Critical phenomena in systems with long-range-correlated quenched disorder

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As a model for a phase transition in an inhomogeneous system, we consider a system where the local transition temperature varies in space, with a correlation function obeying a power law $\sim x^{-a}$ for large separations x. We extend the Harris criterion for this case, finding that for $a<d$ (where d is the spatial dimension) the disorder is irrelevant if $a\nu-2>0$, while if $a>d$ we recover the usual Harris criterion: The disorder is irrelevant if $d\nu-2=-\alpha>0$. An m-vector system of this type is studied with the use of a renormalization-group expansion in $\epsilon=4-d$ *and* $\delta=4-a$. We find a new long-range-disorder fixed point in addition to the short-range-disorder and pure fixed points found previously. The crossover between fixed points is found to follow the extended Harris criterion. The new fixed point has complex eigenvalues, leading to oscillating corrections to scaling, and has a correlation-length exponent $\nu=2/a$. We argue that this new scaling relation is exact and applies more generally than just to the specific model. We show that the extended Harris criterion also applies to percolation with long-range-correlated site or bond-occupation probabilities, so that the scaling law should be obeyed by such systems. Results for the percolation properties of the triangular Ising model are in agreement with these predictions.

I. INTRODUCTION

This paper is concerned with the effect of introducing a small amount of quenched disorder into a system which, when pure, undergoes a second-order phase transition. We restrict our attention to "random-temperature" disorder which arises, for example, from a small amount of bond randomness or from a small density of impurities which cause random variations in the local transition temperature $T_c(\vec{x})$. We do not consider random "magnetic fields" which couple linearly to the order parameter, or random anisotropy fields which break the local symmetry in the order-parameter space.

This work differs from most previous work in that we do not restrict ourselves to randomness with short-range correlations; rather, we wish to consider fluctuations in $T_c(\vec{x})$ arising from "inhomogeneities" in the system which are characterized by a correlation function that falls off relatively slowly with distance.

In the body of this paper we are concerned with a model where the correlation function $\langle T_c(\vec{x})T_c(\vec{y})\rangle_{\rm av}-\langle T_c\rangle_{\rm av}^2$ falls off with distance as a power law $\sim|\vec{x}-\vec{y}|^{-a}$, where a is a constant. (More general situations are considered in the appendices.) A power-law correlation is the simplest possible assumption and has the *possibility* of scale-invariant behavior, with new fixed points of the renormalization group, and new values of critical exponents. In an actual experimental system one would probably not expect to find that fluctuations in $T_c(\vec{x})$ are truly represented by a power law; however, if fluctuations in T_c arise from a number of different causes, with a wide dispersion in characteristic length scales, it may well be that the resulting correlation function is approximated over a number of decades by effective power-law behavior. (For example, it appears that time variations of quantities in a wide variety of situations are characterized by a power spectrum proportional to the inverse of the frequency, although no general explanation for the phenomenon has been found.[1]) Furthermore, we hope that study of the case of a simple power-law correlation function may help develop intuitive insights to understand better the more complicated cases.

If a sample contains straight lines of impurities or straight dislocation lines of *random orientation,* then the quenched disorder may be described by an isotropic correlation function of the power-law form considered in this paper, with power $a=d-1$. Random planes of impurities would give $a=d-2$. We have recently received a report by Boyanovsky and Cardy[2] in which they consider ϵ_d-dimensional "lines" of impurities of a *single orientation,* with perfect correlations in the disorder along the lines and no correlations in the other directions. (The same system was also considered earlier by Dorogovtsev.[3]) As Boyanovsky and Cardy point out, their system is anisotropic since the lines of impurities single out a direction, and they find an *anisotropic* disorder fixed point. Nevertheless, there are a number of similarities between their results and ours.

An early investigation of the effects of quenched disorder on a continuous phase transition was the exact analysis of McCoy and Wu[4,5] on the two-dimensional Ising model with rows of differing bonds. They found a "smeared" transition, which exhibited a smooth specific heat through the transition region. Later studies concentrated on systems in which the disorder of the transition temperature had only short-range correlations. Harris[6] developed a consistency criterion for a random transition-temperature system to undergo a second-order transition with the same exponents and critical properties as the corresponding pure system. The criterion is that the specific-heat exponent α must be negative for the pure system. Later authors,[7–11] with the use of the renormalization group (RG) on the n-vector model in spatial dimension d near 4, found that the Harris criterion correctly predicts the crossover to new behavior when α is positive. Unlike the long-range-correlated McCoy and Wu model, the new behavior is still a second-order transition (described by a new short-range-disorder fixed point), but with new critical exponents. Furthermore, the disordered fixed point has a nonpositive value of α, so that the new fixed point may be said to be consistent with the Harris criterion.

When long-range power-law correlations in the disorder exist the Harris criterion must be modified. In Sec. II we find that the criterion for the long-range disorder to be irrelevant is that $a\nu-2$ must be positive if $a<d$, while $d\nu-2=-\alpha$ must be positive (the normal Harris criterion) if $a>d$, where ν is the exponent for the temperature dependence of the correlation length.

The renormalization-group approach of Grinstein and Luther[7] can be readily extended to the case of long-range quenched disorder, provided that a and d are both close to 4. We find in this region that our extended Harris criterion correctly describes the crossover in critical behavior as the number of order-parameter components m, the spatial dimension d, and the range of the correlations a are varied. Depending on the values of a, d, and m (and in some cases on the amplitude of the imposed disorder) we find one of five possibilities for the asymptotic critical behavior—described by the Gaussian fixed point, the Wilson-Fisher "pure" fixed point, the short-range-disorder fixed point found by previous authors,[7–11] a new long-range-disorder fixed point, or a runaway behavior which carries the system out of the region of applicability of our calculation, but likely indicates some kind of smeared transition.[12] A summary of the regions where the various types of critical behavior occur, for a and d close to 4 and $m>1$, is given in Figs. 1 and 2. We expect that as the range of the correlations is increased the system will eventually exhibit crossover from a second-order transition to a smeared transition. However, in the region of validity of our calculation, $a=4-\delta$ with $\delta=O(\epsilon)$, we do not observe such crossover behavior for $d<4$, while for $d>4$ the runaway, which may be an indication of a smeared transition, appears as a decreases through 4 and then disappears again at a smaller value of a as the range of the correlations increases; so that for still longer-range correlations the system again has a second-order transition. When the long-range-disorder fixed point describes the behavior, the critical exponent ν takes on the value $2/a$ to the accuracy of our calculation, which means that our extended Harris criterion is marginally satisfied. (We argue that the relation $\nu=2/a$ is exact at the fixed point.)

Experimentally, the crossover to behavior described by the short-range-disorder fixed point has not been observed, most likely because α is negative, or only slightly positive, for pure m-vector models in two or three dimensions. Thus, either the disorder is irrelevant or the crossover region is small. However, if a system with long-range correlations could be made, even if the correlations were not exactly power law in form, the results of this study in-

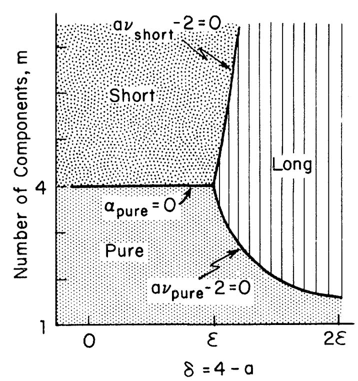

FIG. 1. Regions in the δ-m plane where the various types of critical behavior occur for $d<4$ ($\epsilon=4-d$). Here $\delta=4-a$ where a is the power of the falloff of the disorder correlation function, and m is the number of order-parameter components. The crossover between pure and short-range-disorder behavior is determined by the Harris criterion: The pure is stable for the pure specific-heat exponent $\alpha_{pure}<0$; the crossovers between pure or short-range behavior and long-range-disorder behavior are determined by the extended Harris criterion: The pure or short ranged is stable if $a\nu-2>0$, where ν is the correlation-length exponent ν_{pure} or ν_{short}, respectively.

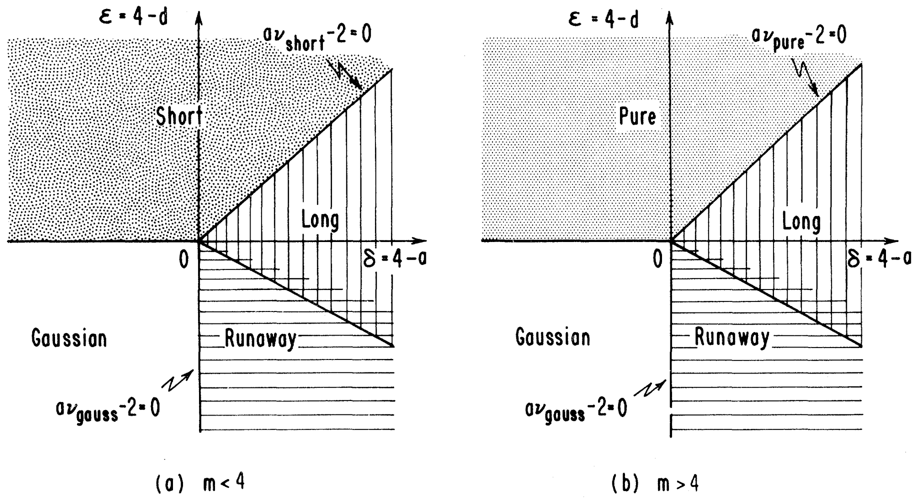

FIG. 2. Regions in the $\delta-\epsilon$ plane where the various types of critical behavior occur for the number of order-parameter components (a) $1<m<4$ and (b) $m>4$. Here $\delta=4-a$ where a is the power of the falloff of the disorder correlation function and $\epsilon=4-d$. The Gaussian, pure, and short-range-disorder behaviors become unstable to the long-range disorder according to the extended Harris criterion: The crossover occurs when $a\nu-2$ goes negative, where ν is the correlation-length exponent ν_{Gauss}, ν_{pure}, or ν_{short}, respectively. In the cross-hatched regions separating the long-range and runaway behaviors the long-range fixed point has a finite domain of attraction; those systems which lie outside of the domain of attraction exhibit runaway behavior. See Sec. IV.

dicate that new behavior could well be observed. If the correlations are sufficiently long ranged, disorder will be relevant for any system, and the crossover exponent $(2\nu^{-1}-a)$ can be large.

Our renormalization-group analysis is carried out for a Landau-Ginzburg-Wilson Hamiltonian of the form

$$\beta H = \int d^d x \{ \tfrac{1}{2}[r+\delta r(\vec{x})] \, |\phi(\vec{x})|^2 + \tfrac{1}{2}c \, |\vec{\nabla}\phi(\vec{x})|^2 + u \, |\phi(\vec{x})|^4 \} , \tag{1.1}$$

where $\phi(\vec{x})$ is the m-component order parameter, $\beta=(k_B T)^{-1}$, and $\delta r(\vec{x})$ represents quenched random-temperature disorder, with

$$\langle \delta r(\vec{x}) \rangle_{\text{av}} = 0 \tag{1.2a}$$

$$\tfrac{1}{8}\langle \delta r(\vec{x}) \delta r(\vec{y}) \rangle_{\text{av}} = g(|\vec{x}-\vec{y}|) , \tag{1.2b}$$

where $\langle \cdots \rangle_{\text{av}}$ denotes an average over the spatially homogeneous and isotropic quenched-disorder probability distribution. We shall take $g(x) \sim x^{-a}$ for large x, so that its Fourier transform obeys

$$\bar{g}(k) \sim v + w k^{-(d-a)} \tag{1.3}$$

for small k. Note that $\bar{g}(k)$ must be positive definite, so if $a>d$, then $v>0$, while if $a<d$, then $w>0$. The case $w=0$ corresponds to the short-range case considered in the past, where g was taken to be a δ function. Also, if $a>d$, then in the long-wavelength limit $\bar{g}(k) \rightarrow$const, as in the short-range case. Thus disorder with long-range correlations which fall off with distance faster than x^{-d} leads to the same critical behavior as that due to short-range correlated disorder. In Appendix A we briefly discuss the case $g(x) \sim \sum A_i x^{-a_i}$, finding, as one might expect, that it is the smallest a_i which determines the critical behavior.

The techniques used to analyze the system are an extension of those used by Grinstein and Luther[7] for the short-range case. We use the replica trick[7,11,13] to obtain a translationally invariant effective Hamiltonian, and keep only the lowest term in its cumulant expansion. We then apply the renormalization group to the effective Hamiltonian, expanding around the Gaussian fixed point. We find that we cannot take a to be arbitrary, it must be near 4. Thus we carry out a double expansion in $\epsilon=4-d$ and $\delta=4-a$, with ϵ and δ of the same order, finding the fixed points and eigenvalues to $O(\epsilon,\delta)$.

As mentioned above, we find the short-range-disorder fixed point found in previous work, and also a new long-range-disorder one, with correlation-length exponent $\nu=2/a$. For much of the range of ϵ and δ the long-range-disorder fixed point has complex eigenvalues in the directions other than the temperature direction. This leads to oscillating corrections to scaling, and to the possibility that the fixed point can go unstable (via a Hopf bifurcation) without any other fixed point becoming stable. This is discussed at greater length in Sec. IV.

The fixed point in fact goes unstable (via a subcritical Hopf bifurcation) when $\delta \approx 1.8 |\epsilon|$ with $\epsilon < 0$, leading to spiralling runaway in the RG flows. This runaway may be a crossover to a smeared transition; *but* the fixed point goes unstable as δ decreases (i.e., as the range *decreases*), not as it increases as we would have expected.

As mentioned above, Boyanovsky and Cardy[2] have considered ϵ_d-dimensional "lines" of impurities of a single orientation. They found that the crossover from the pure behavior to behavior described by an *anisotropic* long-range-disorder fixed point is described by an extension of the Harris criterion similar to that derived here. As in our case, their long-range-disorder fixed point has complex eigenvalues.

The special case of the Ising model ($m=1$) must be dealt with separately. For $m=1$ there is an accidental degeneracy in the recursion relations when only short-range correlations are included, making[7,9,14] the short-range fixed point and its eigenvalues order $\epsilon^{1/2}$ rather than order ϵ. Including long-range correlations, we find that the short-range fixed point is stable until $\delta = O(\epsilon^{1/2})$, when it exchanges stability with the long-range fixed point, consistent with the extended Harris criterion. To actually map out the stability and position of the long-range fixed point we would have to find the recursion relations to higher order in the interactions, a program which has yet to be carried out.

In the final section we consider the percolation problem with long-range correlations in the site- (or bond-) occupation probabilities $\sim x^{-a}$ for large distances. We argue that the extended Harris criterion applies to this problem, and thus that the scaling relation $\nu = 2/a$ should be satisfied when the long-range correlations are relevant. Examining the results of Klein *et al.*[15] on the percolation properties of the triangular Ising model with temperature $T \geq T_c$, we find that our expectations are satisfied. For $T > T_c$ the correlation function falls off exponentially at large distances and normal percolation critical behavior occurs. However, for $T = T_c$ the correlations have power-law decay and the extended Harris criterion indicates that the disorder is relevant. Klein *et al.* in fact find different percolation critical behavior, which satisfies the scaling law $\nu = 2/a$.

The remainder of this paper is organized as follows: In Sec. II we derive the extended Harris criterion mentioned above. The RG recursion relations are derived in Sec. III, and their fixed points and eigenvalues are determined in Sec. IV. We pay particular attention to the crossover of stability between the fixed points, and compare it with the predictions of the extended Harris criterion. Section IV also contains a discussion of the effects of complex eigenvalues since the eigenvalues of the long-range-disorder fixed point are found to be complex. In Sec. V we consider the long-range-correlated percolation problem. In Appendix A we discuss a slightly generalized model in which $\bar{g}(k) \sim v + \sum w_i k^{-(d-a_i)}$. Finally, in Appendix B we consider a more general model where $g(x)$ is an arbitrary correlation function, and we develop renormalization-group recursion relations correct to first order in $\epsilon = 4-d$, and to first order in $\delta(k) \equiv \epsilon - \partial \ln \bar{g}(k)/\partial \ln k$, assuming that $\bar{g}(k)$ and the quartic interaction constant are of order ϵ.

II. EXTENDED HARRIS CRITERION

Consider adding a small amount of disorder to a system which undergoes a normal second-order transition. The disorder may be irrelevant, in which case the system will continue to exhibit the pure critical behavior; or the disorder may be relevant, driving the system to some new behavior, such as a smeared transition or a second-order transition with new exponents. The Harris criterion[6] arises from examining whether it is *consistent* for the disordered system to undergo a second-order transition with the pure exponents, specifically with the pure correlation-length exponent ν. When the criterion fails we expect new behavior to occur. In fact, the results of earlier work[7–11] and of this paper find that the crossover to new behavior occurs exactly at the point at which the pure transition is no longer consistent.

Harris originally derived his criterion for the case of random-temperature disorder, i.e., a system with an effective local critical temperature $T_c(\vec{x})$, with only short-range correlations in the disorder. Thus $g(\vec{x})$ falls off rapidly with distance, where

$$g(\vec{x}-\vec{y}) = \langle T_c(\vec{x}) T_c(\vec{y}) \rangle_{av}^c \equiv \langle T_c(\vec{x}) T_c(\vec{y}) \rangle_{av} - \langle T_c \rangle_{av}^2 , \qquad (2.1)$$

with $\langle \cdots \rangle_{av}^c$ the connected impurity average. We shall extend the argument to the case of long-range correlations in the disorder which falls off as a power law, $g(\vec{x}) \sim |\vec{x}|^{-a}$ for large x. We wish to determine if it is consistent for the system to undergo a second-order transition with the pure correlation-length exponent ν. To this end consider dividing up the system into regions the size of the pure correlation length ξ. We then ask if the variation in the critical temperatures of the regions becomes negligible as T approaches T_c. We expect that spins will be well correlated for distances up to the correlation length ξ, so we take the transition temperature of a region of size ξ to be the average of $T_c(\vec{x})$ over the region. Defining reduced temperature $t=(T-T_c)/T_c$ and local reduced temperature $t(\vec{x})=[T-T_c(\vec{x})]/T_c$, we have $\langle t(\vec{x})\rangle_{av}=t$, and the effective reduced temperature of a region is

$$t_V=\frac{1}{V}\int_V d^dx\, t(\vec{x}) , \tag{2.2}$$

where the integration is over the region of volume $V=\xi^d$. The variance of t_V will be

$$\begin{aligned}\Delta^2 &\equiv \langle (t_V)^2\rangle_{av}^c \\ &=\frac{1}{V^2}\int_V d^dx \int_V d^dy\, \langle t(\vec{x})t(\vec{y})\rangle_{av}^c \\ &=\frac{1}{T_c^2}\frac{1}{V^2}\int_V d^dx \int_V d^dy\, g(\vec{x}-\vec{y}) \\ &\sim \xi^{-d}\int_0^{\xi} x^{d-1}g(x)dx ,\end{aligned} \tag{2.3}$$

where the last step assumes that g is isotropic and that ξ is large. Taking $g(x)\sim x^{-a}$ for large x, and T near T_c so that ξ is large, we find

$$\Delta^2 \sim \xi^{-d}\begin{cases}\text{const}, & a>d\\ \ln\xi, & a=d\\ \xi^{d-a}, & a<d .\end{cases} \tag{2.4}$$

The picture of a pure second-order transition is consistent if Δ^2/t^2 vanishes as $t\to 0$. Now $\xi\sim t^{-\nu}$, so

$$\Delta^2/t^2 \sim \begin{cases} t^{d\nu-2}, & a>d\\ t^{d\nu-2}\ln t^{-\nu}, & a=d\\ t^{a\nu-2}, & a<d .\end{cases} \tag{2.5}$$

Thus for stability of the pure-system critical behavior we must have

$$\begin{aligned} &d\nu-2=-\alpha>0, \quad a\ge d\\ &a\nu-2>0, \quad a<d\end{aligned} \tag{2.6}$$

where in the first relation we have used the scaling law[16] $d\nu=2-\alpha$ to relate ν and α. For $a\ge d$ we have recovered the original short-range Harris criterion; in this case the long-range tail of $g(x)$ is immaterial. However, for $a<d$ the fact that $g(x)$ is long ranged leads to a new requirement for stability of the pure fixed point. Notice that for $a<d$, $a\nu-2<d\nu-2$; so this new requirement is more stringent than the original one.

III. RECURSION RELATIONS

We wish to investigate the properties of a disordered system defined by the Hamiltonian equation (1.1). Because the disorder is quenched, it is the free energy which must be averaged over the disorder.[17] It is convenient to use the replica trick, which results in an effective Hamiltonian H_{eff}, which is a functional of n replications of the original order parameter. In the limit $n\to 0$ we obtain the averaged free energy.[7,11,13] The effective Hamiltonian H_{eff} is translationally invariant, so it is straightforward to apply RG techniques in its analysis, taking $n\to 0$ in the final results.

We write the partition function

$$Z=\text{Tr}_{\{\phi\}}e^{-\beta H(\phi)} . \tag{3.1}$$

Then the averaged free energy can be written

$$\begin{aligned}-F &\equiv \langle \ln Z\rangle_{av}\\ &=\lim_{n\to 0}\frac{1}{n}\ln\langle Z^n\rangle_{av}\\ &=\lim_{n\to 0}\frac{1}{n}\ln \text{Tr}_{\{\phi^\alpha\}}e^{-\beta H_{\text{eff}}(\{\phi^\alpha\})} ,\end{aligned} \tag{3.2}$$

where we have introduced n replicas of ϕ, $\{\phi^\alpha\}$ with the replica index $\alpha=1,2,\dots,n$. The effective Hamiltonian H_{eff}, defined by

$$e^{-\beta H_{\text{eff}}}=\left\langle \prod_\alpha e^{-\beta H(\phi^\alpha)}\right\rangle_{av} , \tag{3.3}$$

can be expressed as a cumulant expansion in $-\beta\sum_\alpha H(\phi^\alpha)$. The first cumulant vanishes since $\langle \delta r\rangle_{av}=0$ [Eq. (1.2a)]. Including the next cumulant, we find

$$\begin{aligned}\beta H_{\text{eff}} = \sum_\alpha \int d^dx[\tfrac{1}{2}r\,|\phi^\alpha(\vec{x})|^2+\tfrac{1}{2}c\,|\vec{\nabla}\phi^\alpha(\vec{x})|^2\\ +u\,|\phi^\alpha(\vec{x})|^4]-\sum_{\alpha,\beta}\int d^dx\, d^dy\, g(\vec{x}-\vec{y})\,|\phi^\alpha(\vec{x})|^2\,|\phi^\beta(\vec{y})|^2 ,\end{aligned} \tag{3.4}$$

where $g(\vec{x}-\vec{y})=\frac{1}{8}\langle \delta r(\vec{x})\delta r(\vec{y})\rangle_{av}$. Notice that the term involving g couples the different replicas. In the case that the correlations in δr are purely short ranged, it is easy to see that higher-order terms in the cumulant expansion are irrelevant[7–11]; also, if δr is Gaussian distributed all of the higher cumulants vanish. Higher-order cumulants may be relevant if the higher-order correlations are sufficiently long ranged; however, the higher cumulants *generated* by the RG are irrelevant.[18] In this paper we shall ignore the higher-order cumulants. Thus our results are applicable to systems with Gaussian disorder, or to systems in which the ranges of the higher-order correlations are sufficiently short.

Upon taking Fourier transforms $\phi(\vec{x})\rightarrow\bar{\phi}(\vec{k})$, we get the usual momentum-independent fourth-order interaction u acting in a single replica, plus an interaction which couples replicas with momentum dependence $-\bar{g}(\vec{k})$, where $\bar{g}$ is the Fourier transform of g. With $g(\vec{x})\sim x^{-a}$ for large x, if $a>d$, then $\bar{g}(k)$ is constant for small momentum k, and we recover the $H_{\rm eff}$ of the short-range case analyzed by previous authors. However, if $a<d$, $\bar{g}(k)\sim v+wk^{-(d-a)}$ for small k, and we have, in addition to the momentum-independent interaction $-v$, the momentum-dependent interaction $-wk^{-(d-a)}$.

We carry out the RG procedure[19] expanding in the interactions u, $-v$, and $-w$. We shall find that at the fixed points of the RG the couplings u, v, and w are $O(\epsilon,\delta)$ [where $\epsilon=4-d$ and $\delta=4-a$ and we take $\delta=O(\epsilon)$]: So in order to calculate fixed points and eigenvalues to order ϵ we must work to first order in the couplings in calculating r' and to second order in calculating u', v', and w'.

The interactions can be represented graphically as in Fig. 3, where we have indicated the replica indices and momentum dependence. Figure 4 shows the contributions to r', and Fig. 5 shows the contributions to u', v', or w'. In Figs. 4 and 5 a dotted line represents either of the three interactions. Since we are interested in the limit $n\rightarrow 0$, graphs which have a free-replica index (which gives rise to a factor of n) do not contribute and are neglected. Thus the graph of the form 4(b) appears only with the u interaction, and graph 5(a) has at least one of the interactions equal to u. Graphs 5(a) and 5(b) with interaction A contribute to A', while graph 5(c) contributes to u' if either A or B are u, and to v' otherwise. Although there is a contribution to c' from the momentum dependence of graph 4(a) with $A=-w$, the contribution is $O[w(d-a)]=O(\epsilon^2)$, so η continues to vanish to order ϵ as in the pure system.

Proceeding in the usual way,[19] we find differential recursion relations

$$\frac{dr}{dl}=2r+\frac{4(m+2)u}{1+r}-\frac{8(v+w)}{1+r}, \qquad (3.5a)$$

$$\frac{du}{dl}=\epsilon u-\frac{4(m+8)u^2}{(1+r)^2}+\frac{48u(v+w)}{(1+r)^2}, \qquad (3.5b)$$

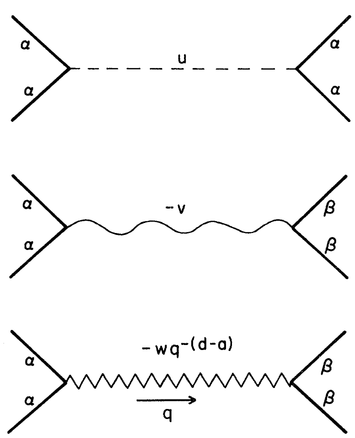

FIG. 3. Interactions u, $-v$, and $-w$. Notice that u acts only within a single replica α, while the other two act between arbitrary replicas α and β, and that the w interaction has momentum dependence $q^{-(d-a)}$.

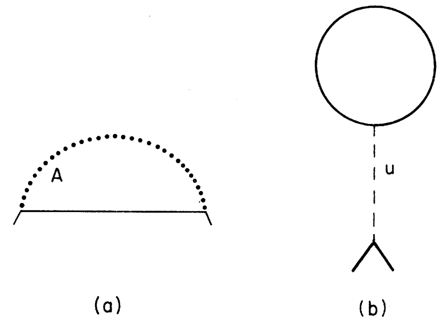

FIG. 4. Graphs that contribute to r'. The dotted interaction labeled A is any of the three interactions.

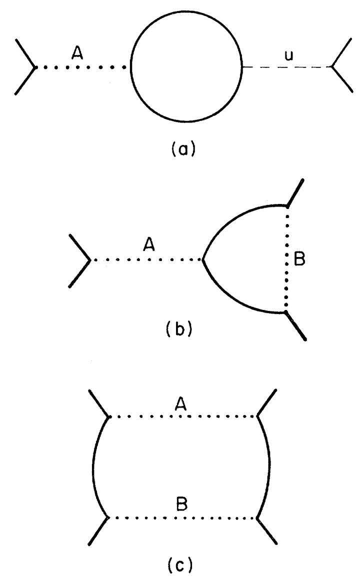

FIG. 5. Graphs that contribute to u', v', and w'. The dotted interaction labeled A or B is any of the three interactions.

$$\frac{dv}{dl}=\epsilon v-\frac{8(m+2)uv}{(1+r)^2}+\frac{16v(v+w)}{(1+r)^2}+\frac{16(v+w)^2}{(1+r)^2}\ , \tag{3.5c}$$

$$\frac{dw}{dl}=\delta w-\frac{8(m+2)uw}{(1+r)^2}+\frac{16w(v+w)}{(1+r)^2}\ , \tag{3.5d}$$

where l is the logarithm of the RG length-rescaling factor. We have scaled the momentum so $c=1$, taken the momentum cutoff to be unity, and absorbed a factor of $S_d/(2\pi)^d$ (S_d is the "surface area" of the d-dimensional unit sphere) into a redefinition of u, v, and w. Notice that $\delta=4-a$ appears in the w recursion relation (3.5d). Thus, only if $\delta=O(\epsilon)$ will the recursion relations have fixed points with a nonzero value of w^*. Also, v is generated by w under the RG [Eq. (3.5c)], so that even if $v=0$, $dv/dl>0$; thus the problem is not simplified by taking an initial Hamiltonian which neglects the v interaction.

The physical region of parameter space is $u\geq 0$, $w\geq 0$, and $v+w\geq 0$. Negative u would lead to an unstable initial Hamiltonian. Now, $\bar{g}(k)=v+wk^{-(d-a)}$ must be non-negative for the momentum up to the cutoff since it is the Fourier transform of the translationally invariant correlation function g. For $a<d$, the w term is dominant at small k, so w must be ≥ 0. At the momentum cutoff, $\bar{g}(1)=v+w$, so $v+w$ must also be ≥ 0. For $a>d$, w is irrelevant and the requirement is that $v\geq 0$. It is easy to see the encouraging result that, at least to this order, the RG flows that start in the physical region never leave it.

IV. FIXED POINTS AND EIGENVALUES

A system that undergoes a second-order transition will, except when the temperature is exactly the critical temperature, be driven away from the critical point toward either high or low temperatures by successive applications of the RG. The critical behavior is thus determined by a fixed point of the RG recursion relations which is stable in all directions in parameter space except for the temperature (r) direction. All of the order-ϵ fixed points we find are unstable in the r direction; in what follows we shall call a fixed point "stable" if it is stable in all but that direction. Crossover in critical behavior (from, for example, pure to disordered behavior as the number of spin components is changed) corresponds to an exchange of stability between fixed points describing the different behaviors. Thus our program is to find the fixed points of the RG recursion relations [Eqs. (3.5)], and then to diagonalize the relations about the points. For most systems studied in the past, the eigenvalues turn out to be real.[20] However, we shall find that one of the fixed points (the long-range-disorder one) has a pair of complex eigenvalues leading to RG flows which spiral around the fixed point. The signs of the real parts of the eigenvalues in all but the r direction determine the stability of the fixed point. Only if all of them are negative is the fixed point stable. The eigenvalue in the r direction is positive for all of the fixed points, and in fact can be seen to be ν^{-1}, where ν is the correlation-length critical exponent. The complex eigenvalues lead to oscillating corrections to scaling.[20]

An important task in the following discussion will be to identify the stable fixed points as we vary the parameters a, d, and m. When the eigenvalues are real, fixed points will (generically) change stability as illustrated schematically in Fig. 6(a). As a parameter (the number of spin components, the dimensionality, or the range of the disorder) is changed, an unstable and a stable fixed point *exchange* stability when they are coincident. This describes normal crossover behavior. Another possible scenario not illustrated in Fig. 6 is that two fixed points, one stable and the other unstable, meet and become complex; since the recursion relations are real these fixed points become inaccessible to the system. In addition, when the eigenvalues are complex, a Hopf bifurcation scenario is also possible.[21] In Fig. 6(b) we show a schematic *supercritical* Hopf bifurcation in which a stable fixed point goes unstable by giving off a stable limit cycle, and in Fig. 6(c) we show a *subcritical* Hopf bifurcation (equivalent to a supercritical Hopf bifurcation run backwards in "time") in which an unstable limit cycle enclosing a stable fixed point shrinks to zero size, making the fixed point unstable.

The *supercritical* Hopf bifurcation scenario would lead to very strange critical behavior described by a stable limit cycle rather than the normal stable fixed point. The scaling behavior would be oscillatory in l, the logarithm of the length rescaling factor. In the *subcritical* Hopf bifurcation scenario there is a contraction of the domain of attraction of the stable fixed point as the unstable limit cycle collapses, until the domain vanishes and the fixed point is unstable. Thus systems will exhibit runaway behavior when they are no longer attracted by the fixed point. Different systems described by different initial positions in parameter space will display runaway behavior at different times as the limit cycle collapses.

The system we are studying displays the normal exchange of stability among fixed points and the subcritical Hopf bifurcation, but not the supercritical Hopf bifurcation. We observe the runaway of

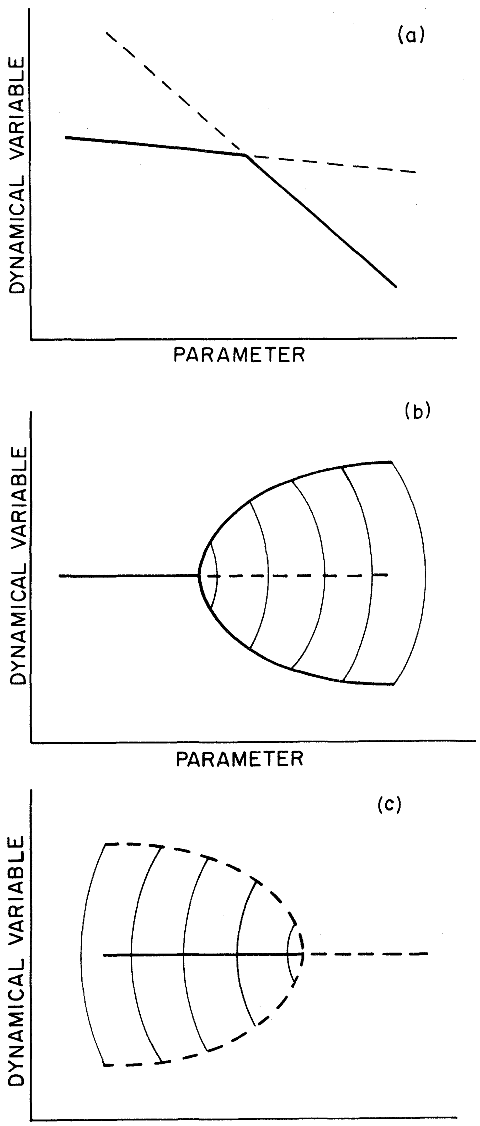

FIG. 6. Three generic scenarios for the change of stability of a fixed point of a dynamical system as a parameter is varied. The solid (dashed) lines represent stable (unstable) fixed points, and the solid (dashed) paraboloids represent stable (unstable) limit cycles. (a) A stable and an unstable fixed point meet and exchange stability. (b) The supercritical Hopf bifurcation. A stable fixed point becomes unstable and gives off a stable limit cycle. (c) The subcritical Hopf bifurcation. An unstable limit cycle collapses around a stable fixed point making it unstable. Notice that in this last case the domain of attraction of the fixed point shrinks as the limit cycle collapses.

RG flows which are outside of the domain of attraction of the long-range-disorder fixed point as the unstable limit cycle collapses (see Fig. 7). The fixed point becomes unstable when $\delta \approx 1.8 | \epsilon |$ with $\epsilon < 0$.

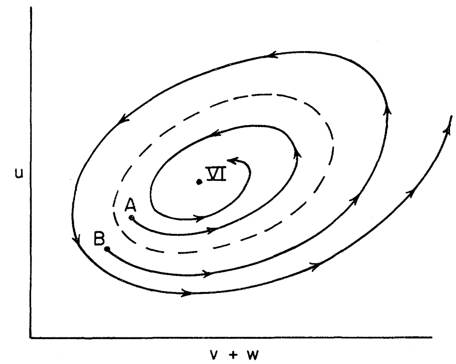

FIG. 7. Schematic view of the shrinking of the domain of attraction of the long-range-disorder fixed point just before it becomes unstable by a subcritical Hopf bifurcation as $\delta \rightarrow 1.8 | \epsilon |$ with $\epsilon < 0$. We show a projection of the RG flows onto the $u,(v+w)$ plane. The point marked VI is the long-range-disorder fixed point. The flow which starts at point A is attracted to the fixed point, while that which starts at point B is outside the domain of attraction and spirals out.

We now find the fixed points of the recursion relations and map their stability. If $m = 1$ there is an accidental degeneracy in the fixed-point equations when $w = 0$. Thus we consider the case $m > 1$ and $m = 1$ separately in the following two sections.

A. Case $m > 1$

In Table I we show the values of the parameters of the Hamiltonian at the six fixed points of the recursion relations (3.5). We find the normal Gaussian and pure fixed points, which determine the behavior of the system with no disorder for $d > 4$ and $d < 4$, respectively. We also find the short-range-disorder and unphysical (III) fixed points of previous authors,[7–11] and in addition we find two new fixed points, another unphysical (V) one, and the long-range-disorder fixed point which describes the behavior of the system when the correlations in the disorder are sufficiently long ranged.

In Table II we show the eigenvalues obtained by diagonalizing the recursion relations about each of the fixed points. The first column gives the eigenvalue with an eigenvector in the r direction; this eigenvalue is equal to ν^{-1}, where ν is the correlation-length exponent. The remaining three eigenvectors with eigenvalues λ_1, λ_2, and λ_3 lie in the three-dimensional u,v,w subspace. The equation for the eigenvalues of the long-range fixed point is unwieldy; in Fig. 8 we plot these eigenvalues for three values of m. Notice that they are complex for much of the range of ϵ and δ.

Fixed point III is unphysical because it has an un-

TABLE I. Fixed points of the recursion relations, Eqs. (3.5).

Fixed point		r^*	u^*	v^*	w^*
I.	Gaussian	0	0	0	0
II.	Pure	$-\frac{m+2}{2(m+8)}\epsilon$	$\frac{\epsilon}{4(m+8)}$	0	0
III.	Unphysical	$-\frac{1}{8}\epsilon$	0	$-\frac{1}{32}\epsilon$	0
IV.	Short-range disorder	$-\frac{3m}{16(m-1)}\epsilon$	$\frac{\epsilon}{16(m-1)}$	$\frac{4-m}{64(m-1)}\epsilon$	0
V.	Unphysical	$-\frac{1}{4}\delta$	0	$\frac{\delta^2}{16(\delta-\epsilon)}$	$-\frac{\delta}{16}\left[\frac{2\delta-\epsilon)}{\delta-\epsilon}\right]$
VI.	Long-range disorder	$-\frac{1}{4}\delta$	$\frac{3\delta-\epsilon}{4(5m+4)}$	$\frac{1}{\delta-\epsilon}\left[\frac{2(m+2)\epsilon-(m+8)\delta}{4(5m+4)}\right]^2$	$\frac{1}{16(\delta-\epsilon)(5m+4)^2}\times[6m(m+2)\epsilon^2-(11m^2+32m-16)\epsilon\delta+4(m+8)(m-1)\delta^2]$

physical negative value of v^* for $\epsilon>0$, whereas for $\epsilon<0$ it lies in the physical region of Hamiltonian space, but is unstable. Similarly, fixed point V has $v^*+w^*=-\delta/16$, which is unphysical for $\delta>0$, while for $\delta<0$ the point is never stable. Thus, in the discussion that follows, we shall neglect these two fixed points.

We now discuss the crossovers between the physically accessible fixed points of Table I. The results of this analysis are summarized in Figs. 1 and 2, which appear in the Introduction.

When $w=0$ (no long-range correlations), the Gaussian fixed point is stable for $\epsilon<0$, while for $\epsilon>0$ we find crossover between the short-range-disorder and pure fixed points at $m=4$. This is consistent with the Harris criterion which states that the pure fixed point is stable against short-range disorder if $d\nu_{\text{pure}}-2>0$. To order ϵ, this becomes

$$\frac{m-4}{m+8}\epsilon>0\ . \tag{4.1}$$

For a model with long-range-correlated disorder ($w>0$) and $d<4$ ($\epsilon>0$) we find (see Table II and Fig. 8) crossover to the long-range-disorder fixed point from (1) the pure fixed point at

$$0=\frac{2(m+2)}{m+8}\epsilon-\delta=a-\frac{2}{\nu_{\text{pure}}} \tag{4.2}$$

when $m>4$, or from (2) the short-range fixed point at

$$0=\frac{3m}{4(m-1)}\epsilon-\delta=a-\frac{2}{\nu_{\text{short}}} \tag{4.3}$$

when $m<4$, where ν_{pure} and ν_{short} are the pure and

TABLE II. Eigenvalues of the fixed points of Table I.

Fixed point		$\lambda_r=1/\nu$	λ_1	λ_2	λ_3
I.	Gaussian	2	ϵ	ϵ	δ
II.	Pure	$2-\frac{m+2}{m+8}\epsilon$	$-\epsilon$	$\frac{4-m}{m+8}\epsilon$	$\delta-\frac{2(m+2)}{m+8}\epsilon$
III.	Unphysical	$2-\frac{1}{4}\epsilon$	$-\frac{1}{2}\epsilon$	$-\epsilon$	$\delta-\frac{1}{2}\epsilon$
IV.	Short-range disorder	$2-\frac{3m}{8(m-1)}\epsilon$	$-\epsilon$	$\frac{m-4}{4(m-1)}\epsilon$	$\delta-\frac{3m}{4(m-1)}\epsilon$
V.	Unphysical	$2-\frac{1}{2}\delta$	$\epsilon-3\delta$	$\frac{1}{2}[\epsilon-4\delta\pm(8\delta^2-4\epsilon\delta+\epsilon^2)^{1/2}]$	
VI.	Long-range disorder	$2-\frac{1}{2}\delta$	See Fig. 6		

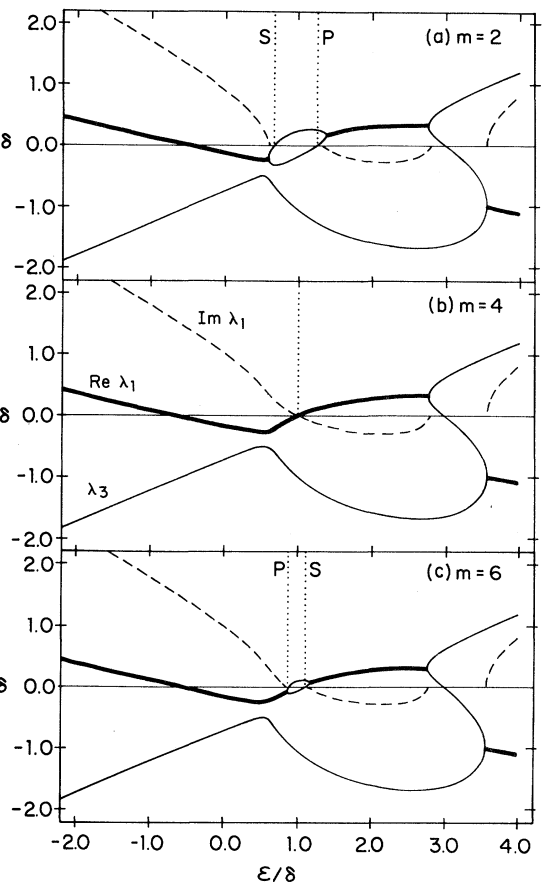

FIG. 8. Eigenvalues λ_1, λ_2, and λ_3 of the long-range-disorder fixed point for (a) $m=2$, (b) $m=4$, and (c) $m=6$. The solid lines are the real parts, the heavy line is the real part of a complex-conjugate pair, and the thin line is the other (real) eigenvalue. The dashed line is the imaginary part of one of the pair. Notice that we plot λ/δ and ϵ/δ, so if $\delta<0$ the plot must be inverted. The values of ϵ/δ marked P and S are those at which the long-range fixed point is coincident with the pure and the short-range fixed points, respectively. For $m=4$ the pure and short-range fixed points are coincident, and the long ranged meets them at $\epsilon/\delta=1$. The long-range fixed point is stable when the real parts of the three eigenvalues are all negative. Notice that for $\epsilon/\delta\approx-0.5$ the real part of the conjugate-pair eigenvalues changes sign. This corresponds to a subcritical Hopf bifurcation (see text).

short-range-disorder correlation-length exponents, respectively. This behavior is consistent with the extended Harris criterion of Sec. II: $a\nu-2>0$ for stability, as we see by the second equalities of Eqs. (4.2) and (4.3), which are true to the order we are working, $O(\epsilon)$. The long-range fixed point is then stable for all δ (of order ϵ) greater than the value which satisfies (4.2) or (4.3). In this calculation we do *not* find a crossover to smeared behavior as the range of the correlations increases for all $\delta=O(\epsilon)$ and $\epsilon>0$.

We now consider the case $d>4$ ($\epsilon<0$). For $\delta>0$, with δ sufficiently large, the long-range-disorder fixed point is stable. As δ decreases to $\delta_H\approx1.8|\epsilon|$ the long-range fixed point goes unstable by a subcritical Hopf bifurcation and there is no stable fixed point remaining in the physical region. The RG flows spiral outwards from the (now unstable) long-range fixed point, reaching large values of u, v, and w where our perturbation expansion cannot be expected to be valid. It seems reasonable to conjecture that this spiral runaway is a sign of a smeared transition, although we cannot say more about it from the results of this calculation. We can see the domain of attraction of the fixed point shrink as δ approaches δ_H from above, as we expect in the subcritical Hopf bifurcation scenario (see above). This is illustrated *schematically* in Fig. 7; the actual RG flows are too close together to allow for a clear reproduction since the real part of the eigenvalue which determines the decay or growth of the spiral is small.

As δ decreases to 0 the Gaussian fixed point, which was previously unstable, interchanges stability with the unphysical (V) fixed point which was in the unphysical region. The Gaussian fixed point remains stable for all $\delta<0$ and $\epsilon<0$, consistent with the extended Harris criterion $a\nu_{\text{Gauss}}-2>0$, satisfied for $a>4$ ($\delta<0$) since $\nu_{\text{Gauss}}=\frac{1}{2}$.

Thus for $\epsilon<0$ we find rather surprising behavior. For correlations of sufficiently long range the long-range-disorder fixed point is stable and describes the transition. It is as the range of the correlation is *decreased* that the subcritical Hopf bifurcation occurs, leading to spiral runaway of the RG flows, which we would like to interpret as signaling a smeared transition. The runaway occurs until δ goes negative, at which point the normal Gaussian fixed point becomes stable.

$$\nu=\frac{2}{a}\ , \tag{4.4}$$

to $O(\delta)$. In fact, we believe that this result is exact. In Appendix A we shall find that the long-range fixed point arising from correlations $\sim r^{-a}$ is unstable to perturbation by a correlation $\sim r^{-a'}$ for any $a'<a$. More generally, for any random-temperature-disordered system which undergoes a continuous transition we expect that if the long-range nature of the correlations in the disorder is relevant, then it is the *longest*-range part which determines the asymptotic critical behavior. Thus, any system which exhibits behavior determined by the power of the falloff of the long-range correlations a will be unstable to perturbation by any $a'<a$, but stable against perturbation by any $a''>a$. Applying the extended Harris criterion to the system, we expect $a'\nu-2<0$, but $a''\nu-2>0$ for all $a'<a<a''$. These inequalities can be satisfied only if $a\nu-2=0$, which gives (4.4). In Sec. V we argue that this is also true for long-range-correlated percolation, and find that the scaling relation (4.4) is satisfied by the percolation behavior of the triangular Ising model at $T=T_c$.

The other exponents of the long-range-disorder fixed point are determined by scaling relations[16] using the exact result (4.4) and the fact mentioned above that $\eta=O(\epsilon^2)$. We find

$$\alpha=2(a-d)/a\ ,$$
$$\beta=(2-\epsilon)/a+O(\epsilon^2)\ , \quad (4.5)$$
$$\gamma=4/a+O(\epsilon^2)\ .$$

For much of the range of ϵ and δ for which the long-range fixed point is stable, its eigenvalues in directions other than the temperature direction are complex. As mentioned above, this will lead to oscillating corrections to scaling.[20]

B. Case $m=1$

The analysis of the system with $m=1$ is complicated by the fact that there is an accidental degeneracy in the recursion relations when $w=0$. This leads to a short-range-disorder fixed point with u and v proportional to $\epsilon^{1/2}$, rather than of order ϵ as found above for $m>1$. Other than the short-range fixed point, the fixed points of the recursion relations found for the case $m>1$ remain well behaved for $m=1$ and are given in Table I.

For $d>4$ ($\epsilon<0$), the short-range-disorder fixed point is complex and does not affect the behavior of the system. The behavior is determined by the crossover between the Gaussian and long-range-disorder fixed points exactly as in the case $m>1$.

However, for $d<4$ ($\epsilon>0$), the short-range-disorder fixed point of order $\epsilon^{1/2}$ describes the behavior of the system with short-range-correlated disorder. As the range of the correlations is increased, the short-range-disorder fixed point remains stable (and the long-range fixed point unstable) for all δ of order ϵ; crossover to the long-range fixed point occurs when δ is of order $\epsilon^{1/2}$. We shall find that this behavior is consistent with the extended Harris criterion since $\nu_{\text{short}}=\frac{1}{2}+O(\epsilon^{1/2})$.

We now write down the recursion relations for the case $m=1$ and $\delta=O(\epsilon^{1/2})$. We will find that for this case the fixed points of the recursion relations have u^*, v^*, and r^* of order $\epsilon^{1/2}$, while $4v^*-3u^*$ and w^* are of order ϵ. Because of the degeneracy in the equations, we must include terms of third order in u and v in order to determine the fixed-point values. The recursion relations are

$$\frac{dr}{dl}=2r+\frac{4(m+2)u}{1+r}-\frac{8v}{1+r}+O(\epsilon)\ , \quad (4.6a)$$

$$\frac{du}{dl}=48uv-36u^2+\epsilon u+48uw+816u^3$$
$$-2208u^2v+1312uv^2+O(\epsilon^2)\ , \quad (4.6b)$$

$$\frac{dv}{dl}=32v^2-24uv+\epsilon v+48vw+16w^2$$
$$+672v^3-1056uv^2+240u^2v+O(\epsilon^2)\ , \quad (4.6c)$$

$$\frac{dw}{dl}=\delta w-24uw+16vw+16w^2+O(\epsilon^2)\ , \quad (4.6d)$$

where the coefficients of the terms of third order in u and v are derived from previous work of other investigators.[7] Notice that Eqs. (4.6b) and (4.6c) are degenerate to lowest order when $w=0$ in that the combination $4v-3u$ appears in both expressions. This leads to the apparent divergence at $m=1$ of the short-range-disorder fixed point found for $m>1$, which appears in Table I. The short-range fixed point is $O(\epsilon^{1/2})$, with values of

$$r^*=-\tfrac{1}{2}(6\epsilon/53)^{1/2}\ ,$$
$$u^*=\tfrac{1}{6}(6\epsilon/53)^{1/2}\ , \quad (4.7)$$
$$v^*=\tfrac{1}{8}(6\epsilon/53)^{1/2}\ ,$$
$$w^*=0\ ,$$

and correlation-length exponent $\nu_{\text{short}}=\frac{1}{2}+\frac{1}{4}(6\epsilon/53)^{1/2}$. This fixed point is stable in the u and v directions,[14] and in the w direction it has eigenvalue $\delta-2(6\epsilon/53)^{1/2}$. Thus the short-range fixed point becomes unstable (exchanging stability with the long-range-disorder fixed point) when

$$O=2(6\epsilon/53)^{1/2}-\delta=a-2/\nu_{\text{short}}\ , \quad (4.8)$$

which is consistent with the extended Harris criterion by the second equality, which is valid to $O(\epsilon^{1/2})$.

Taking $w\neq0$ breaks the degeneracy, so that the long-range-disorder fixed point of Table I is a solution of Eqs. (4.6). However, for $\delta=O(\epsilon)$ the long-range fixed point is unstable; only when $\delta>2(6\epsilon/53)^{1/2}$ is it stable, having exchanged stability with the short-range-disorder fixed point according to the extended Harris criterion. For $\delta=O(\epsilon^{1/2})$, the long-range fixed point has the values of

$$r^*=-\delta/4\ ,$$
$$u^*=\delta/12\ , \quad (4.9)$$
$$v^*=\delta/16\ ,$$
$$w^*=O(\epsilon)\ ,$$

to $O(\epsilon^{1/2})$. The long-range correlation-length exponent is still $\nu_{\text{long}}=2/a$, so that the extended Harris criterion is marginal when the long-range nature of the correlations is relevant, as we expect. One of the eigenvalues in the u,v,w subspace is $-\delta$, while the other two vanish to $O(\epsilon^{1/2})$.

We are confident that the above picture of the crossover between the fixed points is correct since it is in accord with the extended Harris criterion. However, working only to $O(\epsilon^{1/2})$ we have not proved it. We would need $O(\epsilon)$ results to check that when the long-range fixed point is stable, it in fact is in the physical region (i.e., it has $w^*>0$). Also, we would have to work to higher order to map out the stability of the long-range fixed point for all δ of order $\epsilon^{1/2}$; there may be crossover, via a Hopf bifurcation, to some new behavior as the range of the correlations is increased.

V. LONG-RANGE-CORRELATED PERCOLATION

In this section we argue that the extended Harris criterion can be applied to percolation problems[22] in which there are long-range correlations in the site- (or bond-[23]) occupation probabilities. In addition, since the extended Harris criterion is applicable, we expect the scaling relation $\nu=2/a$ to hold when the long-range nature of the correlations is relevant. We shall find that the results of Klein *et al.*[15] on the percolation properties of the triangular Ising model are in agreement with our analysis.

For conciseness, consider a site-percolation problem. It is described by a variable θ_i at each site i which takes on the value 1 or 0 if the site is occupied or vacant, respectively. Then, the probability that a site is occupied is $p=\langle\theta_i\rangle$ where $\langle\cdots\rangle$ is an ensemble average, which we take to be homogeneous and isotropic. The system percolates when $p\geq p_c$. We define the connected site-occupation correlation function

$$g(i,j)=\langle\theta_i\theta_j\rangle^c\equiv\langle\theta_i\theta_j\rangle-\langle\theta_i\rangle^2\,. \qquad (5.1)$$

In the normal case of no correlations between sites,

$$g(i,j)=p(1-p)\delta_{ij}\,, \qquad (5.2)$$

which vanishes unless $i=j$. We wish to consider the case of power-law falloff of correlations

$$g(i,j)\sim|\vec{x}_i-\vec{x}_j|^{-a}\,. \qquad (5.3)$$

We can derive the extended Harris criterion for this case following Sec. II very closely. Dividing the system up into regions the size of the pure percolation correlation length ξ, we argue that each region will have an effective occupation probability p_V which is the average of the θ_i over the region. For the picture of a pure transition to be consistent, the variance in p_V must be negligible compared to p_c-p as $p\to p_c$. The resulting equations are identical to those of Sec. II, and we regain the extended Harris criterion, Eq. (2.6) for this case.

For percolation, the correlation-length exponent ν is defined by

$$\xi\sim(p_c-p)^{-\nu}\,, \qquad (5.4)$$

where ξ is the percolation correlation length. The values of ν for short-range-correlations in θ_i are[24] $\nu=1.33$, 0.84, 0.66, and $\frac{1}{2}$ for $d=2$, 3, 4, and ≥ 6, respectively. Thus $d\nu-2>0$, and the Harris criterion states that short-range-correlation disorder is irrelevant. This is what we expect since the percolation problem, in a sense, already has short-range disorder with the correlation function (5.2). For small enough a, however, $a\nu-2$ will be negative, so that the long-range correlations become relevant. By analogy with the results for spin models in Sec. IV, we anticipate that when the long-range correlations are relevant, the percolation exponents will be modified, and that they will depend explicitly on the parameter a. As in the preceding section, the argument that the percolation exponents should be determined by the longest-range contribution to the correlation function $g(i,j)$, together with the extended Harris criterion, suggests that

$$\nu=2/a \qquad (5.5)$$

in the regime where the percolation exponents vary with a. (This implies that an addition to the correlation function (5.3) of a term proportional to $|\vec{x}_i-\vec{x}_j|^{-a'}$ will be relevant if and only if $a'<a$.)

Klein *et al.*[15] studied the percolation of clusters of "down" spins in the triangular Ising model in an external magnetic field h for $T\geq T_c$. For this problem $\theta_i=\frac{1}{2}(1-s_i)$, where $s_i=\pm 1$ is an Ising spin at site i. Thus $p=\frac{1}{2}(1-m)$, where $m=\langle s_i\rangle$, and the site-occupation correlation function $g(i,j)=\frac{1}{4}g_I(i,j)$, where $g_I(i,j)=\langle s_is_j\rangle^c$ is the Ising connected spin-correlation function. For the triangular site percolation problem $p_c=\frac{1}{2}$, so the percolation threshold is at $m=0$ and $p_c-p=\frac{1}{2}m$. Fixing $T\geq T_c$, m can be smoothly varied by applying an external field h, approaching the percolation threshold as $h\to 0$.

For $T>T_c$ the site-occupation correlation function is short ranged,

$$g(i,j)=\tfrac{1}{4}g_I(i,j)\sim e^{-|\vec{x}_i-\vec{x}_j|/\xi_I}\,, \qquad (5.6)$$

where $\xi_I(h,T)$ is the Ising correlation length. Thus we predict that for $T>T_c$ the site-occupation correlations are irrelevant and the system should exhibit normal percolation critical exponents.

However, for $T=T_c$ the correlations are long ranged,

$$g(i,j)=\tfrac{1}{4}g_I(i,j)\sim|\vec{x}_i-\vec{x}_j|^{-\eta_I} \tag{5.7}$$

(with $\eta_I=\frac{1}{4}\equiv a$) for distances $|\vec{x}_i-\vec{x}_j|$ less than the correlation length $\xi_I(h,T_c)$, with $\xi_I\to\infty$ as $h\to 0$. It is the form of the site-occupation correlation function $g(i,j)$ for lengths *less* than the correlation length ξ which is used in the derivation of the extended Harris criterion sketched above. Thus the criterion should hold so long as the percolation correlation length ξ is at most of the order of the Ising percolation length ξ_I as $h\to 0$ and thus $p\to p_c$. For $a=\frac{1}{4}$ the extended Harris criterion indicates that the long-range correlations are relevant, so the scaling relation (5.5) should hold,

$$\xi\sim(p_c-p)^{-2/a}\ . \tag{5.8}$$

For the Ising model at T_c we have $\xi_I\sim m^{-(\nu_I/\beta_I)}$, where ν_I and β_I are the exponents for the correlation length and the magnetization, respectively, at $h=0$ and $T<T_c$. The scaling relation[16] $2\beta_I=\nu_I(d-2+\eta_I)$ for the Ising critical exponents implies that $(\nu_I/\beta_I)=2/\eta_I$, so

$$\xi_I\sim m^{-2/\eta_I}\ . \tag{5.9}$$

For our percolation problem $(p_c-p)=\frac{1}{2}m$ and $a=\eta_I$. Thus ξ is of order ξ_I and it was consistent to apply the arguments leading to the extended Harris criterion.

The above results are identical to those found by Klein *et al.* using real-space RG techniques. They found that for $T>T_c$ the system had normal percolation critical behavior, while at $T=T_c$ they found different percolation critical behavior, with the percolation correlation length ξ proportional to the Ising correlation length ξ_I.

ACKNOWLEDGMENTS

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APPENDIX A: SUM OF POWER-LAW FORM FOR THE CORRELATION FUNCTION $g(x)$

In this section we consider a system with more general correlation in its random-temperature disorder of the form

$$g(x)\sim\sum_{i=1}^{N}A_i x^{-a_i}\ . \tag{A1}$$

This form leads to a Fourier-transformed correlation function

$$\bar{g}(k)\sim v+\sum_i w_i k^{-(d-a_i)} \tag{A2}$$

for small k. We write down the recursion relations and find the fixed points, finding that the long-range critical behavior is determined by a fixed point at which all of the $w_i=0$ except for the coefficient of the most divergent term of $\bar{g}(k)$, i.e., the one with the smallest a_i.

With the use of arguments very similar to those of Sec. III, the recursion relations are found to be

$$\frac{dr}{dl}=2r+\frac{4(m+2)u}{1+r}-\frac{8\Omega}{1+r}\ , \tag{A3a}$$

$$\frac{du}{dl}=\epsilon u-\frac{4(m+8)u^2}{(1+r)^2}+\frac{48u\Omega}{(1+r)^2}\ , \tag{A3b}$$

$$\frac{dv}{dl}=\epsilon v-\frac{8(m+2)uv}{(1+r)^2}+\frac{16v\Omega}{(1+r)^2}+\frac{16\Omega^2}{(1+r)^2}\ , \tag{A3c}$$

$$\frac{dw_i}{dl}=\delta_i w_i-\frac{8(m+2)uw_i}{(1+r)^2}+\frac{16w_i\Omega}{(1+r)^2}\ , \tag{A3d}$$

where $\Omega\equiv\bar{g}(1)=v+\sum w_i$ and $\delta_i=4-a_i$. These recursion relations have the same fixed points I through IV as before (see Table I) with all of the $w_i=0$; and then they have $2N$ fixed points with all but one $w_i=0$ (say $w_j\neq 0$) of the same form as the unphysical V and long-range-disorder fixed points with δ replaced by δ_j. Diagonalizing around such a long-range fixed point, we find eigenvalues in the w_k direction,

$$\lambda_k=\delta_k-\delta_j\ . \tag{A4}$$

Thus, each long-range fixed point is unstable to a larger δ_i. There is only one stable long-range fixed point, the one with the *maximum* δ_i, i.e., the *minimum* a_i, call it a_1. At this fixed point all of the $w_i=0$ for $i>1$, and we can ignore them. The system's critical properties are the same as those of the system considered in the body of this paper with $\bar{g}(k)\sim v+w_1k^{-(d-a_1)}$.

APPENDIX B: QUENCHED DISORDER WITH GENERAL ISOTROPIC LONG-RANGE CORRELATIONS

In this appendix we develop recursion relations for a system with arbitrarily correlated random-temperature disorder, expanding to second order in the normal four-point coupling u and in the Fourier-transformed correlation function $g(k)$. (For clarity we shall neglect the overbar on the Fourier-transformed correlation function in this appendix). We continue to assume, however, that the system is isotropic so that $g(k)$ depends only on the magnitude of k. Unlike the cases considered above, we do not assume that the correlation function $g(k)$ can be characterized by a finite number of parameters, so we must find recursion relations for $g(k)$ for all k less than the cutoff, which we take to be unity. Since we do not assume that $g(k)$ behaves asymptotically as a power law for small k, we do not find scale-invariant critical behavior characterized by fixed points of the RG recursion relations. However, the recursion relations we derive could be used to investigate the behavior of systems with perhaps more realistic choices of $g(k)$.

As in Sec. III, we replicate the system and find an effective Hamiltonian with a momentum-dependent interaction of strength $g(k)$ acting between replicas. Rescaling the Hamiltonian in the normal way with length rescaling factor $b=e^l$, we find that the rescaled interaction $g(k)$ is $b^\epsilon g(k'/b)$, where we ignore η because it will be of higher order, as in Sec. III. Taking the derivative of the rescaled $g(k)$ with respect to l at $l=0$, we get $\delta(k)g(k)$, where

$$\delta(k)\equiv\epsilon-\frac{\partial \ln g(k)}{\partial \ln k} . \quad \text{(B1)}$$

Expanding in u and $g(k)$, we obtain diagrams identical to those of Figs. 4 and 5, and find the recursion relations

$$\frac{dr}{dl}=2r+\frac{4(m+2)u}{1+r}-\frac{8g(1)}{1+r} , \quad \text{(B2a)}$$

$$\frac{du}{dl}=\epsilon u-\frac{4(m+8)u^2}{(1+r)^2}+\frac{48ug(1)}{(1+r)^2} , \quad \text{(B2b)}$$

$$\frac{dg(k)}{dl}=\delta(k)g(k)-\frac{8(m+2)ug(k)}{(1+r)^2}+\frac{16g(k)g(1)}{(1+r)^2}+\frac{16g(1)^2}{(1+r)^2} . \quad \text{(B2c)}$$

The recursion relations (B2) can be seen to be equivalent to those found for the case $g(k)\sim v+wk^{-(d-a)}$ in Sec. III for v and w if one considers Eq. (B2c) for $dg(k)/dl$ in the two limits $k=1$ and $k\to 0$. In general, the recursion relations (B2) must be integrated numerically. Functions such as the susceptibility, the order-parameter correlation function $C(q,T)$, or the free energy can be obtained in the usual way[25] by stopping the integration at the appropriate value of l, such that the renormalized $|r|$ is of order unity, or such that qe^l is of order unity. Thus, within our approximations (where $\eta=0$), we find for the susceptibility χ and correlation length ξ above T_c

$$\chi=\xi^2=e^{2l^*} , \quad \text{(B3)}$$

where l^* is the value of l such that $r=1$.

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[12] By a "smeared transition" we mean a situation where various criteria would lead to different choices for the transition temperature. For example, in the model considered by McCoy and Wu (Refs. 4 and 5), there is a range of temperature when the susceptibility is infinite; the onset of nonzero magnetization does not coincide with the highest temperature of infinite susceptibility and the specific heat is infinitely differentiable at all temperatures.
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[16] See, for example, S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, 1976), p. 114.
[17] See, for example, Refs. 11 and 16.
[18] If the jth cumulant $\langle \delta r(\vec{x}_1)\delta r(\vec{x}_2)\cdots \delta r(\vec{x}_j)\rangle_{\text{av}}^c$ scales as $(\text{length})^{-a_j}$, then the scaling power of the corre-

sponding term in the cumulant expansion for $H_{\rm eff}$ is $(2-\eta)j-a_j$. Thus, ignoring η near $d=4$, if $a_j<a_c(j)=2j$, then the jth cumulant will be relevant. However, cumulants generated by the RG have $a_j=(j-1)a$, where a is the power of the second cumulant as in the body of the paper. Thus the generated jth cumulant only becomes relevant for $a<2j/(j-1)$, and all $j>2$ generated cumulants are irrelevant for a near 4.

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