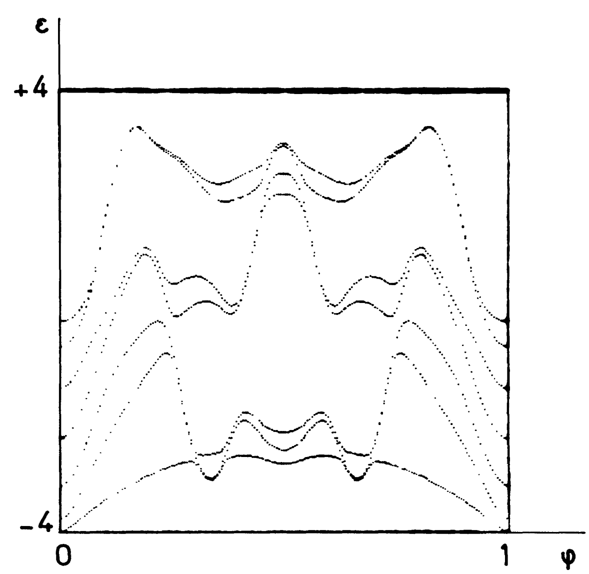

FIG. 2. For $N = 7$ and wire network boundary conditions (see text), a section of the spectrum at momentum $k = 0$. The energy ϵ is plotted as a function of the magnetic flux ϕ.

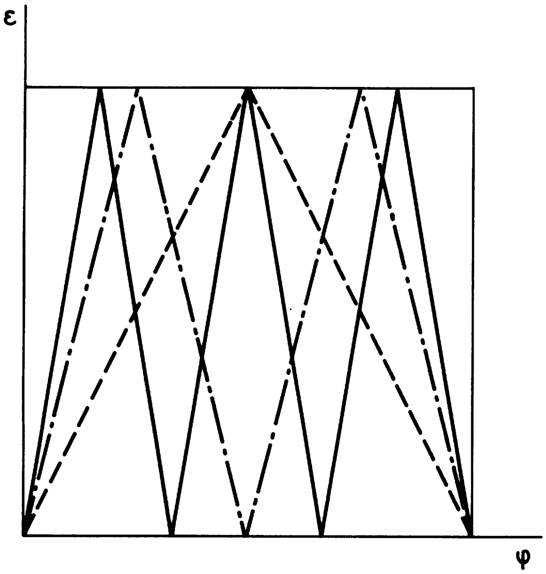

FIG. 3. Same coordinates as in Fig. 2. Sketch of the three folded curves given by $\epsilon+4=8p\phi$, $p=2$ (dashed line), $p=4$ (dot-dashed line), $p=6$ (full line). The foldings occur at $\epsilon=\pm4$. These three curves provide a skeleton of Fig. 2, through a succession of level anticrossings.

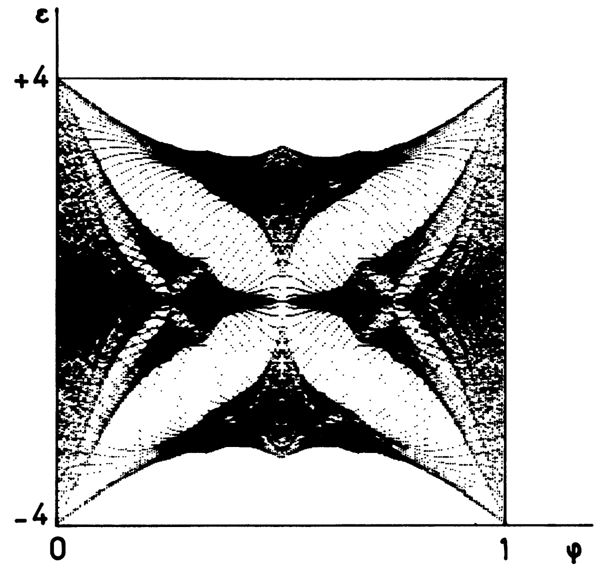

FIG. 4. Spectrum for a strip of width $N=13$ and tight-binding boundary conditions. Same coordinates as in Fig. 2 and 3. The numerical mesh is $\Delta\phi=5\times10^{-3}$ and $\Delta k=5\pi\times10^{-2}$.

skeptical reader is invited to use a piece of transparent paper to superpose the curves of Fig. 3 on Fig. 2). For $N=9$, another curve [corresponding to $p=8$ in (5)] introduces into the spectrum a further oscillating component with frequency 4 in ϕ, and so forth for increasing N. Strips with even N have half-odd integer frequencies [$p=1,3,5,\ldots$ in (5)]. All these frequencies can be recognized in the asymptotic spectrum[6] by keeping track of the gaps as a function of ϕ.

Figure 4 shows the complete spectrum for $N=13$ and tight-binding boundary conditions. In comparison with the asymptotic spectrum,[6] the most notable difference is that the gaps are now pseudogaps filled with edge states. These states ensure the continuity of the eigenvalues as a function of ϕ. Further observation shows that in the largest gaps, which have period 2 in ϕ, there is only one pair of Fermi momenta $(+k,-k)$, solutions of the equation $\epsilon(k,\phi)=\epsilon_F$ for given flux ϕ and Fermi level ϵ_F. In the next gaps (period 1 in ϕ), there are two pairs of solutions. Generally, we have found numerically that (i) there are p pairs of solutions in the gaps which have period $2/p$ in ϕ, (ii) these solutions correspond to eigenstates localized on the edges, (iii) the integer p is precisely equal to the modulus of the integer value of σ_H (expressed in units of e^2/h).

Each pair of edge states contributes equally and additively to the density of the states in the gap and to the value of σ_H, as given by (2), in the asymptotic limit $N\to\infty$. This finding, which generalizes a result of Halperin[2] for electrons without a periodic potential, holds for either of the boundary conditions presented above. The principal difference between the two sets of boundary conditions is that, for the superconducting case, there are edge states with energies below the lowest bulk level and above the highest, while for the tight-binding conditions, all the edge states lie in gaps between the bulk bands.

For spinless noninteracting electrons, the electronic density is given by the integrated density of states below the Fermi energy ϵ_F. The Středa[4]-Widom[10] formula (2), supplemented by Wannier's[11] result on the linear variation of the integrated density of states below any gap, allows then for a simple determination of the Hall integer, in agreement with the derivation of Thouless,[5] for any gap of the asymptotic spectrum. We discuss now the interpretation of this formula in terms of edge states.

For a strip geometry, the electronic density is a linear function of the Fermi momenta $k_i (i=1,\ldots p)$. Then the right-hand side of Eq. (2) can be expressed, for a finite system, by

$$ec\frac{\partial n}{\partial H}=\frac{e^2}{h}\frac{1}{\pi(N-1)}\sum_{i=1}^{p}\left.\frac{\partial k_i}{\partial\phi}\right|_{\epsilon_F}\mathrm{sgn}\left[\frac{\partial\epsilon}{\partial k_i}\right], \quad (6)$$

where the summation is over all pairs of Fermi momenta. Naturally, the sign of the contribution from a state at the Fermi level comes out to depend on the direction of its group velocity $v_g\sim\partial\epsilon/\partial k$.

Observation shows that the edge states in the largest gaps (with period 2 in ϕ) originate from the highest bare Landau levels [$p=N-1$ in (5)]. Their energy tends to be a piecewise function of the composite variable $(k \pm \pi(N-1)\phi)$ and, furthermore, a linear function of this variable, in accordance with (5), valid for $k \sim 0$. These edge states give rise to the steep flat slopes around the deep troughs observable in Fig. 1. Such a functional form,

$$\epsilon(k,\phi)=\epsilon(k-\pi(N-1)\phi) \quad , \tag{7}$$

implies

$$\left.\frac{\partial k}{\partial \phi}\right|_{\epsilon}=\pi(N-1) \quad . \tag{8}$$

Using Eqs. (2) and (6), one thus obtains for large N the quantized Hall conduction formula, that σ_H is an integer multiple of e^2/h. Then, it remains to provide the physical reason for the functional form (7).

The explanation is basically again gauge invariance. For a state localized on the right edge (say),

$$k-\pi(N-1)\phi=k-\frac{e}{\hbar c}Aa \quad , \tag{9}$$

where A is the constant vector potential seen by the localized eigenstate. Thus, for a state localized on one edge of the strip, the magnetic flux through the strip is equivalent to a Bohm-Aharonov field, like the solenoid flux in the Laughlin-Halperin geomertry.

Therefore the quantization of σ_H, as given by formulas (2), (6), and (8), directly originates from the transverse localization of the edge states. The more localized they are, the more accurate the quantization is. A look at Fig. 1 shows that for most eigenstates, which are bulk states, (7) does not hold at all. Thus the quantized Hall conductance obtains when the Fermi level is in a gap of the bulk bands, and only edge states occur at the Fermi level. Besides their interest for Hall conductance, formulas (2) and (6) provide a test for localization of the Landau levels at a given energy.

In higher gaps (gaps with smaller period in ϕ), there are edge states coming from lower bare Landau levels [$p < N-1$ in (5)] and the quantization is more slowly reached as N increases. A quantitative analysis for periodic and disordered potentials will be given in a forthcoming publication where some of the statements presented here will be substantiated.

In conclusion, we have shown that formula (2), which is very efficient for computing the quantized Hall conductance in the gaps of a noninteracting electron gas, can be simply understood in terms of the properties of edge states, without requiring the introduction of a ring geometry. Furthermore, this expression has been shown to provide a sensitive test of the localized character of the Landau levels at any Fermi energy.

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