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Nonlinear scaling fields and corrections to scaling near criticality

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Thermodynamic functions in the vicinity of an ordinary critical point are expected to obey asymptotic scaling laws as $t=(T-T_c)/T_c$ and the ordering field, h, approach zero. However, the optimal scaling variables are the *nonlinear scaling fields*, $g_t=t+b_th^2+c_tt^2+\cdots$ and $g_h=h(1+c_ht+\cdots)$. The nonlinearities yield correction factors to the leading power-law (and scaling) variation of thermodynamic quantities, L, of the form $(1+a_Lt+b_Lt^2+\cdots)$, etc., where the correction amplitudes $a_L, b_L, \ldots,$ are uniquely determined by the nonlinear scaling-field coefficients. It follows that "analytic" corrections to, e.g., the susceptibility, are directly related to those for the free energy and magnetization (in zero field). The term b_th^2 also generates *nonanalytic* contributions such as an additive, energylike term, varying as $|t|^{1-\alpha}$ in the zero-field susceptibility, and factors like $(1+c_L|h|^{2-1/\Delta})$ on the critical isotherm, $t=0$. Irrelevant scaling fields yield further, in general distinct, nonanalytic corrections, and cause shifts in T_c and various amplitudes although "universal" ratios remain constant.

I. INTRODUCTION

As the critical point of a ferromagnet, fluid, etc., is approached, the asymptotic behavior of a quantity $L(T)$, as $T\rightarrow T_c^{\pm}$, can usually be characterized as

$$L(T)=L_0|t|^{\lambda}\left[1+\sum_i a_{L,i}^{\pm}|t|^{\theta_i}\right], \qquad (1.1)$$

where L_0 and $\{a_{L,i}^{\pm}\}$ are constants and $t=(T-T_c)/T_c$. The + (−) refers to $T>T_c$ ($T<T_c$). The expression in parentheses in (1.1) represents the correction to scaling factor to the asymptotic power law with exponent λ. It has been evident for some time[1] that many real experiments, even when of the highest precision,[2] do not attain the truly asymptotic regime. Thus one rather observes an effective exponent[1,2]

$$\lambda_{\rm eff}=\frac{\partial \ln L}{\partial \ln|t|}\approx\lambda+\sum_i a_{L,i}^{\pm}\theta_i|\bar{t}|^{\theta_i}, \qquad (1.2)$$

where $\bar{t}$ is a suitable average of t over the range of measurement.[3] The correction factor also plays an important role in analysis aimed at estimating λ accurately from series expansions for model systems.[4] Accordingly, the nature and origin of the leading correction terms, for quantities such as the free energy F, the spontaneous magnetization M, and the suceptibility-compressibility χ (with exponents $2-\alpha$, β, and $-\gamma$, respectively) are matters of continuing significance to theory and experiment.

A commonly discussed source for corrections to scaling arises, in the renormalization-group viewpoint,[5–9] from the leading irrelevant variables. Recently we have emphasized[10] that additional analytic corrections arise from the nonlinearity of the scaling fields. In Ref. 10 we discussed the leading correction for the planar Ising model and showed that it arises only from this latter source. The aim of the present paper is to give a general discussion of the corrections due to the nonlinearity of the scaling fields. In particular, we calculate in detail higher correction terms and discuss relations among them.

In Sec. II we present a simple general scaling analysis in which we ignore all the irrelevant variables and concentrate on the case of pure power laws, without logarithmic corrections. The theory with a logarithmic specific heat, relevant, e.g., to the planar Ising model, is discussed in Sec. III. Irrelevant variables are then introduced in Sec. IV, and conclusions are drawn in Sec. V.

II. SIMPLE POWER-LAW SCALING

Ignoring irrelevant variables and possible logarithmic corrections, the singular part of the free energy has the scaling form[5–8]

$$F_s(T,h)=|g_t|^{2-\alpha}Y_{\pm}(g_h/|g_t|^{\Delta}), \qquad (2.1)$$

where $\Delta=\beta+\gamma$. The nonlinear scaling fields g_t and

g_h represent, in leading order, the temperature T and the ordering field h at an ordinary critical point. In the absence of irrelevant fields, g_t and g_h are analytic functions of t and h, and we may expand them as

$$g_t = t + b_t h^2 + c_t t^2 + d_t t^3 + e_t t h^2 + f_t h^4 + O(t^4, t^2 h^2) , \quad (2.2)$$

$$g_h = h[1 + c_h t + d_h t^2 + e_h h^2 + O(t^3, t h^2)] , \quad (2.3)$$

where we have assumed symmetry in h so that odd powers of h do not appear in the correction factors. Many of the results stressed in the present paper arise directly from the second term $b_t h^2$ in (2.2). We shall show that its presence is responsible for various new, nonanalytic, correction terms which might not, as first sight, be expected.

In the renormalization-group theory, g_t and g_h are variables for which the recursion relations become exactly linear[7]; thus if b is the rescaling factor and a prime denotes renormalization, one has

$$g_t' = b^{\lambda_t} g_t, \quad g_h' = b^{\lambda_h} g_h , \quad (2.4)$$

which leads to the pure power-law scaling form (2.1).

The functions $Y_+(y)$ and $Y_-(y)$ describe the scaling properties of F_s for $g_t > 0$ and $g_t < 0$, respectively. It should be noted here that the free energy should be analytic in g_t when the ordering field is nonzero, or $g_h \neq 0$. In the limit $y^{-1} = |g_t|^{\Delta}/g_h \to 0$, the two components of the scaling function, $Y_+(y)$ and $Y_-(y)$, must join smoothly to form a function of $x = g_t / |g_h|^{1/\Delta}$, analytic for small arguments. We shall return to this point later.

The various thermodynamic quantities now follow as derivatives of F_s with respect to h or T. For example, the order parameter is

$$M \propto \frac{\partial F}{\partial h} = Y'_{\pm} \left[\frac{\partial g_h}{\partial h} \right] |g_t|^{2-\alpha-\Delta} \mp \Delta g_h Y'_{\pm} \left[\frac{\partial g_t}{\partial h} \right] |g_t|^{1-\alpha-\Delta} \pm (2-\alpha) Y_{\pm} \left[\frac{\partial g_t}{\partial h} \right] |g_t|^{1-\alpha} , \quad (2.5)$$

where the prime here indicates a derivative of the function with respect to its argument. Similarly, if, for brevity, we write $(\partial g_h/\partial h) \equiv g_{h;h}$, $(\partial^2 g_t/\partial h^2) \equiv g_{t;h,h}$, etc., the susceptibility is

$$\chi \propto \frac{\partial M}{\partial h} = Y''_{\pm}(g_{h;h})^2 |g_t|^{2-\alpha-2\Delta} + Y'_{\pm} g_{h;h,h} |g_t|^{2-\alpha-\Delta} + (2-\alpha)(1-\alpha) Y_{\pm} (g_{t;h})^2 |g_t|^{-\alpha}$$
$$\pm (2-\alpha) Y_{\pm} g_{t;h,h} |g_t|^{1-\alpha} \pm 2(2-\alpha-\Delta) Y'_{\pm} g_{t;h} g_{h;h} |g_t|^{1-\alpha-\Delta}$$
$$+ \Delta^2 Y''_{\pm} (g_{t,h})^2 g_h |g_t|^{-\alpha-2\Delta} \mp Y''_{\pm} g_{t;h} g_{h;h} g_h |g_t|^{1-\alpha-2\Delta}$$
$$\mp \Delta Y'_{\pm} g_{t;h,h} g_h |g_t|^{1-\alpha-\Delta} - \Delta(3-2\alpha-\Delta) Y'_{\pm} (g_{t;h})^2 g_h |g_t|^{-\alpha-\Delta} . \quad (2.6)$$

Substituting (2.2) and (2.3), and expanding in powers of t and h, leads to explicit expressions for the correction terms. At $h=0$ we find

$$F = A_F^{\pm} |t|^{2-\alpha} (1 + a_F t + b_F t^2 + \cdots) + \mathscr{A}_F(t) , \quad (2.7)$$

$$M_0 = B |t|^{\beta} (1 + a_M t + b_M t^2 + \cdots) \quad (t<0) , \quad (2.8)$$

$$\chi = C^{\pm} |t|^{-\gamma} (1 + a_\chi t + b_\chi t^2 + \cdots) + D^{\pm} |t|^{1-\alpha} (1 + a_{\chi\mathscr{E}} t + \cdots) + \mathscr{A}_\chi(t) , \quad (2.9)$$

where $A_F^{\pm} = Y_{\pm}(0)$, $B \propto Y'_-(0)$, and $C^{\pm} \propto Y''_{\pm}(0)$, while $\mathscr{A}_F(t)$ and $\mathscr{A}_\chi(t)$ stand for terms analytic for small t. The correction amplitudes are given by

$$a_F = (2-\alpha) c_t , \quad (2.10)$$

$$b_F = (2-\alpha) d_t + \tfrac{1}{2}(2-\alpha)(1-\alpha) c_t^2 , \quad (2.11)$$

$$a_M = \beta c_t + c_h , \quad (2.12)$$

$$b_M = \beta d_t + \beta c_t c_h + \tfrac{1}{2}\beta(\beta-1) c_t^2 + d_h , \quad (2.13)$$

$$a_\chi = -\gamma c_t + 2 c_h , \quad (2.14)$$

$$b_\chi = -\gamma d_t - 2\gamma c_t c_h + \tfrac{1}{2}\gamma(\gamma+1) c_t^2 + 2 d_h + c_h^2 . \quad (2.15)$$

Finally, the additive correction terms in the susceptibility are fixed by

$$D^{\pm} = \pm 2(2-\alpha) b_t A_F^{\pm} , \quad (2.16)$$

and

$$a_{\chi\mathscr{E}} = (e_t/b_t) + (1-\alpha) c_t . \quad (2.17)$$

The lowest-order relations (2.10), (2.12), and (2.14) were presented in Ref. 10. It is interesting to note that the term $b_t h^2$ in the temperaturelike nonlinear scaling field generates an energylike term, varying as $|t|^{1-\alpha}$ in the susceptibility. Although such a term is expected as a result of short-range correlations,[11] its amplitude has not previously been related to the coefficient b_t.

We emphasized in Ref. 10 the relation

$$a_F - 2a_M + a_\chi = 0 , \qquad (2.18)$$

which follows directly from (2.10), (2.12), and (2.14) by eliminating c_t and c_h. Thus only two of the amplitudes a_F, a_M, and a_χ should be independent. Similarly, we now find that

$$b_F - 2b_M + b_\chi = (a_F - a_M)^2 = [(\gamma+\beta)c_t - c_h]^2 = b_2 . \qquad (2.19)$$

Similar relations can be found for higher-order terms. Knowledge of the analytic correction terms in any two of F, M_0, or χ thus determines those for the third one to all orders. Similar statements clearly apply to the energy and specific heat.

Using the definition (1.2), we can now see that

$$\alpha'_{\text{eff}} + 2\beta_{\text{eff}} + \gamma'_{\text{eff}} = 2 + b_2 |\bar{t}|^2 , \qquad (2.20)$$

where b_2 is defined in (2.19). For relatively small $\bar{t}$, in the absence of irrelevant variables, the effective exponents thus obey scaling to a good approximation.

We now turn to the critical isotherm $T = T_c$ or $t=0$. In the limit $|g_t|^\Delta / g_h \to 0$ we may rewrite (2.1) in the form

$$F_s(T,h) = |g_h|^{(2-\alpha)/\Delta} Y_0(g_t / |g_h|^{1/\Delta}) , \qquad (2.21)$$

where $Y_0(x)$ is analytic at $x \to 0$, and can be directly related to Y_+ and Y_-. For $t \to 0$ we have

$$g_t = b_t h^2 + f_t h^4 + O(h^6) \qquad (2.22)$$

and

$$g_h = h[1 + e_h h^2 + O(h^4)] , \qquad (2.23)$$

which yields

$$F_s(T_c,h) = |h|^{(2-\alpha)/\Delta} \left[1 + \frac{2-\alpha}{\Delta} e_h h^2 + \cdots \right] [Y_0(0) + Y_0'(0) b_t |h|^{2-1/\Delta} + \cdots] , \qquad (2.24)$$

where we have expanded $Y_0(x)$ for small x.

The magnetization and the susceptibility can now be found by taking derivatives with respect to h. We thus see that the "analytic" nonlinear scaling fields generate nonanalytic correction to F_s, M, etc. For example, we have

$$|M| = B_c |h|^{1/\delta} (1 + x_M |h|^{2-1/\Delta} + y_M |h|^2 + \cdots) , \qquad (2.25)$$

where $1/\delta = -1 + (2-\alpha)/\Delta$, B_c is related to $Y_0(0)$, the correction amplitudes are given by

$$x_M = [(1-\alpha+2\Delta)/(2-\alpha)][Y_0'(0)/Y_0(0)] b_t , \qquad (2.26)$$

$$y_M = (2-\alpha+2\Delta) e_h / \Delta , \qquad (2.27)$$

and an odd analytic function of h has been dropped. Similar expressions apply to F_s and to χ. Note that b_t determines amplitudes of both the "nonanalytic" corrections at $t=0$ and the energylike term in the susceptibility, through (2.16).

The specific heat at $t=0$ can now be found by differentiating (2.21) with respect to t and then setting $t=0$. The result contains, in addition to the leading power $|h|^{-\alpha/\Delta}$, terms like $|h|^0$, $|h|^{(1-\alpha)/\Delta}$, $|h|^{(2-\alpha)/\Delta}$, and $|h|^{2-\alpha/\Delta}$. Thus, in addition to $|h|^2$ in the correction factor, one should encounter new nonanalytic correction terms of order $|h|^{1/\Delta}$, $|h|^{2/\Delta}$, etc.; these can be traced directly to the analytic dependence of F on t for finite h. One may, in analogy to (2.20), derive effective exponent relations for exponents defined at T_c.

Thus far, we have concentrated only on the special axes $h=0$ and $t=0$. In fact, the "analytic" corrections are meaningful for the whole equation of state. Usually, the scaled equation of state is written so that $F_s/|t|^{2-\alpha}$ (or $M/|t|^\beta$) is, asymptotically, a function of the single variable $h/|t|^\Delta$, for $M,t,h \to 0$. However, such a representation may be misleading when $b_t \neq 0$. Although g_h may always be replaced by h in the limit $t,h \to 0$, the value of g_t depends in a significant way on the manner in which the critical point $t=h=1$ is approached in the (t,h) plane.[12] It is only in the limit $|b_t h^2| \ll |t| \ll 1$ that one can recover the "usual" scaling dependence of $F_s/|t|^{2-\alpha}$ on $h/|t|^\Delta$. Thus expanding (2.1) in powers of h^2/t we find

$$F_s(T,h) \approx |t|^{2-\alpha} Y_\pm(h/|t|^\Delta) \times [1 \pm b_t |t|^{2\Delta-1} y_\pm(h/|t|^\Delta) + t\tilde{y}_\pm(h/|t|^\Delta)] , \qquad (2.28)$$

for $|b_t h^2| \ll |t| \ll 1$, where

$$y_\pm(x) = [2-\alpha-\Delta Y'_\pm(x)/Y_\pm(x)]x^3\ , \quad (2.29)$$

$$\tilde{y}_\pm(x) = (2-\alpha)c_t + (c_h - \Delta c_t)xY'_\pm(x)/Y_\pm(x)\ . \quad (2.30)$$

Usually one has $\Delta > 1$, so that the new nonanalytic correction term $|t|^{2\Delta-1}$ may be neglected in comparison with the leading "analytic" term t. Note that all the correction terms are of the scaling form $|t|^{\theta_i} y_i(h/|t|^\Delta)$; however, they are limited to the range $|b_t h^2| \ll |t|$. If $|b_t h^2| \gg |t|$, one must switch back to the original form (2.21). It is then easy to check that $F_s(T,h)$ can be expanded in powers of t, with coefficients whose dependence on h is similar to (2.24).

III. SCALING WITH A LOGARITHMIC SPECIFIC-HEAT SINGULARITY

The specific heat of planar Ising models displays a logarithmic specific-heat singularity.[13] Consequently, the free energy in zero field, although formally described by $\alpha=0$, contains a term of the form $|t|^2 \ln|t|^{-1}$, which does not follow directly from (2.1). [The way in which such a term appears "naturally," as a continuous parameter, say the dimensionality d, is varied, involves the "analytic background" in a crucial way (see, e.g., Ref. 14).] In general, the scaling form (2.1) must then be replaced by[15]

$$F_s(T,h) \approx g_t^2(\ln|g_t|^{-1})\tilde{Y}_\pm(g_h/|g_t|^\Delta) + g_t^2 Y_\pm(g_h/|g_t|^\Delta)\ , \quad (3.1)$$

when $\alpha=0$. The two scaling functions $\tilde{Y}_\pm(y)$ and $Y_\pm(y)$ must match appropriately when $y = g_h/|g_t|^\Delta \to \infty$, so that, in the limit $g_t \to 0$, the $\ln|g_t|^{-1}$ term cancels, to be replaced by a $\ln|g_h|$ term.[15] (Note that this cancellation cannot occur if, for example, $\ln|g_t|^{-1}$ is replaced by $\ln|t|^{-1}$, even though this would still reproduce the desired zero-field results.)

For the planar Ising models we also know[13,16] that $M_0/|t|^\beta$ and $\chi|t|^\gamma$, with $\beta=\frac{1}{8}$ and $\gamma=1\frac{3}{4}$, approach finite, nonzero limits B and $C^\pm$, as $t\to 0^\pm$ for $h=0$. In other words, M_0 and χ have, in zero field, *no leading logarithmic factors.* It follows from this that the first two derivatives of $\tilde{Y}_\pm(y)$ must vanish at $y=0$. In fact, we suspect strongly that a similar result will hold for *all* derivatives of $\tilde{Y}_\pm(y)$, i.e., no leading factor $\ln|t|^{-1}$ appears in *any* field derivatives of the free energy of the planar Ising model in the limit $h\to 0$. This belief is based on the known structure of the correlation functions and the fact that the true inverse correlation length $\xi^{-1} \equiv \kappa(T)$ is an analytic function of t so that $\nu=1$ and there are no logarithmic terms. If we accept this "minimal logarithms" conclusion generally (which will certainly suffice for our present discussion, which will not go beyond M and χ), we may replace $\tilde{Y}_\pm(y)$ in (3.1) by a *constant amplitude*

$$A_F \equiv \tilde{Y}_\pm(0) \quad (3.2)$$

(see also Ref. 7). Note that the requirements of analyticity in t for $h\neq 0$ force the equality of the scaling amplitudes for $T \gtrless T_c$, i.e., $\tilde{Y}_+(0) = \tilde{Y}_-(0)$.

Given the scaling form we can now follow the procedures of Sec. II and examine the interesting thermodynamic functions. When $h\to 0$ the second term in (3.1) becomes analytic in t and we obtain

$$F_s(T,0) \approx A_F t^2(\ln|t|^{-1}) \times (1 + a_F t + b_F t^2 + \cdots) + \mathscr{A}_F(t)\ , \quad (3.3)$$

where a_F and b_F are given by (2.10) and (2.11) with $\alpha=0$. The singular terms in the magnetization are likewise found to be

$$M \propto \frac{\partial F}{\partial h} \approx 2A_F\left[\frac{\partial g_t}{\partial h}\right] g_t(\ln|g_t|^{-1}) + 2Y_\pm\left[\frac{\partial g_t}{\partial h}\right] g_t \mp \Delta Y'_\pm\left[\frac{\partial g_t}{\partial h}\right] g_h|g_t|^{1-\Delta} + Y'_\pm\left[\frac{\partial g_h}{\partial h}\right]|g_t|^{2-\Delta}\ . \quad (3.4)$$

Only the last term survives when $h\to 0$, and we recover the asymptotic form (2.8) with correction amplitudes given by (2.12) and (2.13) (with, for planar Ising models, $\beta = 2-\Delta = \frac{1}{8}$). The behavior of the susceptibility follows similarly, but (2.9) is replaced by

$$\chi = C^\pm|t|^{-\gamma}(1 + a_\chi t + b_\chi t^2 + \cdots) + D_0|t|(\ln|t|^{-1})(1 + a_{\chi\mathscr{g}} t + \cdots) + \mathscr{A}_\chi(t)\ , \quad (3.5)$$

with

$$D_0 = 4b_t A_F\ , \quad (3.6)$$

although the expressions (2.14)–(2.17) for the correction amplitudes remain valid with $\alpha=0$ (and, for planar Ising models, with $\gamma=1\frac{3}{4}$).

Consider now the critical isotherm $t\to 0$. Since, as mentioned, $F_s(T,h)$ must be analytic in T through T_c for nonzero. h, the scaling function must behave as

$$Y_{\pm}(y) \approx Y_0^c y^{2/\Delta} \pm Y_1^c y^{1/\Delta} + Y_l^c \ln y + Y_2^c \pm Y_3^c y^{-1/\Delta} + Y_4^c y^{-2/\Delta} \pm \cdots , \quad (3.7)$$

where one must have

$$Y_l^c = -A_F/\Delta \quad (3.8)$$

in order that the logarithmic terms in g_t cancel identically. With the use of this result one finds

$$|M| = B_c |h|^{1/\delta}(1 + x_M |h|^{2-1/\Delta} + y_M h^2 + \cdots) + E_1 h + E_l h^3 \ln |h|^{-1} + \cdots , \quad (3.9)$$

where x_M and y_M are given by (2.26) and (2.27) with $\alpha = 0$, while

$$E_l = 4A_F b_t^2/\Delta . \quad (3.10)$$

Note that a similar, energy-related term varying as $h^{3-\alpha/\Delta}$ arises in (2.25) but was not displayed since, as is clear here, it is comparatively of very high order.

The specific heat on the critical isotherm follows similarly as

$$C_H \propto \frac{\partial^2 F}{\partial t^2} = 2(A_F/\Delta)\ln |h|^{-1} + A_0^c + A_1^c |h|^{1/\Delta} + A_2^c |h|^{2-1/\Delta} + \cdots , \quad (3.11)$$

where

$$A_1^c/B_c = (\beta + \gamma)c_t + c_h = \tfrac{1}{2}\gamma a_F + a_M , \quad (3.12)$$

while higher-order terms proportional to $h^2 \ln |h|^{-1}$, to h^2, etc. also appear.

IV. ROLE OF IRRELEVANT VARIABLES

Thus far we assumed that the ordinary critical point is fully described by the two relevant fields t and h. Renormalization-group theory[5–8] predicts that, in general, there will exist many irrelevant fields, whose effects become negligible very close to the critical point as $t, h \to 0$. Consider one such field, say the least irrelevant field u. If the associated nonlinear scaling field is denoted by g_u, then (2.1) must be extended to[7]

$$F_s(T,h,u) = |g_t|^{2-\alpha} Y_{\pm}(g_h/|g_t|^{\Delta}, g_u |g_t|^{\theta_u}) , \quad (4.1)$$

where the leading irrelevant variable exponent is $\theta_u > 0$.

If u represents a perturbation, even with respect to the order parameter, as found in the ϵ expression,[6,8] one has

$$g_u = u + O(t,h^2) . \quad (4.2)$$

In addition, u must also enter into the nonlinear scaling fields g_t and g_h, in the form

$$g_t = t + a_t u + O(h^2,t^2,u^2,ut,uh^2) , \quad (4.3)$$

$$g_h = h[1 + a_h u + c_h t + O(h^2,t^2,u^2,ut,uh^2)] . \quad (4.4)$$

The term $a_t u$ in (4.3), and the higher-order terms of order u^2, u^3, etc., imply a shift in T_c relative to a system with $u = 0$. If T_c is identified as the value of T for which t vanishes [in the leading singular term of Eq. (1.1)], then we may absorb these terms in a redefined (shifted) variable t. Similarly, the terms of order ut, $u^2 t$, etc. represent a (multiplicative) change in scale, so t becomes $A(u)t$. Such a change will imply that amplitudes like L_0 in (1.1) are nonuniversal, containing factors like $[A(u)]^{\lambda}$. All such factors, however, should disappear from properly constructed amplitude ratios, which will then be universal.[17] Finally, terms like ut^2, $u^2 t^2$, etc. may be absorbed in a redefinition of c_t, while uth^2 serves to redefine e_t, and so on. Similarly, the term $a_h u$ in (4.4) represents a rescaling of the variable h, and further u-dependent terms represent modifications to the coefficients c_h, d_h, etc. in (2.3). Although one must accept the fact that all the amplitudes we have discussed in the previous sections are u dependent, the relations among them, e.g., (2.10)–(2.17) and (2.23)–(2.24), nonetheless remain true.

We are thus left with the explicit dependence of F_s on g_u, as exhibited in (4.1). Assuming that $Y_{\pm}(y,z)$ is analytic in z when $z \to 0$,[18] we now find[5,7]

$$F_s(T,h,u) = |g_t|^{2-\alpha} Y_{\pm}(g_h/|g_t|^{\Delta}) \times [1 + g_u |g_t|^{\theta_u} f_u^{\pm}(g_h/|g_t|^{\Delta}) + \cdots] , \quad (4.5)$$

where

$$f_u^{\pm}(y) = \frac{\partial}{\partial z} \ln \tilde{Y}_{\pm}(y,z)|_{z=0} .$$

Usually, the exponent θ_u is nonintegral. The resulting nonanalytic corrections, of order $|t|^{\theta_u}$, can then, in principle be distinguished from the "analytic" corrections discussed previously. Note that the terms of order t and h^2 in (4.2) for g_u, yield corrections of order $|t|^{\theta_u+1}$ or $h^2|t|^{\theta_u}$, which will always be smaller than $|t|^{\theta_u}$ and thus represent higher-order terms. Finally, Eqs. (2.7) and (2.8) now become

$$F = A_F^{\pm} |t|^{2-\alpha}(1 + a_F t + a_F^u |t|^{\theta_u} + b_F t^2 + b_F^u |t|^{2\theta_u} + c_F^u |t|^{\theta_u+1} + \cdots) + \mathscr{A}_F(t) , \quad (4.6)$$

$$M_0=B\,|t\,|^{\beta}(1+a_M t+a_M^u\,|t\,|^{\theta_u}+b_M t^2+b_M^u\,|t\,|^{2\theta_u}+c_M^u\,|t\,|^{\theta_u+1}+\cdots)\,. \tag{4.7}$$

It has recently been shown[19] that the dependence on u cancels in ratios like a_F^u/a_M^u or b_F^u/b_M^u, which are thus also universal.

Generally, one must analyze data including all the terms in (4.6) and (4.7), and possible additional significant terms due to further irrelevant fields. The situation is likely to be complicated if some θ_u, or an integral multiple of θ_u, happens to be an integer. For example, if one has an irrelevant variable u with $\theta_u=1$, one would in general be unable to distinguish between the "analytic" term $a_F t$ and the "nonanalytic" one $a_F^u\,|t\,|$. It is therefore important to note here their different relations: The "analytic" coefficients a_F, a_M, and a_χ always obey (2.18), whereas the "nonanalytic" coefficients a_F^u, a_M^u, and a_χ^u usually do not.[20] This may enable one to distinguish two distinct origins for such corrections. Nonetheless, one should also note the likelihood of corrections like $t^2\ln|t\,|$ appearing when a multiple of θ_u is an integer (e.g., $3\theta_u=2$ for this example).[5,7]

In the planar Ising model we have shown[10] that the coefficients of the linear correction terms always obey (2.18), even for general lattice anisotropy $\kappa=J_x/J_y$. If $\theta_u=1$ (and it cannot be smaller), this means that

$$a_F^u-2a_M^u+a_\chi^u\equiv 0\,, \tag{4.8}$$

which imposes a strong restriction on the scaling function $Y_\pm(y,z)$. It thus seems more reasonable to conclude that the planar Ising models do not have any irrelevant fields with $\theta_u=1$.[10] (Note, even so, that the lattice anisotropy κ represents a marginal variable with $\theta_\kappa=0$, which changes critical amplitudes but not exponents or most "universal" amplitude ratios; it does, however, change those otherwise "universal" ratios that measure the spatial isotropy of the critical-point decay of correlations.[13])

V. CONCLUSIONS

We have shown that the nonlinear scaling fields are directly responsible for the leading corrections to scaling in the case of planar Ising models. It is thus crucial to include such "analytic" corrections in the analysis of two-dimensional experimental and numerical data. In most three-dimensional systems θ_u is approximately 0.5,[4,7] so that the "analytic" corrections will usually be dominated by those due to the leading irrelevant fields. However, in some cases (e.g., the spin-$\frac{1}{2}$ Ising model) u appears to be relatively small,[4] so that $a_F^u\,|t\,|^{\theta_u}$ may be small compared to $a_F t$. In such cases, and in all circumstances where the precision allows the determination of more than one correction term,[3] one must include the "analytic" terms.

It is also worth emphasizing again the various nonanalytic terms, which arise from the term $b_t h^2$ in the nonlinear thermal field g_t. Even though it is relatively small, it would certainly be of interest to try to detect the energylike term $|t\,|^{1-\alpha}$ in the susceptibility χ [see (2.4)]. The nonanalytic corrections on the critical isotherm [see, e.g., (2.25)] are also of interest.

We hope that this note will prove useful as a basis in future analyses of precise experimental data near critical points. Further considerations, however, are clearly needed in practice[3] and it is worth mentioning that a *parametric representation*[21] of our results would also be of practical utility in examining the full equation of state.

Finally, we emphasize again that all our results were based on the assumption of symmetry in h, so that g_t (g_h) contains only even (odd) powers of h. In many cases this symmetry is absent, and both g_t and g_h will contain linear terms in t and h. As near a bicritical point, or near a liquid-gas transition, one should then be careful in choosing the correct scaling axes.[12] Inappropriate choices will yield new nonanalytic correction terms. A new example of such a situation concerns the percolation problem,[22] in which there exists no evident symmetry in the "ghost" field (h), and therefore the scaling fields may well mix the concentration p and h already at linear order.

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