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Exact solution of a one-dimensional Ising model in a random magnetic field

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Analytic results for the free energy, magnetic structure factor, and Edwards-Anderson order parameter of a one-dimensional ferromagnetic Ising model in a random magnetic field are obtained. The structure factor consists of both Lorentzian and Lorentzian-squared terms at all temperatures (T) greater than zero; the Lorentzian-squared terms vanish at $T=0$. The calculated correlation length agrees with predictions of the Imry-Ma domain argument.

The low-temperature (T) behavior of the Ising model in a random magnetic field (RFIM) is currently the source of considerable controversy. One of the important issues is the determination of the lower critical dimension d_c, below which the model cannot exhibit ferromagnetism at any $T>0$, even for arbitrarily small random fields, h. There exist two competing groups of theories which predict $d_c=2$ (Ref. 1) and $d_c=3$ (Ref. 2), respectively. The experimental results are at present insufficiently clean to rule out either of these possibilities unambiguously.[3,4] Other quantities which are of interest are, for example, the spin-spin correlation function and its behavior at large distances, the dependence of the correlation length $\xi(T,\Delta)$ on T and the average square of the random field $\Delta \equiv [h^2]_{\text{av}}$, and the Edwards-Anderson order parameter. It would be useful to find exactly soluble models for which these thermodynamic quantities can be calculated.

One of the few aspects of the RFIM on which there is unanimous agreement is that $d_c>1$: The one-dimensional (1D) RFIM is always magnetically disordered, even at $T=0$. A detailed solution of the 1D model can nevertheless help to clarify the following unresolved issues asssociated with the random-field Ising problem:

(1) A simple renormalization-group scaling argument based on interface representations of the RFIM[1(b),2(b)–2(d)] suggests that, having determined d_c, one can compute, for any $d<d_c$, the largest linear size ξ_Δ which a ferromagnetically ordered cluster of spins can attain in the RFIM at low temperatures. This length, which naturally serves as the limiting low-temperature ferromagnetic correlation length of the system, is simply $\xi_\Delta \sim \Delta^{-(d_c-d)^{-1}}$ in the limit $\Delta \to 0$. Hence in 1D, $\xi_\Delta \sim \Delta^{-(d_c-1)^{-1}}$, an expression one can match against the exact 1D solution to infer d_c. Since the argument that yields $\xi_\Delta \sim \Delta^{-(d_c-d)^{-1}}$ is plausible but not rigorous, one cannot claim to obtain an unimpeachable result for d_c from the 1D result, however.

(2) Part of the uncertainty about the RFIM results from our inability to calculate the static magnetic structure factor (measured in scattering experiments) at low temperatures. There are indications from theory[1(a),2(c),2(d),5] that, at least for temperatures above the ferromagnetic critical temperature $T_c^{(0)}$ of the associated pure Ising system, the structure factor is a sum of Lorentzian and Lorentzian-squared terms.[6] Structure factor data on 2D and 3D realizations of the RFIM do *not*[3,4] seem to exhibit an appreciable Lorentzian-squared term for $T>T_c^{(0)}$; they are, however, fitted quite well by a sum of Lorentzian and Lorentzian-squared terms for some range of temperatures below $T_c^{(0)}$, though other forms fit equally nicely.[3(c),4] It is therefore instructive to check explicitly for the presence of a Lorentzian-squared term in the 1D structure factor.

(3) The 2D and 3D magnetic experimental realizations of the RFIM are somewhat more complicated than the model,[3,7] typically possessing random exchange in addition to random fields, for example. It is difficult, in the absence of a definitive calculation of the structure factor of the RFIM in 2D and 3D, to separate, in analyzing the data, random-field effects from these other complications. If one can compute the structure factor exactly in the 1D RFIM, one can compare the results to scattering experiments performed on 1D magnetic realizations of the model. Since the 1D realizations possess the same undesirable extra features as do the 2D and 3D systems, such comparison would help differentiate random-field features from other effects in the observed behavior. The insight obtained could, it is hoped, be applied to the analysis of the higher-dimensional experiments.

In this paper we exhibit the exact solution of a 1D

RFIM. The model is defined by the Hamiltonian

$$H=-J\sum_{\langle ij\rangle}\sigma_i\sigma_j-\sum_i h_i\sigma_i \quad (1)$$

(where $J>0$ is the exchange constant, $\sigma_i=\pm 1$ for all sites i, $\sum_{\langle ij\rangle}$ denotes a sum over nearest neighbors, and h_i is the random field on the ith site) and by the probability distribution: For each site, $h_i=+\infty$, $-\infty$, or 0 with probability $p/2$, $p/2$, and $(1-p)$, respectively; here $0\leq p\leq 1$. This model is, to our knowledge, the first 1D RFIM for which the structure factor is analytically calculable.[8] The central results we obtain are that $\xi_\Delta\sim\Delta^{-1}$ as $\Delta\rightarrow 0$, consistent with the domain argument of Imry and Ma[1(a)] and with $d_c=2$ in the formula $\xi\sim\Delta^{-(d_c-1)^{-1}}$, and that the structure factor *does* possess a Lorentzian-squared term at all $T>0$ but only Lorentzian terms at $T=0$. We find that for any T the actual correlation length ξ is determined by ξ_Δ and the thermal correlation length ξ_T of the pure 1D chain through $\xi^{-1}=\xi_T^{-1}+\xi_\Delta^{-1}$.

Before displaying the results in detail we comment briefly on the relationship of the field distribution of our model to the more standard distributions,[1(a),5]

$$p(h_i)\sim e^{-h_i^2/\Delta}$$

or

$$p(h_i)\sim\delta(h_i-h)+\delta(h_i+h)\ ,$$

where Δ and h^2 are typically taken $<<J^2$. Our model has random fields only on a fraction p of sites but the fields are infinitely strong. This is not as unphysical as it first seems. At $T=0$, for example, any field h_i whose magnitude exceeds $2J$ in the 1D RFIM is effectively infinite in that it will constrain the spin σ_i to point in the direction of h_i. At low T, then, "infinite" really means "bigger than $2J$." Nevertheless such large fields seem contrary to the principle that the random-field strength should be small compared to J if ferromagnetism is to be possible in *any* dimension. In our model, however, the typical distance $1/p$ between fields is large for small p. It turns out that $2p$ plays the role of Δ/J^2 in the usual models. To see this, note that a spin on a site with an infinite field must always point in the direction of that field. Hence each spin neighboring such a pinned spin feels an effective magnetic field of magnitude J: The energy of the neighbor changes by $2J$ whenever it changes direction. For small p, when the probability of adjacent spins both being pinned by fields is negligible, a total fraction $2p$ of the spins feels this effective field. Hence $\Delta\equiv[h^2]_{\rm av}=2pJ^2$ in 1D or $\Delta=2dpJ^2$ for a hypercubic lattice in d dimensions. As further verification that the physics of our particular random-field distribution is the same as in the usual models, we note that the venerable domain argument,[1(a)] original source of the prediction $d_c=2$, can be simply applied to our model. The result is identical to that for the standard models, provided one again makes the identification $\Delta\sim pJ^2$.

It seems to us very reasonable, therefore, that our model accurately represents the usual model. In 1D, however, it is conderably more manageable than its small-h counterpart. One need only solve the trivial problem of a finite length of Ising chain in zero field, with boundary conditions established by the infinite pinning fields, to compute all thermodynamic and correlation functions. We note that our result $\xi_\Delta\sim\Delta^{-1}$ can also be shown to hold[9] in the standard models at $T=0$. It seems likely that the Lorentzian-squared term in the structure factor for $T>0$ is likewise a general feature of the 1D RFIM and not an artifact of our model. It is worth recalling, in support of this belief, that[7] application of a uniform magnetic field to a randomly dilute antiferromagnet generates an effective random field. Moreover, in 1D the dilute Ising antiferromagnet in a uniform field constitutes an exactly soluble model. It is straightforward to show that the structure factor for this model does indeed contain a Lorentzian-squared piece for all $T>0$. For purposes of studying the limiting effect of random fields on correlations in 1D the model is, unfortunately, not terribly interesting; even in zero field (i.e., in the absence of random fields) correlations in the dilute antiferromagnetic system are limited by the gaps in the chain caused by missing spins. The main effect of the applied field is to shorten the correlations between any two spins *not* separated by such a gap.

We now present our results for some of the thermodynamic functions associated with the model (1). Let $f_\pm(N)$ be the reduced free energy density of a chain of length $(N+1)$ with periodic (+) or antiperiodic (−) boundary conditions. The free energy density $F(p,T)$ associated with the Hamiltonian (1) is obtained by averaging $f_\pm(N)$ over all possible chain lengths $(N+1)$,

$$\beta F(p,T)=-\sum_{N=1}^{\infty}\tfrac{1}{2}W(p)[f_+(N)+f_-(N)]\ , \quad (2)$$

where $W(p)=p^2(1-p)^{N-1}$ is the probability of having a chain of length $(N+1)$. Using the standard transfer matrix method[10] we find

$$f_\pm(N)=\ln[(c^N\pm s^N)/2]\ ,$$

where $c=2\cosh\beta J$, $s=2\sinh\beta J$, and $\beta^{-1}=k_B T$. The free energy density may be rewritten as

$$\beta F=-(1-p)\ln 2+\tfrac{1}{2}\ln(1-z^2)-\tfrac{1}{2}p^2\sum_{N=1}^{\infty}(1-p)^{N-1}\ln(1-z^{2N})\ , \quad (3)$$

where $z=\tanh\beta J$. In the high-temperature limit $z<<1$, we find

$$\beta F=-(1-p)\ln 2-\tfrac{1}{2}(1+p^2)z^2+O(z^4)\ , \quad (4)$$

while at $T=0$ $(z=1)$ one has $F=-J(1-p)$. Note that while this last expression is nonsingular in the limit $p\to 0$, $F(p,T)$ has the usual essential singularity (i.e., terms of the form $e^{-\beta J}$) at $T=0$ for any p.

Consider now the correlation functions $S(l)=[\langle\sigma_n\sigma_{n+l}\rangle]_{\rm av}$ and

$$\chi(l)=[\langle\sigma_n\sigma_{n+l}\rangle-\langle\sigma_n\rangle\langle\sigma_{n+l}\rangle]_{\rm av}\ , \tag{5}$$

where $\langle\cdots\rangle$ denotes thermodynamic average. It is easy to see that the only configurations which contribute to $S(l)$ and $\chi(l)$ are those which involve no more than one site between n and $(n+l)$ with a magnetic field on it. Let $\langle\sigma_n\rangle_N^{\pm}$ be the average magnetization at site n, $n=0,\ldots,N$, associated with a finite chain with boundary conditions $\sigma_0=1$ and $\sigma_N=\pm 1$. One has

$$\langle\sigma_n\rangle_N^{\pm}=\frac{z^n\pm z^{N-n}}{1\pm z^N}\ . \tag{6a}$$

Similarly one finds

$$\langle\sigma_n\sigma_{n+l}\rangle_N^{\pm}=\frac{z^l\pm z^{N-l}}{1\pm z^N},\quad 0\le n<n+l\le N\ . \tag{6b}$$

The correlation function $S(l)$ is given by

$$S(l)=\sum_{N=0}^{\infty}(N+1)p^2(1-p)^{N+l-1}\tfrac{1}{2}(\langle\sigma_n\sigma_{n+l}\rangle_{N+l}^{+}+\langle\sigma_n\sigma_{n+l}\rangle_{N+l}^{-})$$
$$+\sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\tfrac{1}{4}p^3(1-p)^{N+M+l-2}\sum_{n=1}^{l-1}(\langle\sigma_n\rangle_{N+n}^{+}+\langle\sigma_n\rangle_{N+n}^{-})(\langle\sigma_{l-n}\rangle_{M+l-n}^{+}+\langle\sigma_{l-n}\rangle_{M+l-n}^{-})\ , \tag{7}$$

where the first sum takes into account configurations with no magnetic field on sites between n and $(n+l)$, while the second sum averages over configurations with a magnetic field on one intermediate site. In the second sum we have made use of the fact that for configurations which involve a field on an intermediate site, one may replace $\langle\sigma_n\sigma_{n+l}\rangle$ by $\langle\sigma_n\rangle\langle\sigma_{n+l}\rangle$.

Using Eqs. (6) and (7) we find that for large l, satisfying $z^l\ll 1$, the leading terms in $S(l)$ are

$$S(l)=Al(1-p)^lz^l+B(1-p)^lz^l+O(l(1-p)^lz^{3l})\ , \tag{8a}$$

where

$$A=p(1-z^2)^2/[1-(1-p)z^2]^2\ , \tag{8b}$$

$$B=\{2p^2(1-z^2)^2/[1-(1-p)z^2]\}\times\frac{\sum_{k=1}^{\infty}z^{4k}}{(1-z^{2k})[1-(1-p)z^{2k}][1-(1-p)z^{2(k+1)}]}-A+C\ , \tag{8c}$$

$$C=\{1/[1-(1-p)z^2]^2\}[1+p-2z^2+(1-p)z^4] \tag{8d}$$

Note that in the limit $z=0$ one has $A/B=p$. The inverse correlation length is given by $\xi^{-1}=\xi_T^{-1}+\xi_\Delta^{-1}$, where $\xi_T^{-1}=-\ln z$ and $\xi_\Delta^{-1}=-\ln(1-p)$ are the thermal and the random-field inverse correlation lengths, respectively. We also note that $S(l)$ is *not* a purely exponential function, as is the case for the nonrandom chain, but rather it exhibits a logarithmic correction. Taking the Fourier transform of Eq. (8) we find

$$S(q)\cong\frac{4A\xi^{-2}}{(\xi^{-2}+q^2)^2}+\frac{2(B\xi^{-1}-A)}{(\xi^{-2}+q^2)}\ , \tag{9}$$

which has the anticipated Lorentzian plus Lorentzian-squared structure. The expression (8) is not valid at $T=0$ $(z=1)$, since, in deriving it, we assumed $lz^{2l}\ll 1$. The $T=0$ correlation function may be calculated directly from Eq. (7). We find that in the limit $(1-p)^l\ll 1$ the correlation function is purely exponential, namely, $S(l)\sim\bar{c}(1-p)^l$ for some constant $\bar{c}$.

Consider now $\chi(l)$. In calculating this correlation function, the only configurations which have to be taken into account are those which have no magnetic field on the sites between n and $(n+1)$. $\chi(l)$ is given by

$$\chi(l)=\sum_{N=0}^{\infty}p^2(1-p)^{N+l-1}\tfrac{1}{2}\sum_{n=0}^{N}(\langle\sigma_n\sigma_{n+l}\rangle_{N+l}^{+}+\langle\sigma_n\sigma_{n+l}\rangle_{N+l}^{-}-\langle\sigma_n\rangle_{N+l}^{+}\langle\sigma_{n+l}\rangle_{N+l}^{+}-\langle\sigma_n\rangle_{N+l}^{-}\langle\sigma_{n+l}\rangle_{N+l}^{-})\ . \tag{10}$$

For large l, satisfying $z^l\ll 1$, we find that to leading order in z^l

$$\chi(l)\cong Dz^l(1-p^l)[1+O(z^{2l})]\ , \tag{11a}$$

$$D=\frac{(1-p)(1-z^2)^2}{[1-(1-p)z^2]^2}\ . \tag{11b}$$

This correlation function is purely exponential. Again Eq. (11) is not valid at $T=0$. We have calculated $\chi(l)$ at $T=0$ and found that $\chi(l)$ does not have logarithmic corrections even in this limit. Therefore the Fourier transform of this correlation function does not have the Lorentzian-squared term, in perfect agreement[5] with mean-field-theory results for Landau-Ginzburg versions of the RFIM.

Finally, we calculate the Edwards-Anderson order parameter $Q\equiv[\langle\sigma_n\rangle^2]_{\rm av}$. We find $Q\sim p+O(z^2)$ for $z\ll 1$, while for $z=1$ $Q(T=0,p=0)=1$, while $\lim_{p\to 0}Q(T=0,p)=\frac{2}{3}$. Therefore $Q(T=0,p)$ exhibits a discontinuity at $p=0$, indicating the existence of a phase transition at $T=p=0$. This discontinuity is removed if one applies an infinitesimal nonrandom magnetic field.

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