Quantized motion of three two-dimensional electrons in a strong magnetic field

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We have found a simple, exact solution of the Schrödinger equation for three two-dimensional electrons in a strong magnetic field, given the assumption that they lie in a single Landau level. We find that the interelectronic spacing has characteristic values, not dependent on the form of the interaction, which change discontinuously as pressure is applied, and that the system has characteristic excitation energies of approximately $0.03e^2/a_0$, where a_0 is the magnetic length.

Tsui, Störmer, and Gossard[1] have recently discovered that the quantum Hall effect can occur at the fractional value $\frac{1}{3}(e^2/h)$. Anomalous behavior in parallel conductance consistent with their observation has also been reported by Ebert, von Klitzing, Probst, and Ploog.[2] Unlike the ordinary quantum Hall effect,[3,4] in which the Hall conductance of a two-dimensional electron gas is accurately quantized to integral multiples of e^2/h, the new effect is caused by the condensation of the system into a macroscopic collective ground state reminiscent of that of a superconductor, in that it can carry current with diminished,[1] or, conceivably, no resistive loss. While occurrence of the effect within a spin-polarized Landau level and at a particular electron density preclude its being ordinary spin-pairing superconductivity, there has been speculation that it might be a form of Fröhlich superconductivity. It has been known for some time[5] that the electron gas under conditions present in the experiment is unstable to the formation of a charge-density wave with a period comparable to the magnetic length

$$a_0=\left[\frac{\hbar c}{eH_0}\right]^{1/2},$$

where H_0 is the magnetic field strength. It is intuitively appealing that a charge-density inhomogeneity should be the relevant order parameter, since it is effective at separating the electrons and thus minimizing their repulsive Coulomb energy. On the other hand, Yoshioka's[6] recent Hartree-Fock calculations of the charge-density wave, while yielding a condensation energy of roughly e^2/a_0 per particle, has not produced a significant reason for $\frac{1}{3}$ to be preferred. The Hartree-Fock theory is a mean-field theory, and thus has the usual mean-field difficulties in two dimensions, and can also exclude important physics. However, the difficulty of obtaining a commensurability energy from the Hartree-Fock ground state is a sufficiently serious problem to have cast doubt on the charge-density-wave hypothesis.

The purpose of this paper is to report the results of calculations we believe shed some light on the nature of this effect. We have succeeded in diagonalizing exactly the problem of three two-dimensional electrons in a strong magnetic field. In contrast to the Hartree-Fock solution of the same problem, we find quantization of the interelectronic spacing in a manner independent of the interaction. We also find incompressibility, and characteristic excitation energies of $0.03e^2/a_0$, both of which are consistent with experiment.[1]

We consider electrons confined to the x-y plane and subjected to a magnetic field $H_0\hat{z}$. We have a single-body Hamiltonian of the form

$$H=\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla}-\frac{e}{c}\vec{A}\right]^2, \tag{1}$$

with the symmetric gauge vector potential given by

$$\vec{A}=\frac{H_0}{2}(y\hat{x}-x\hat{y}). \tag{2}$$

We adopt energy and length scales in which the cyclotron energy

$$\hbar\omega_c=\hbar\frac{eH_0}{mc}$$

and the magnetic length

$$a_0=\left[\frac{\hbar}{m\omega_c}\right]^{1/2}=\left[\frac{\hbar c}{eH_0}\right]^{1/2}$$

are 1. The eigenstates of H are

$$|m,n\rangle=\frac{1}{(2^{m+n+1}\pi m!n!)^{1/2}}e^{(1/4)(x^2+y^2)}\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right]^m\left[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right]^n e^{-(1/2)(x^2+y^2)}, \tag{3}$$

with

$$H\,|\,m,n\,\rangle=(n+\tfrac{1}{2})\,|\,m,n\,\rangle\ . \tag{4}$$

$|\,m,n\,\rangle$ is also an eigenstate of angular momentum with eigenvalue $(m-n)$. The manifold of states with energy $(n+\frac{1}{2})$ constitutes the nth Landau level. We shall be considering primarily the states of the lowest Landau level, written more succinctly as

$$|\,m\,\rangle\equiv|\,m,0\rangle=\frac{1}{(2^{m+1}\pi m!)^{1/2}}z^m e^{-(1/4)|z|^2}\ , \tag{5}$$

with $z=x-iy$. These are described qualitatively as states of cyclotron motion about the origin with angular momentum m and orbit radius $\sqrt{2m}$, since

$$\langle m\,|\,r^2\,|\,m\,\rangle=2(m+1)\ . \tag{6}$$

We observe also that

$$\left\langle m\left|\frac{1}{r}\right|m\right\rangle=\frac{(2m)!}{2^{2m+1}(m!)^2}\sqrt{2\pi}\ , \tag{7}$$

which limits to $1/\sqrt{2m}$ with large m.

The physical origin of the quantization of electron separation in our picture is that particles of like sign in a strong magnetic field do not repel one another, but orbit about their center of mass with a speed proportional to the Coulombic force between them. We consider two electrons described by the Hamiltonian

$$H=\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla}_1-\frac{e}{c}\vec{A}_1\right]^2+\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla}_2-\frac{e}{c}\vec{A}_2\right]^2+\frac{e^2}{r_{12}}\ , \tag{8}$$

where $\vec{\nabla}_1$ denotes differentiation with respect to the first coordinate and $\vec{A}_1$ denotes the vector potential evaluated at the position of the first particle. This problem separates in the usual way: We adopt center-of-mass and internal coordinates given by

$$\bar{z}=\frac{z_1+z_2}{2},\quad z_a=\frac{z_1-z_2}{\sqrt{2}}\ , \tag{9}$$

and obtain a Hamiltonian for the internal motion of the form[7]

$$H_{\text{internal}}=\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla}_a-\frac{e}{c}\vec{A}_a\right]^2+\frac{1}{\sqrt{2}r_a}\ . \tag{10}$$

The potential conserves angular momentum, and thus, with m odd to comply with antisymmetry, the eigenstates take the form

$$\Psi(x,y)=R(r)e^{-im\phi}\ , \tag{11}$$

with $x_a-iy_a=re^{-i\phi}$ and

$$-\frac{1}{2}\left[\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{m^2}{r^2}\right]-\frac{m}{2}R+\frac{r^2}{8}R+\frac{e^2}{\sqrt{2}r}R=ER\ , \tag{12}$$

where E is the energy eigenvalue. This is the radial equation of a two-dimensional harmonic oscillator with an added repulsive core of strength e^2. If this core is sufficiently weak, the eigenstates are approximated by the free-particle states $|\,m\,\rangle$, and we have

$$H_{\text{internal}}\,|\,m\,\rangle\simeq(E_m+\tfrac{1}{2})\,|\,m\,\rangle\ , \tag{13}$$

with

$$E_m=\frac{e^2}{\sqrt{2}a_0}\left\langle m\left|\frac{1}{r}\right|m\right\rangle\ . \tag{14}$$

This approximation, which improves with increasing m, is extremely good under the conditions of the experiment. In Fig. 1 we compare a numerical solution of Eq. (12) for $m=1$ with $|\,1\,\rangle$ for the case $e^2/a_0=\hbar\omega_c$. We find a difference of 20% in the position of the peak in Ψ and 7% in the energy eigenvalue. If we assume a value of 13 for the dielectric constant of GaAs and an effective mass of 0.07, we find that this corresponds to an experimental magnetic field of 6 T, which is a representative value.[1,2] Thus we have a physical picture in which the particles orbit one another at distance r, as they would in the absence of the Coulomb interaction, but with a *negative* binding energy e^2/r. Landau-

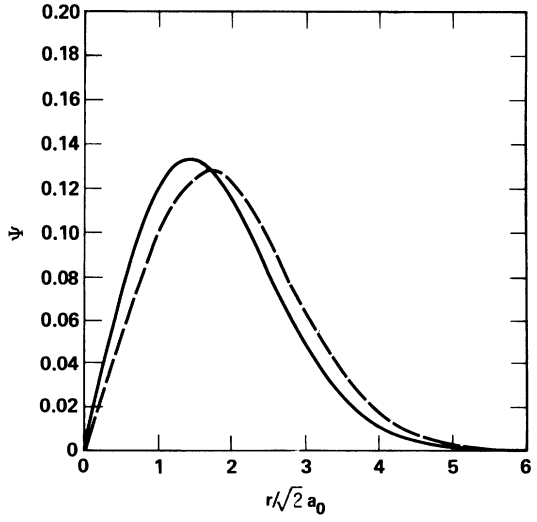

FIG. 1. Comparison of the free-particle state $|\,1\,\rangle$ (solid line) with the correct nodeless solution of Eq. (12) for $m=1$ (dashed line).

level mixing is a small perturbation. Also, restriction of the wave function to the lowest Landau level turns an otherwise effective one-body problem into a zero-body problem, in that the free-particle solutions diagonalize H_{internal}. We know, therefore, that the three-particle problem will reduce to an effective one-body problem. This is the reason it can be solved simply.

We now consider three electrons, described by the Hamiltonian

$$H=\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla}_1-\frac{e}{c}\vec{A}_1\right]^2+\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla}_2-\frac{e}{c}\vec{A}_2\right]^2+\left[\frac{\hbar}{i}\vec{\nabla}_3-\frac{e}{c}\vec{A}_3\right]^2+\frac{e^2}{r_{12}}+\frac{e^2}{r_{23}}+\frac{e^2}{r_{31}}\ . \tag{15}$$

As before, we separate H with a center-of-mass coordinate transformation, given by

$$\begin{aligned}\bar{z}&=\frac{z_1+z_2+z_3}{3}\ ,\\ z_a&=\left(\tfrac{2}{3}\right)^{1/2}\left[\left[\frac{z_1+z_2}{2}\right]-z_3\right]\ ,\\ z_b&=\frac{1}{\sqrt{2}}(z_1-z_2)\ .\end{aligned} \tag{16}$$

The internal Hamiltonian becomes

$$\begin{aligned}H_{\text{internal}}=&\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla}_a-\frac{e}{c}\vec{A}_a\right]^2+\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla}_b-\frac{e}{c}\vec{A}_b\right]^2\\ &+\frac{e^2}{\sqrt{2}}\left[\frac{1}{|z_b|}+\frac{1}{\left|-\frac{1}{2}z_b+\frac{\sqrt{3}}{2}z_a\right|}+\frac{1}{\left|-\frac{1}{2}z_b-\frac{\sqrt{3}}{2}z_a\right|}\right]\ .\end{aligned} \tag{17}$$

The antisymmetry condition is that the wave function be odd in z_b and symmetric under rotation by $\pm 120°$ in the a-b plane. A complete orthonormal basis which satisfies these conditions is given by the functions

$$|m,n\rangle=\frac{1}{[2^{6m+4n+1}(3m+n)!n!\pi^2]^{1/2}}\left[\frac{(z_a+iz_b)^{3m}-(z_a-iz_b)^{3m}}{2i}\right](z_a^2+z_b^2)^n e^{-(1/4)(|z_a|^2+|z_b|^2)}\ . \tag{18}$$

Orthonormality is most easily verified if the coordinate tranformation

$$\tilde{z}_1=\frac{z_a+iz_b}{\sqrt{2}},\quad \tilde{z}_2=\frac{z_a-iz_b}{\sqrt{2}} \tag{19}$$

is first performed. $|m,n\rangle$ is an eigenfunction of total angular momentum, with eigenvalue $M=3m+2n$. Since H_{internal} conserves M, one need consider only matrix elements of the Coulomb interaction between states of the same M. We evaluate these numerically by first expanding the polynomials in the manner

$$|m,n\rangle=\frac{1}{[2^{6m+4n+1}(3m+n)!n!\pi^2]^{1/2}}\sum_{k=1}^{M}a_k^{(m,n)}z_b^k z_a^{M-k}e^{-(1/4)(|z_a|^2+|z_b|^2)}\ , \tag{20}$$

and then applying Eq. (7) term by term, to obtain

$$\left\langle m,n\left|\frac{1}{|z_b|}\right|m',n'\right\rangle=\frac{1}{2^M}\left[\frac{2\pi}{(3m+n)!(3m'+n')!n!n'}\right]^{1/2}\sum_{k=1}^{M}a_k^{(m,n)}a_k^{(m',n')*}\left[\frac{(2k)!(M-k)!}{2^{2k}k!}\right]\ . \tag{21}$$

Invariance of the wave functions under rotation by $\pm 120°$ in the a-b plane causes the Coulomb matrix element to be $3/\sqrt{2}$ times this number. The matrix elements

$$\left\langle m,n\left|\frac{1}{|z_b|}\right|m',n'\right\rangle$$

are tabulated for $M<22$ in Table I. For small M,

TABLE I. Coulomb matrix elements across the states $|m,n\rangle$ defined by Eq. (18) in units of $(3/\sqrt{2})/(e^2/a_0)$. Quantum numbers m,n are indicated in parenthesis. $M=3m+2n$ is the total angular momentum. There are no states of $M=0$, 1, 2, or 4.

$M=3$	(1,0)	$5.679\,0797\times10^{-1}$			
$M=5$	(1,1)	$4.978\,3743\times10^{-1}$			
$M=6$	(2,0)	$4.201\,1726\times10^{-1}$			
$M=7$	(1,2)	$4.471\,2999\times10^{-1}$			
$M=8$	(2,1)	$4.032\,3072\times10^{-1}$			
$M=9$	(3,0)	$3.401\,7834\times10^{-1}$	$1.306\,1401\times10^{-2}$		
	(1,3)	$1.306\,1401\times10^{-2}$	$4.087\,2620\times10^{-1}$		
$M=10$	(2,2)	$3.868\,3028\times10^{-1}$			
$M=11$	(3,1)	$3.327\,5419\times10^{-1}$	$1.995\,4144\times10^{-2}$		
	(1,4)	$1.995\,4144\times10^{-2}$	$3.784\,8378\times10^{-1}$		
$M=12$	(4,0)	$2.927\,5434\times10^{-1}$	$8.410\,4645\times10^{-3}$		
	(2,3)	$8.410\,4645\times10^{-3}$	$3.714\,6562\times10^{-1}$		
$M=13$	(3,2)	$3.256\,0275\times10^{-1}$	$2.485\,1395\times10^{-2}$		
	(1,5)	$2.485\,1395\times10^{-2}$	$3.539\,2739\times10^{-1}$		
$M=14$	(4,1)	$2.878\,7081\times10^{-1}$	$1.424\,1562\times10^{-2}$		
	(2,4)	$1.424\,1563\times10^{-2}$	$3.572\,9604\times10^{-1}$		
$M=15$	(5,0)	$2.609\,2388\times10^{-1}$	$5.064\,2663\times10^{-3}$	$9.232\,9895\times10^{-4}$	
	(3,3)	$5.064\,2663\times10^{-3}$	$3.186\,7162\times10^{-1}$	$2.836\,9523\times10^{-2}$	
	(1,6)	$9.232\,9895\times10^{-4}$	$2.836\,9523\times10^{-2}$	$3.335\,0100\times10^{-1}$	
$M=16$	(4,2)	$2.833\,3270\times10^{-1}$	$1.926\,8913\times10^{-2}$		
	(2,5)	$1.926\,8913\times10^{-2}$	$3.443\,0872\times10^{-1}$		
$M=17$	(5,1)	$2.572\,6473\times10^{-1}$	$9.043\,9649\times10^{-3}$	$1.805\,9974\times10^{-3}$	
	(3,4)	$9.043\,9649\times10^{-3}$	$3.119\,5955\times10^{-1}$	$3.090\,5383\times10^{-2}$	
	(1,7)	$1.805\,9974\times10^{-3}$	$3.090\,5383\times10^{-2}$	$3.161\,7842\times10^{-1}$	
$M=18$	(6,0)	$2.376\,8325\times10^{-1}$	$3.264\,9505\times10^{-3}$	$5.959\,0829\times10^{-4}$	
	(4,3)	$3.264\,9505\times10^{-3}$	$2.790\,5413\times10^{-1}$	$2.354\,6263\times10^{-2}$	
	(2,6)	$5.959\,0829\times10^{-4}$	$2.354\,6263\times10^{-2}$	$3.324\,2217\times10^{-1}$	
$M=19$	(5,2)	$2.538\,6185\times10^{-1}$	$1.283\,0525\times10^{-2}$	$2.734\,2388\times10^{-3}$	
	(3,5)	$1.283\,0525\times10^{-2}$	$3.054\,7790\times10^{-1}$	$3.273\,1460\times10^{-2}$	
	(1,8)	$2.734\,2388\times10^{-3}$	$3.273\,1460\times10^{-2}$	$3.012\,5463\times10^{-1}$	
$M=20$	(6,1)	$2.348\,0284\times10^{-1}$	$5.982\,0064\times10^{-3}$	$1.259\,3072\times10^{-3}$	
	(4,4)	$5.982\,0064\times10^{-3}$	$2.749\,7763\times10^{-1}$	$2.715\,2470\times10^{-2}$	
	(2,7)	$1.259\,3072\times10^{-3}$	$2.715\,2470\times10^{-2}$	$3.215\,3298\times10^{-1}$	
$M=21$	(7,0)	$2.197\,4229\times10^{-1}$	$2.266\,4334\times10^{-3}$	$3.284\,2353\times10^{-4}$	$8.049\,7269\times10^{-5}$
	(5,3)	$2.266\,4334\times10^{-3}$	$2.506\,7438\times10^{-1}$	$1.635\,464\times10^{-2}$	$3.658\,0450\times10^{-3}$
	(3,6)	$3.284\,2353\times10^{-4}$	$1.635\,4641\times10^{-2}$	$2.332\,3702\times10^{-1}$	$3.403\,7984\times10^{-2}$
	(1,9)	$8.049\,7269\times10^{-5}$	$3.658\,0450\times10^{-3}$	$3.403\,7984\times10^{-2}$	$2.882\,2825\times10^{-1}$

the angular-momentum degeneracy is often 1, and in this case $|m,n\rangle$ is an eigenstate. However, even when the degeneracy is not 1, the largest off-diagonal matrix elements are typically 10 times smaller than differences in the diagonal ones. Thus the $|m,n\rangle$ tend to be good approximations to the eigenstates generally. For completeness we list all the energy eigenvalues for $M<39$ in Table II.

The first degenerate angular momentum is $M=9$ which is 3 times the minimum value of $M=3$ and thus corresponds roughly to a packing density of $\frac{1}{3}$. In this case, as is typical when there is degeneracy, the energy difference between the states of the same M is much greater than the smallest difference between states of adjacent M. This is because the n quantum number is energetically very unfavorable. In Fig. 2 we compare plots of charge density for the states $|3,0\rangle$ and $|1,3\rangle$, given that the center of mass lies at the origin and that one electron lies at $y=-3$. Fixing the position of one electron is necessary because the absolute charge densities are cylindrically symmetric. The evident tendency of the n quantum number to force charge onto the fixed electron can also be seen at other values of M.

The positive eigenvalues we calculate can be interpreted as binding energies if we apply pressure to the electrons by placing them in an external potential of the form

TABLE II. Eigenvalues of the Coulomb matrix in units of $(3/\sqrt{2})/(e^2/a_0)$.

$M=3$	0.567 907 97					
$M=5$	0.497 837 43					
$M=6$	0.420 117 26					
$M=7$	0.447 129 99					
$M=8$	0.403 230 72					
$M=9$	0.337 773 90	0.411 130 63				
$M=10$	0.386 830 28					
$M=11$	0.325 271 55	0.385 966 42				
$M=12$	0.291 865 70	0.372 354 27				
$M=13$	0.311 161 52	0.368 368 62				
$M=14$	0.285 062 93	0.360 103 92				
$M=15$	0.260 442 18	0.297 089 52	0.355 563 79			
$M=16$	0.277 753 97	0.349 887 45				
$M=17$	0.255 586 18	0.284 131 89	0.345 684 63			
$M=18$	0.237 408 16	0.270 397 53	0.341 353 86			
$M=19$	0.250 144 23	0.272 767 39	0.337 682 77			
$M=20$	0.233 829 64	0.263 329 82	0.334 153 98			
$M=21$	0.219 565 91	0.244 189 64	0.263 142 95	0.330 983 45		
$M=22$	0.229 940 05	0.256 759 49	0.327 998 95			
$M=23$	0.216 764 99	0.237 837 59	0.255 195 80	0.325 248 04		
$M=24$	0.205 223 87	0.225 839 85	0.250 786 81	0.322 667 16		
$M=25$	0.213 732 63	0.231 324 74	0.248 671 01	0.320 258 19		
$M=26$	0.202 957 58	0.221 635 21	0.245 430 48	0.317 992 78		
$M=27$	0.193 369 01	0.210 503 23	0.224 951 86	0.243 223 76	0.315 862 53	
$M=28$	0.200 526 82	0.217 432 57	0.240 655 51	0.313 851 62		
$M=29$	0.191 485 29	0.207 095 89	0.218 967 32	0.238 559 04	0.311 950 78	
$M=30$	0.183 356 81	0.197 960 42	0.213 329 26	0.236 399 25	0.310 149 52	
$M=31$	0.189 477 23	0.203 513 63	0.213 531 94	0.234 475 98	0.308 439 81	
$M=32$	0.181 758 76	0.195 287 54	0.209 404 16	0.232 591 27	0.306 813 94	
$M=33$	0.174 754 55	0.187 362 26	0.199 760 91	0.208 728 78	0.230 845 36	0.305 265 36
$M=34$	0.180 064 60	0.192 539 15	0.257 712 05	0.229 165 28	0.303 788 06	
$M=35$	0.173 376 43	0.185 155 47	0.195 881 88	0.204 557 25	0.227 580 88	0.302 376 77
$M=36$	0.167 259 62	0.178 286 78	0.189 749 77	0.202 282 39	0.226 064 00	0.301 026 69
$M=37$	0.171 922 12	0.182 867 74	0.191 978 23	0.200 936 58	0.224 621 08	0.299 733 52
$M=38$	0.166 055 14	0.176 437 22	0.186 957 84	0.199 122 05	0.223 239 86	0.298 493 37

$$V=\frac{\alpha}{2}(|z_1|^2+|z_2|^2+|z_3|^2)$$
$$=\tfrac{3}{2}\alpha|\bar{z}|^2+\frac{\alpha}{2}(|z_a|^2+|z_b|^2)\,. \quad (22)$$

Disregarding the center-of-mass binding, we have

$$\left\langle m,n\left|\frac{\alpha}{2}(|z_a|^2+|z_b|^2)\right|m',n'\right\rangle$$
$$=\delta_{mm'}\delta_{nn'}\alpha(M+2)\quad . \quad (23)$$

Thus, the form of pressure has no effect on the wave functions, but lowers the small-M states energetically relative to those of large M. In Fig. 3 we plot the total angular momentum of the ground state as a function of $1/\alpha$. We see discontinuous jumps to successive integral multiples of 3 as the pressure is lowered. Multiples of 3 occur because at any given pressure, the lowest energy state has $n=0$. Also in Fig. 3 we plot the root-mean-square area of the electrons. The area of the triangle whose vertices are the positions z_1, z_2, and z_3 is an operator given by

$$A=\tfrac{1}{2}\mathrm{Im}\left[\left[\frac{z_1+z_2}{2}-z_3\right](z_1-z_2)^*\right]$$
$$=\frac{\sqrt{3}}{4i}(z_a z_b^*-z_b z_a^*)\,, \quad (24)$$

with matrix elements given by

$$\langle m,n\,|\,A\,|\,m',n'\rangle=0 \quad (25)$$

and

$$\langle m,n\,|\,A^2\,|\,m',n'\rangle=\delta_{mm'}\delta_{nn'}\tfrac{3}{4}[(3m)^2+(M+2)]\,. \quad (26)$$

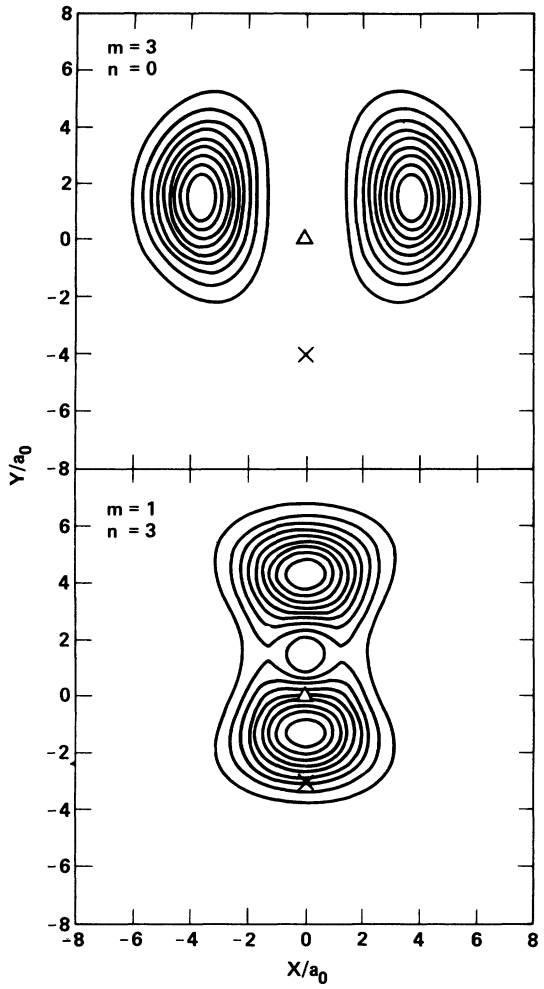

FIG. 2. Comparison of the charge densities associated with states of the same angular momentum ($M=9$), given that the center of mass lies at the origin ($\triangle$) and that one of the electrons lies at $y=-3$ ($\times$). Contours lie at integral multiples of 0.1 times the maximum charge density given the two constraints.

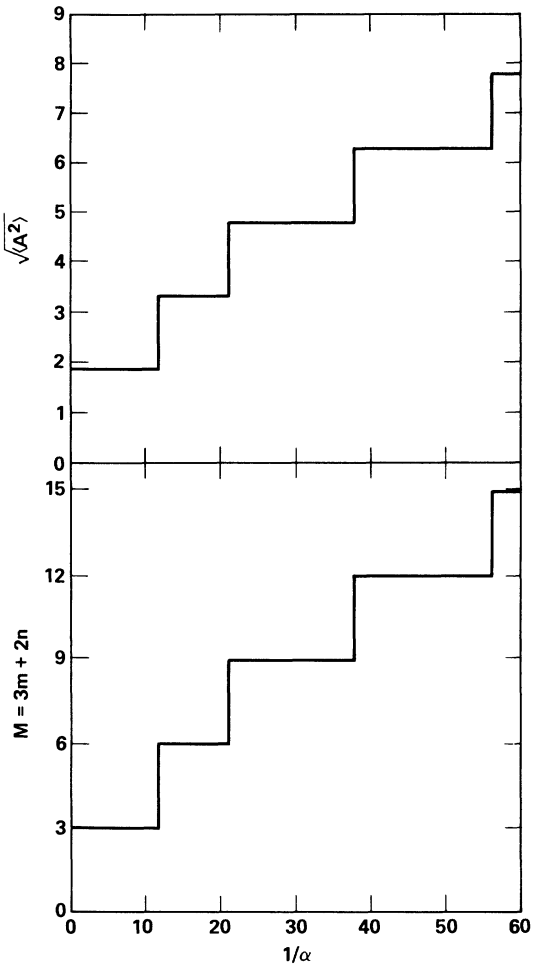

FIG. 3. Total angular momentum and reduced areas of the electron cluster as a function of inverse pressure $1/\alpha$. Units of α are $\sqrt{3/2}(e^2/a_0^3)$ and the units of A are a_0^2. Area operator is $1/\sqrt{3}$ times that defined by Eq. (24).

Thus the expected area also changes discontinuously with pressure. We interpret this as meaning that the cluster is incompressible. As this is due in part to the finiteness of the cluster, it is difficult to predict whether the property will persist as the number of particles is increased. However, if it does, it could enable the macroscopic ground state to flow past obstacles without generating collective excitations.

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[7]Note that since Eq. (2) is linear, $\vec{A}$ and $\vec{\nabla}$ transform as co- and contravariant vectors, respectively, under the transformation described by Eq. (9), e.g.,

$$\vec{A}_a = \frac{H_0}{2}\left[\frac{y_1-y_2}{2}\hat{x} - \frac{x_1-x_2}{2}\hat{y}\right] = \frac{H_0}{2}(y_a\hat{x} - x_a\hat{y}) .$$

Note also that this transformation has a Jacobian different from 1.