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Nonanalytic supercurrents in ^{3}He-A

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The energy gap of the A phase of superfluid ^{3}He has two singular points on the Fermi surface. These singular points lead to various theoretical puzzles, including one recently discovered by Volovik and Mineev. These authors, using a generalized—gauge-transformation approximation, find a term in the $T=0$ supercurrent which is nonanalytic in gradients of the order parameter. We use the quasiclassical equations invented by Eilenberger to check the validity of their approximation; we find that the generalized—gauge-transformation approximation fails, in that terms it leaves out make important (in fact, divergent) contributions to the nonanalytic supercurrent. We then formulate a new way to calculate the leading nonanalytic terms in the supercurrent.

I. INTRODUCTION

The order parameter of the A phase of superfluid ^{3}He has a structure which has led to several related theoretical puzzles. These include:

(1) A coefficient in the gradient free-energy density which diverges at low temperature as $\ln T$.[1]

(2) A strange term in the supercurrent, the significance of which eludes understanding.[1,2]

(3) A term in the zero-temperature supercurrent which is nonanalytic in gradients of the order parameter.[3]

These puzzles all originate from the fact that as a function of $\hat{k}$ (the direction in momentum space on the Fermi surface) the A-phase energy gap has two singular points. The gap vanishes at these two points, and its phase changes by 2π upon circling one of them. It is important to realize that these singularities are topologically stable[4]; they cannot be removed by a small deformation of the gap's functional form.

In this paper we discuss puzzle (3). Volovik and Mineev[3] (VM) have discovered a term in the zero-temperature A-phase supercurrent which is nonanalytic in gradients of the order parameter. To do their calculation, they rely on a generalized—gauge-transformation approximation. We use the method of quasiclassical Green's functions to assess the validity of their calculation, and find their approximation leaves out important effects. We then calculate the current in a different way, in an effort to avoid the pitfalls of the gauge-transformation scheme. Our conclusion is that the $T=0$ supercurrent does indeed contain nonanalytic terms; one of the VM type appears, but with a different numerical coefficient.

The leading terms in the supercurrent are analytic, and are first order in gradients. The nonanalytic terms are second order in gradients and are smaller than the leading terms by a factor of $(\xi_0 \vec{\nabla}_R)$, where ξ_0 is the zero-temperature coherence length. Hence, the nonanalytic terms are primarily of theoretical interest; the breakdown of the power-series expansion in gradients at so early a stage means that the usual hydrodynamic method for deducing the current fails if pushed beyond the leading terms.

Nonanalytic terms in the gradient free energy may well be more experimentally interesting, since, as mentioned in puzzle (1), a leading term in the gradient expansion has a diverging coefficient. We hope to study this question in the future.

The plan of this paper is as follows. In Sec. II we review the VM calculation. In Sec. III we use the quasiclassical method to check the approximation involved in their calculation; we do this by considering the terms not included at their level of approximation. An infinite set of new terms turns out to contribute to the leading nonanalyticities. In order to sum up all leading terms, a different way of organizing the calculation is needed. In Sec. IV we explain a new way of extracting the leading nonanalytic terms in the supercurrent. Section V contains a summary and discussion. For the sake of comparison, in the Appendix we discuss nonanalyticities which can arise in the polar phase at zero temperature.

II. VOLOVIK-MINEEV CALCULATION

In this section we review the VM calculation.[3] The A-phase gap is of the form

$$\Delta_{\alpha\beta}(\vec{R},\hat{k})=i\vec{d}\cdot\vec{\sigma}_{\alpha\mu}\sigma^{y}_{\mu\beta}\Delta(\vec{R},\hat{k}) , \tag{1}$$

$$\Delta(\vec{R},\hat{k})=\Delta_0(\hat{m}+i\hat{n})\cdot\hat{k} , \tag{2}$$

where $\vec{d}$ is a real unit vector and $\hat{m}$, $\hat{n}$, and $\hat{l}=\hat{m}\times\hat{n}$ form an $\vec{R}$-dependent orthonormal triad. We will always keep the spin part of the gap fixed, so that $\vec{d}$ is independent of $\vec{R}$.

A crucial concept, which we now explain, is the $\hat{k}$-dependent phase of the gap.[3,5] We write $\Delta(\vec{R},\hat{k})$ in the following form:

$$\Delta(\vec{R},\hat{k})=D(\vec{R},\hat{k})e^{i\phi(\vec{R},\hat{k})} \tag{3}$$

with

$$D=\Delta_0[(\hat{m}\cdot\hat{k})^2+(\hat{n}\cdot\hat{k})^2]^{1/2} \tag{4}$$

and

$$\phi=\arctan\left[\frac{\hat{n}\cdot\hat{k}}{\hat{m}\cdot\hat{k}}\right] . \tag{5}$$

We also introduce the quantity $\vec{w}(\vec{R},\hat{k})$, defined by

$$\vec{w}(\vec{R},\hat{k})=\frac{\hbar}{2M}\vec{\nabla}_R\phi . \tag{6}$$

Thus $\phi(\vec{R},\hat{k})$ is the phase of the gap at each point on the Fermi surface $\hat{k}$ and at each point in space $\vec{R}$; we can think of $\vec{w}$ as a generalized, $\hat{k}$-dependent superfluid velocity. $\vec{w}$ can be written as the sum of two terms,

$$\vec{w}(\vec{R},\hat{k})=\vec{v}_s(\vec{R})+\vec{u}(\vec{R},\hat{k}) , \tag{7}$$

where $\vec{v}_s=(\hbar/2M)\hat{m}\vec{\nabla}\hat{n}$ is the usual superfluid velocity, and $\vec{u}$ is given by

$$u_j(\vec{R},\hat{k})=\frac{\hbar}{2M}\frac{(\hat{k}\cdot\hat{l})[(\hat{k}\times\hat{l})\cdot\vec{\nabla}_j\hat{l}]}{(\hat{k}\times\hat{l})^2} . \tag{8}$$

One should note that $\vec{u}$ diverges as $\hat{k}\to\pm\hat{l}$. The points $\hat{k}=\pm\hat{l}$ are the two singular points referred to in the Introduction; the divergence of $\vec{u}$ reflects the fact that the phase of the gap changes by 2π if these points are circled.

The phase $\phi(\vec{R},\hat{k})$ of the order parameter can now be eliminated by a gauge transformation. An approximate solution to the gauge-transformed Gorkov's equations yields the following expression for the supercurrent:

$$\vec{J}=\frac{2N(0)}{M}p_F^2\int\frac{d\hat{k}}{4\pi}\hat{k}(\hat{k}\cdot\vec{w}) + \frac{2N(0)}{M}p_F\int\frac{d\hat{k}}{4\pi}\int d\epsilon F(E+p_F\hat{k}\cdot\vec{w})\hat{k} . \tag{9}$$

Here $F(x)$ is the Fermi function, $E=(\epsilon^2+|\Delta|^2)^{1/2}$, and $N(0)$ is the density of states at the Fermi surface for one spin population. The first term can be viewed as the condensate current, and the second as the current of excitations with the normal fluid velocity $\vec{v}_n$ equal to zero.

Note that the argument of the Fermi function in (9) contains a term $p_F\hat{k}\cdot\vec{u}$. The quantity $\hat{k}\cdot\vec{u}$ diverges as $\hat{k}\to\pm\hat{l}$, as long as $(\hat{l}\cdot\vec{\nabla})\hat{l}$ is nonzero. Thus it is not surprising that the nonanalytic term in $\vec{J}$ will turn out to be proportional to $|\hat{l}\cdot\vec{\nabla}\vec{l}|$. Moreover, the term in the gradient free energy which has a divergent coefficient, as mentioned in puzzle (1), is precisely $(\hat{l}\cdot\vec{\nabla}\hat{l})^2$.

It is simple to check that (9), when evaluated to first order in spatial gradients, agrees with the result for $\vec{J}$ obtained by Cross.[1] VM use (9) to evaluate $\vec{J}$ beyond first order. To do this, they expand the Fermi function to first order in $\vec{v}_s$, but keep all powers of $\vec{u}$,

$$F(E+p_F\hat{k}\cdot\vec{w})\simeq F(E+p_F\hat{k}\cdot\vec{u}) + F'(E+p_F\hat{k}\cdot\vec{u})p_F\hat{k}\cdot\vec{v}_s . \tag{10}$$

At zero temperature we have

$$F'(E+p_F\hat{k}\cdot\vec{u})=-\delta(E+p_F\hat{k}\cdot\vec{u}) . \tag{11}$$

Thus at $T=0$ the term in $\vec{J}$ arising from the term in (10) linear in $\vec{v}_s$ is

$$\vec{J}=\frac{2N(0)}{M}p_F^2\int\frac{d\hat{k}}{4\pi}\int d\epsilon\,\hat{k}(\hat{k}\cdot\vec{v}_s) \times\delta(E+p_F\hat{k}\cdot\vec{u}) , \tag{12}$$

which leads to a term in $\vec{J}$ of[6]

$$\vec{J}=-\frac{3}{4}\rho\frac{\hbar v_F}{\Delta_0}\hat{l}(\hat{l}\cdot\vec{v}_s)\,|\hat{l}\cdot\vec{\nabla}\hat{l}| . \tag{13}$$

In the next section we will show how the expression (9) for $\vec{J}$ emerges from an approximate solution to the gauge-transformed quasiclassical equation. In this framework it is possible to examine the validity of the approximations leading to (9). We will see that corrections to (9) are indeed important, and then suggest an alternative way to calculate the complete leading nonanalyticities.

III. QUASICLASSICAL ANALYSIS

We now check the VM approximation using the quasiclassical theory. For expositions of this theory, see Refs. 7 and 8. Since we are keeping $\vec{d}$ fixed we can calculate the quasiclassical propagator $\hat{g}$ for one spin population and then multiply by 2 to allow for the other. Hence we can regard $\hat{g}$ as a 2×2 matrix in particle-hole space; $\hat{g}$ is a function of position $\vec{R}$, momentum direction $\hat{k}$, and Matsubara frequency ϵ_n. The current is given by

$$\vec{J}(\vec{R})=N(0)v_F T\sum_{\epsilon_n}\int\frac{d\hat{k}}{4\pi}\mathrm{Tr}[\hat{\tau}_3\hat{g}(\hat{k},\vec{R};\epsilon_n)\hat{k}] \ . \tag{14}$$

To compute $\hat{g}$ as a function of the order-parameter matrix $\hat{\Delta}$ we use the equation of motion[9]

$$[i\epsilon_n\hat{\tau}_3-\hat{\Delta},\hat{g}]+iv_F\hat{k}\cdot\vec{\nabla}_R\hat{g}=0 \tag{15}$$

along with the normalization condition

$$\hat{g}\hat{g}=-\pi^2\mathbf{1} \ . \tag{16}$$

The matrix $\hat{\Delta}(\hat{k},\vec{R})$ is given by

$$\hat{\Delta}=\begin{bmatrix} 0 & \Delta(\hat{k},\vec{R}) \\ \Delta^*(-\hat{k},\vec{R}) & 0\end{bmatrix} . \tag{17}$$

To solve (15) via the gauge-transformation approximation, we introduce the gauge-transformed Green's function by

$$\bar{g}\equiv\exp\left[-i\frac{\phi}{2}\hat{\tau}_3\right]\hat{g}\exp\left[+i\frac{\phi}{2}\hat{\tau}_3\right] . \tag{18}$$

Equation (15) then becomes

$$[i\alpha_n\hat{\tau}_3-iD\hat{\tau}_2,\bar{g}]+iv_F\hat{k}\cdot\vec{\nabla}_R\bar{g}=0 \tag{19}$$

with

$$\alpha_n\equiv\epsilon_n+iv_f\hat{k}\cdot\vec{w} \ . \tag{20}$$

Note that both (14) and (16) are still valid for $\bar{g}$.

So far (19) is still exact. To get an approximate solution we proceed by iteration, treating the $\vec{\nabla}_R\bar{g}$ term as a perturbation. The zeroth-order term represents the VM approximation. To get a formal perturbation expansion beyond the VM terms we set

$$\bar{g}=\bar{g}^{(0)}+\bar{g}^{(1)}+\bar{g}^{(2)}+\cdots \ . \tag{21}$$

$\bar{g}^{(0)}$ satisfies

$$[i\alpha_n\hat{\tau}_3-iD\hat{\tau}_2,\bar{g}^{(0)}]=0 \ , \tag{22}$$

$$\bar{g}^{(0)}\bar{g}^{(0)}=-\pi^2\mathbf{1} \ , \tag{23}$$

$\bar{g}^{(1)}$ satisfies

$$[i\alpha_n\hat{\tau}_3-iD\hat{\tau}_2,\bar{g}^{(1)}]+iv_F\hat{k}\cdot\vec{\nabla}_R\bar{g}^{(0)}=0 \ , \tag{24}$$

$$\bar{g}^{(1)}\bar{g}^{(0)}+\bar{g}^{(0)}\bar{g}^{(1)}=0 \ , \tag{25}$$

$\bar{g}^{(2)}$ satisfies

$$[i\alpha_n\hat{\tau}_3-iD\hat{\tau}_2,\bar{g}^{(2)}]+iv_F\hat{k}\cdot\vec{\nabla}_R\bar{g}^{(1)}=0 \ , \tag{26}$$

$$\bar{g}^{(0)}\bar{g}^{(2)}+\bar{g}^{(2)}\bar{g}^{(0)}+\bar{g}^{(1)}\bar{g}^{(1)}=0 \ , \tag{27}$$

etc.

Note that this procedure for solving (19) is not a straightforward gradient expansion; α_n already contains a gradient of the order parameter, and $\bar{g}^{(0)}$ contains all power of α_n. The gauge transformation allows us to fully incorporate the effects of $\vec{w}$ into $\bar{g}^{(0)}$. Effects due to $\vec{\nabla}\vec{w},\vec{\nabla}D,\ldots$ appear in higher-order terms.

Solving the above equation yields

$$\bar{g}^{(0)}=\frac{-\pi(i\alpha_n\hat{\tau}_3-iD\hat{\tau}_2)}{(\alpha_n^2+D^2)^{1/2}} \ , \tag{28}$$

$$\bar{g}^{(1)}=\hat{\tau}_1\frac{\pi}{2}v_F\frac{\hat{k}\cdot(D\vec{\nabla}_R\alpha_n-\alpha_n\vec{\nabla}_R D)}{(\alpha_n^2+D^2)^{3/2}} \ , \tag{29}$$

$$\bar{g}^{(2)}=c_2\hat{\tau}_2+c_3\hat{\tau}_3 \ , \tag{30}$$

$$\begin{aligned}c_2=\frac{\pi}{8i}v_F^2\frac{\hat{k}_i\hat{k}_j}{(\alpha_n^2+D^2)^{7/2}}[&-(6D\alpha_n^2+D^3)\nabla_i\alpha_n\nabla_j\alpha_n+5D\alpha_n^2\nabla_iD\nabla_jD+(6\alpha_n^3-4\alpha_nD^2)\nabla_i\alpha_n\nabla_jD\\&+(2D\alpha_n^3+2D^3\alpha_n)\nabla_i\nabla_j\alpha_n-(2\alpha_n^4+2D^2\alpha_n^2)\nabla_i\nabla_jD] \ ,\end{aligned} \tag{31}$$

$$\begin{aligned}c_3=\frac{\pi}{8i}v_F^2\frac{\hat{k}_i\hat{k}_j}{(\alpha_n^2+D^2)^{7/2}}[&-5D^2\alpha_n\nabla_i\alpha_n\nabla_j\alpha_n+(\alpha_n^3+6D^2\alpha_n)\nabla_iD\nabla_jD+(4D\alpha_n^2-6D^3)\nabla_i\alpha_n\nabla_jD\\&+(2D^2\alpha_n^2+2D^4)\nabla_i\nabla_j\alpha_n-(2D\alpha_n^3+2D^3\alpha_n)\nabla_i\nabla_jD] \ .\end{aligned} \tag{32}$$

It is simple to check that $\bar{g}^{(0)}$, when substituted into (14) leads to the VM expression for $\vec{J}$, Eq. (9). $\bar{g}^{(1)}$ contains no $\hat{\tau}_3$ component, so $\bar{g}^{(2)}$ contains the first correction to the VM current. Before examining $\bar{g}^{(2)}$, we pause to illustrate how to extract nonanalytic pieces from $\bar{g}^{(0)}$ at $T=0$.

We will call $\vec{J}^{(0)}$ the supercurrent arising from $\bar{g}^{(0)}$. Performing the frequency sum at zero temperature yields

$$\vec{J}^{(0)}=2N(0)v_F\int\frac{d\hat{k}}{4\pi}I(\hat{k})\hat{k} \tag{33}$$

with

$$I(\hat{k})\equiv q-q(1-D^2/q^2)^{1/2}\Theta(q^2-D^2)\ . \tag{34}$$

We have defined

$$q\equiv p_F\hat{k}\cdot\vec{w}\ . \tag{35}$$

The first term in $I(\hat{k})$ leads to the well-known, first order in gradients, analytic expression

$$\vec{J}_A^{(0)}=\rho\vec{v}_s+\frac{\hbar}{4M}\rho\vec{\nabla}\times\hat{l}-\frac{\hbar}{2M}\rho\hat{l}(\hat{l}\cdot\vec{\nabla}\times\hat{l})\ . \tag{36}$$

The second term in $I(\hat{k})$ produces the nonanalytic behavior.

To examine the consequences of this second term, we consider the following special case: At a given point $\vec{R}$ we have $\partial_i\hat{l}_j=A\hat{z}_i\hat{x}_j,\hat{l}=\hat{z}$, and a given small $\vec{v}_s$. Note that A has dimensions of (length)$^{-1}$, but can have either sign. We use spherical coordinates (θ,ϕ) to do the $\hat{k}$ integral, so that $D^2=\Delta_0^2\sin^2\theta$. For small A the only contribution to the angular integral comes from the regions close to $\theta=0$ and $\theta=\pi$. Near $\theta=0$ we approximate

$$D^2\simeq\Delta_0^2\theta^2 \tag{37}$$

and

$$q\simeq p_F\vec{v}_s\cdot\hat{z}+\frac{\hbar v_F A\sin\phi}{2\theta}\ . \tag{38}$$

Then we get

$$\vec{J}^{(0)}=\vec{J}_A^{(0)}-\frac{N(0)\hbar v_F^2|A|}{4\pi\Delta_0}\int_0^{2\pi}d\phi\int_0^1 dt[\hat{z}p_F\vec{v}_s\cdot\hat{z}\,|\sin\phi|/(1-t^2)^{1/2}+\hat{y}\hbar v_F A\,|\sin\phi|^3(1-t^2)^{1/2}/2] \tag{39}$$

or

$$\vec{J}^{(0)}=\vec{J}_A^{(0)}-\frac{3}{4}\rho\left[\frac{hv_F}{\Delta_0}\right]|A|\,\hat{z}(\vec{v}_s\cdot\hat{z})-\frac{1}{8}\rho\frac{A|A|}{\Delta_0 M}\hbar^2 v_F\hat{y}\ . \tag{40}$$

This means that $\vec{J}^{(0)}$ contain the two nonanalytic terms

$$\vec{J}_{\rm NA}^{(0)}=-\frac{3}{4}\rho\frac{\hbar v_F}{\Delta_0}\hat{l}(\hat{l}\cdot\vec{v}_s)\,|\hat{l}\cdot\vec{\nabla}\hat{l}|-\frac{1}{8}\rho\frac{\hbar^2 v_F}{\Delta_0 M}|\hat{l}\cdot\vec{\nabla}\hat{l}|(\vec{\nabla}\times\hat{l}-\hat{l}\hat{l}\cdot\vec{\nabla}\times\hat{l})\ . \tag{41}$$

The first term is the current discovered by VM; the second is implicit in their theory but has escaped their attention. They were interested in nonanalytic corrections to ρ_s, and the second term does not contribute to ρ_s. Further, we should also mention that the texture we are considering is not the most general one possible, so that Eq. (41) does not include all the nonanalytic terms that could be generated by $\bar{g}^{(0)}$. However, since a nonzero value for $|\hat{l}\cdot\vec{\nabla}\hat{l}|$ is the essential ingredient in producing nonanalyticities, we can understand the essential physics by a careful study of this special texture.

Having recovered the VM current in the first term of our "perturbation expansion," we now check if this term captures the leading nonanalyticities. We might hope that the correction terms would contain (besides analytic pieces) nonanalytic terms of higher order than those in Eq. (41). This hope will be unfulfilled.

The first correction term that contributes to the current is $\bar{g}^{(2)}$. It would be very laborious to explicitly evaluate the current arising from it, so we adopt a more roundabout plan. We give simple order of magnitude estimates for the integrals by estimating the relevant range of θ and ϵ integrations. This allows us to conclude that $\bar{g}^{(2)}$ (and higher terms) will contribute to the nonanalytic part of $\vec{J}$ in the same order as the "leading term" $\bar{g}^{(0)}$. Hence $\bar{g}^{(2)}$ will generate terms like those in (41). These simple estimates do not allow us to determine the numerical coefficients of the nonanalytic terms arising from $\bar{g}^{(2)}$. A more thorough investigation of the term like the second one in (41) reveals that it, in fact, has a divergent coefficient. Thus the gauge-transformation strategy appears to fail. This forces us to develop a new plan to calculate the leading nonanalytic terms in $\vec{J}$.

We complete this section by giving more details on the order of magnitude analysis. We define a length scale by $L \equiv 1/|A|$. The crucial regions of ϵ and θ are given by (at $T=0$ ϵ_n becomes the continuous variable ϵ)

$$\epsilon \sim \Delta_0 \left[\frac{v_F}{\Delta_0 L}\right]^{1/2} \sim \Delta_0 \left[\frac{\xi_0}{L}\right]^{1/2}, \tag{42}$$

$$\theta \sim \left[\frac{\xi_0}{L}\right]^{1/2}, \pi - \left[\frac{\xi_0}{L}\right]^{1/2}. \tag{43}$$

We are then led to define a small parameter s by

$$s \sim \left[\frac{\xi_0}{L}\right]^{1/2}. \tag{44}$$

To estimate the nonanalytic currents arising from $\bar{g}^{(0)}$ and $\bar{g}^{(2)}$ we then use the following:

$$\begin{aligned} &\theta \sim s, \quad \epsilon \sim \Delta_0 s, \\ &\frac{\Delta(\hat{k})}{\Delta_0} \sim s, \quad (\xi_0 \vec{\nabla}_R) \sim s, \\ &\int d\theta \sim s, \quad \int d\epsilon \sim \Delta_0 s. \end{aligned} \tag{45}$$

We can explain the estimate for $(\xi_0 \vec{\nabla}_R)$ quite simply. It adds a factor of ξ_0/L, which gives s^2; however, the function being differentiated has its power of s reduced by 1. Combining these two effects yields $(s^2)(s^{-1})=s$.

First, consider the case $\vec{v}_s=0$. Using the above estimates in evaluating (14) we can conclude that both $\bar{g}^{(0)}$ and $\bar{g}^{(2)}$ give, for the special case we are considering, a current of the order

$$|\vec{J}| \sim s^4. \tag{46}$$

This means that both $\bar{g}^{(0)}$ and $\bar{g}^{(2)}$ give rise to a nonanalytic current of the form

$$\vec{J}_{NA} \sim |(\hat{l}\cdot\vec{\nabla})\hat{l}|(\vec{\nabla}\times\hat{l} - \hat{l}\hat{l}\cdot\vec{\nabla}\times\hat{l}). \tag{47}$$

We now permit a nonzero $\vec{v}_s$. The first term in (41) arises when

$$\frac{p_F(\vec{v}_s\cdot\hat{l})}{\Delta_0} \sim s. \tag{48}$$

Using this estimate, we can conclude that $\bar{g}^{(2)}$ also leads to a term of the form

$$\vec{J}_{NA} \sim \hat{l}(\hat{l}\cdot\vec{v}_s)|\hat{l}\cdot\vec{\nabla}\hat{l}|. \tag{49}$$

IV. SCALING THEORY

In this section we develop a new method of calculating the leading nonanalytic terms in the current density $\vec{J}$. The essential idea is to locate the (very small) region in $\hat{k},\epsilon,\vec{R}$ space that determines the nonanalyticities, and to solve the quasiclassical equations exactly in that region. By an appropriate scaling of the $\hat{k},\epsilon,\vec{R}$ variables, we enlarge the region of interest and eliminate the irrelevant parts of $\hat{k},\epsilon,\vec{R}$ space. The solution of the rescaled quasiclassical equations allows us to express the nonanalytic contributions to $\vec{J}$ in terms of two well-defined scaling functions $\psi_{\parallel}(x)$ and $\psi_{\perp}(x)$.

Using the insight gained in Sec. III, we recognize that the important regions of the θ and ϵ integrals for contributing to the nonanalytic currents are near the nodes of the gap,

$$0 \le \theta \lesssim \sqrt{\xi_0/L}$$
and
$$0 \le \pi - \theta \lesssim \sqrt{\xi_0/L}, \tag{50}$$

and near zero energy,

$$-\Delta_0\sqrt{\xi_0/L} \lesssim \epsilon \lesssim \Delta_0\sqrt{\xi_0/L}. \tag{51}$$

L is the typical length scale of the order parameter texture. In order to investigate nonanalytic terms like those arising in the preceding section, we consider the following texture:

$$\Delta(\hat{k},\vec{R}) = \Delta_0 e^{i\chi(z)}[\hat{a}(z) + i\hat{b}]\cdot\hat{k}, \tag{52}$$

$$\chi(z) = 2mv_s z/\hbar, \tag{53}$$

$$\hat{a}(z) = \hat{x}\cos(z/L) + \hat{z}\sin(z/L), \tag{54}$$

$$\hat{b} = \hat{y}. \tag{55}$$

The gap depends on z, and we are interested in the current at $z=0$. At this point $\hat{l}=\hat{z}$, $\vec{v}_s = v_s\hat{z}$, and $(\hat{l}\cdot\vec{\nabla})\hat{l} = -\hat{x}/L$. We set

$$\hat{k} = \cos\theta\,\hat{z} + \sin\theta\sin\phi\,\hat{y} + \sin\theta\cos\phi\,\hat{x}.$$

The quasiclassical differential equation (15) connects the point of interest $z=0$ with its neighborhood. The important range of influence is

$$-\sqrt{v_F L/\Delta_0} \lesssim z \lesssim \sqrt{v_F L/\Delta_0}\,. \quad (56)$$

Thus we define the following dimensionless variables

$$\bar{\epsilon}=\epsilon\sqrt{v_F\Delta_0/L}\,, \quad (57)$$

$$\bar{\theta}=\theta/\sqrt{v_F/L\Delta_0}\,, \quad (58)$$

$$\bar{z}=z/\sqrt{v_F L/\Delta_0}\,, \quad (59)$$

$$\bar{Q}=p_F v_s/\sqrt{v_F\Delta_0/L}\,. \quad (60)$$

The typical range for the variables with a bar is of order 1, so we can approximate

$$\cos(z/L)=\cos(\bar{z}\sqrt{v_F/\Delta_0 L})\simeq 1\,, \quad (61)$$

$$\sin(z/L)=\sin(\bar{z}\sqrt{v_F/\Delta_0 L}) \simeq \bar{z}\sqrt{v_F/\Delta_0 L}\,, \quad (62)$$

and find the dimensionless quasiclassical differential equation[10]

$$[(i\bar{\epsilon}-\bar{Q})\hat{\tau}_3 - i(\bar{\theta}\sin\phi\hat{\tau}_1+\bar{\theta}\cos\phi\hat{\tau}_2+\bar{z}\hat{\tau}_2),\hat{g}] + i\frac{\partial}{\partial\bar{z}}\hat{g}=0\,. \quad (63)$$

Equation (63) is the central equation of our theory. It correctly keeps the leading terms in an expansion in $\sqrt{v_F/\Delta_0 L}$. Formally, it is a system of simple first-order (ordinary) differential equations whose solution, subject to the proper normalization and boundary conditions, yields the matrix function $\hat{g}(\bar{z})$. $\hat{g}(\bar{z})$ depends on the parameters $\bar{\epsilon}$, $\bar{\theta}$, ϕ, and $\bar{Q}$. Let us denote by $g(\bar{\epsilon},\bar{\theta},\phi,\bar{Q})$ the $\hat{\tau}_3$ component of $\hat{g}(\bar{z})$ at $\bar{z}=0$. Thus

$$g(\bar{\epsilon},\bar{\theta},\phi,\bar{Q})\equiv\tfrac{1}{2}\mathrm{Tr}\hat{\tau}_3\hat{g}(\bar{\epsilon},\bar{\theta},\phi,\bar{Q},\bar{z}=0)\,. \quad (64)$$

We can express the current at $z=0$ in terms of g. However, we must be careful. To extract only the nonanalytic terms, and to prevent a divergence at large $\bar{\theta}$ when we integrate, we must subtract out the piece of g leading to the first order in gradients analytic term. A study of the asymptotic ($\bar{\epsilon},\bar{\theta}\to\infty$) solution of Eq. (63) reveals that after this subtraction we are indeed left with convergent integrals.

For the particular choice of gap we have made, Eq. (52), there will be two nonanalytic terms in $\vec{J}$, corresponding to the two terms of (41). These are

$$\vec{J}_{\rm NA}=\hat{z}N(0)\Delta_0 v_F\left[\frac{v_F}{\Delta_0 L}\right]^{3/2}\psi_{\parallel}(p_F v_s/\sqrt{\Delta_0 v_F/L})+\hat{y}N(0)\Delta_0 v_F\left[\frac{v_F}{\Delta_0 L}\right]^{2}\psi_{\perp}(p_F v_s/\sqrt{\Delta_0 v_F/L})\,. \quad (65)$$

The two scaling functions are defined by the integrals

$$\psi_{\parallel}(\bar{Q})=\frac{1}{2\pi^2}\int_{-\infty}^{+\infty}d\bar{\epsilon}\int_0^{\infty}d\bar{\theta}\int_0^{2\pi}d\phi\,\bar{\theta}[g(\bar{\epsilon},\bar{\theta},\phi,\bar{Q})-h(\bar{\epsilon},\bar{\theta},\phi,\bar{Q})]\,, \quad (66)$$

$$\psi_{\perp}(\bar{Q})=\frac{1}{2\pi^2}\int_{-\infty}^{+\infty}d\bar{\epsilon}\int_0^{\infty}d\bar{\theta}\int_0^{2\pi}d\phi\,\bar{\theta}^2\sin\phi[g(\bar{\epsilon},\bar{\theta},\psi,\bar{Q})-h(\bar{\epsilon},\bar{\theta},\phi,\bar{Q})]\,. \quad (67)$$

The subtraction term h is given by

$$h=\frac{-i\pi(\bar{\epsilon}+i\bar{Q})}{[(\bar{\epsilon}+i\bar{Q})^2+\bar{\theta}^2]^{1/2}}-\frac{\pi\bar{\theta}\sin\phi}{2[(\bar{\epsilon}+i\bar{Q})^2+\bar{\theta}^2]^{3/2}}\,. \quad (68)$$

Equation (65) is the main result of this section. We have expressed the current $\vec{J}_{\rm NA}$ in terms of two parameter-free mathematical functions $\psi_{\parallel}(\bar{Q})$ and $\psi_{\perp}(\bar{Q})$. To our knowledge, these functions cannot be written in terms of standard functions of the mathematical literature. A discussion of $\psi_{\parallel}$ and $\psi_{\perp}$ must start from the defining differential equation (63) for the matrix function $\hat{g}$, and the integrals (66) and (67) over the $\hat{\tau}_3$ component of $\hat{g}$. Several limits are relatively easy to discuss. For small v_s (more precisely, $p_F v_s \ll \sqrt{v_F\Delta_0/L}$) we can expand $\psi_{\parallel}$ and $\psi_{\perp}$ around zero, and find

$$\vec{J}_{\rm NA}=\hat{z}N(0)\frac{v_F^2 p_F}{\Delta_0}\psi_{\parallel}'(0)v_s\left|\frac{1}{L}\right|+\hat{y}N(0)\frac{v_F^3}{\Delta_0}\psi_{\perp}(0)\left|\frac{1}{L}\right|^2$$

$$=\frac{3}{2}\frac{\hbar v_F}{\Delta_0}\psi_{\parallel}'(0)\hat{l}(\hat{l}\cdot\vec{v}_s)\,|\hat{l}\cdot\vec{\nabla}\hat{l}|-\frac{3}{2}\rho\frac{\hbar^2 v_F}{\Delta_0 M}\psi_{\perp}(0)\,|\hat{l}\cdot\vec{\nabla}\hat{l}|\,(\vec{\nabla}\times\hat{l}-\hat{l}\hat{l}\cdot\vec{\nabla}\times\hat{l})\,. \quad (69)$$

The current has in this limit the same structure as found by the gauge-transformation approximation. However, the numerical coefficients have changed.

V. SUMMARY

We have used the quasiclassical theory to discuss the generalized—gauge-transformation approximation. As Volovik and Mineev[3] discovered, this approximation leads to nonanalytic terms in the $T=0$ A-phase supercurrent. While the approximation is instructive in revealing nonanalytic possibilities, a more comprehensive way of doing the calculation proved to be necessary, as discussed in Sec. III. The failure of the gauge-transformation approximation is perhaps not surprising. It relies on the small expansion parameter $\hbar v_F |\vec{\nabla}\Delta|/|\Delta|^2$ ($\approx q\xi_k$), with q a typical wavelength of the texture and $\xi_k=\hbar v_F/|\Delta(\hat{k})|$ the zero-temperature coherence length. At $\hat{k}=\pm\hat{l}$ the gap $|\Delta|$ vanishes and ξ_k diverges. Hence there is a small neighborhood around $\hat{k}=\pm\hat{l}$ where the expansion parameter ceases to be small. This region contributes decisively to the nonanalytic terms and has to be treated carefully.

In Sec. IV we developed a scaling method of studying the nonanalytic terms. This method involved an appropriate rescaling of the variables in the quasiclassical equations such that the size of the decisive region becomes of order 1. After rescaling we could take the limit $q\to 0$, and find a dimensionless differential equation for the propagator in the region of interest. The propagator gives us the leading nonanalytic terms in the current. Of course, the propagator carries more than this information (excitation spectrum, textural energies, etc.); we have not explored these possibilities here. In general, we believe the scaling method establishes a theoretical framework that allows a detailed study of physical effects that originate from "singularities" of the order parameter in $\vec{k}$ space.

Throughout this paper we have assumed that the gap is of the A-phase form. Since the application of a magnetic field can cause the A phase to exist at low temperatures, the calculations presented here are not just of theoretical interest. However, a more important point can be raised, of whether the nonanalytic terms in $\vec{J}$ are in fact a signal that the gap will distort and leave the A-phase form. This is difficult to answer at present; a crucial consideration is that the singularities responsible for the nonanalytic behavior are topologically stable, and so cannot be cured by a small change in $\Delta(\vec{R},\hat{k})$.[4] We hope that a study of the free-energy density will shed some light on this question.

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APPENDIX

The A-phase, $T=0$ supercurrent has no nonanalytic terms in the presence of a uniform $\vec{v}_s$ if the $\hat{l}$ vector is also spatially uniform. In contrast to this, a polar-phase gap does lead to nonanalyticities in $\vec{J}$ in the presence of a constant $\vec{v}_s$; this is because the polar-phase gap vanishes along a line on the Fermi surface, whereas the A-phase gap vanishes at just two points.

The polar-phase gap is of the form (1), with

$$\Delta(\hat{k},\vec{R})=\Delta_0(\hat{k}\cdot\hat{c})e^{i\psi} . \tag{A1}$$

Here $\hat{c}$ is an $\vec{R}$-dependent unit vector and ψ an $\vec{R}$-dependent real number. The superfluid velocity is given by

$$\vec{v}_s=\frac{\hbar}{2M}\vec{\nabla}_R\psi . \tag{A2}$$

We will examine a special case,

$$\Delta(\vec{R},\hat{k})=\Delta_0(\hat{k}\cdot\hat{z})\exp\left[i\frac{2Mv_s x}{\hbar}\right] . \tag{A3}$$

Then, both $\hat{c}=\hat{z}$ and $\vec{v}_s=v_s\hat{x}$ are spatially uniform. In this case a gauge transformation leads to an exact solution of the quasiclassical equation. Define

$$\bar{g}\equiv\exp\left[-i\frac{\psi}{2}\hat{\tau}_3\right]\hat{g}\exp\left[+i\frac{\psi}{2}\hat{\tau}_3\right] , \tag{A4}$$

$$\alpha_n\equiv\epsilon_n+ip_F\hat{k}\cdot\hat{x}v_s , \tag{A5}$$

$$D\equiv\Delta_0\hat{k}\cdot\hat{z} . \tag{A6}$$

Then $\bar{g}$ is given by

$$\bar{g}\equiv\frac{-\pi(i\alpha_n\hat{\tau}_3-iD\hat{\tau}_2)}{(\alpha_n^2+D^2)^{1/2}} , \tag{A7}$$

and this leads to a supercurrent of

$$\vec{J}=\rho v_s\hat{x}-\rho\frac{p_F}{\Delta_0}v_s|v_s|\hat{x} . \tag{A8}$$

We then see that $\vec{J}$ contains a term which is nonanalytic, and second order in $\vec{v}_s$.

If we study the weak-coupling gap equation using

$\bar{g}$, we find that a gap of the form (A3) is indeed a solution. The amplitude Δ_0 is reduced by $\vec{v}_s$; if we denote the amplitude when $\vec{v}_s=0$ by Δ_{00}, we find

$$\Delta_0=\Delta_{00}\exp\left[-\frac{p_F^3\,|\,v_s\,|^3}{3\Delta_0^3}\right]. \tag{A9}$$

Of course, the fact that a gap of the form (A3) is a solution of the weak-coupling gap equation does not mean it is the solution with the lowest energy. Finally, we should stress that Eqs. (A8) and (A9) are derived assuming that the quantity $(p_F\,|\,v_s\,|\,/\Delta_0)$ is small.

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