Derivation of an expression for the conductivity of superconductors in terms of the normal-state conductivity

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Recent experiments performed by Karecki *et al.* show that an expression for the conductivity of superconductors containing impurities, developed by the author but never published, describes well the behavior of stong-coupling superconductors. We present here a derivation of that expression.

In this paper we present the derivation of a general expression for the conductivity of a superconductor in terms of its normal-state conductivity. The reason for presenting this unpublished work[1] now is the recent publication of an article by Karecki *et al.* on the far-infrared transmission by a strong-coupling superconducting NbN film.[2] In that article the authors show that their experimental results are best fitted by our theory, and that in general this formulation is the only one to describe simply the optical properties of superconductors with strong coupling and finite mean free path.

It is also interesting to notice that the theory describes well strong-coupling superconductors, although it has been developed in the framework of the Bardeen-Cooper-Schrieffer (BCS) theory[3] which is usually thought to be limited to weak-coupling systems. An expression for the conductivity $\sigma^s(\omega,q,l)$ of a superconductor containing impurities is obtained by studying the response of the superconductor to an imposed vector potential

$$\vec{A}=\hat{y}A_0e^{i(qx-\omega t)} \quad . \tag{1}$$

In general the conductivity is complex:

$$\sigma^s(\omega,q,l)=\sigma_1^s(\omega,q,l)+i\sigma_2^s(\omega,q,l) \quad , \tag{2}$$

where l is the mean free path.

In this paper we derive a formula for $\sigma_1^s(\omega,q,l)$. $\sigma_2^s(\omega,q,l)$ can then be obtained from σ_1^s by using Kramer-Kronig relations[4] [see Eqs. (21) and (22)]. To obtain σ_1^s we compare two different expressions for the power lost by the electromagnetic field to the electrons of the system:

$$P=\frac{2\omega^2}{c^2}\sigma_1(\omega,q,l)|A_0|^2=\hbar\omega W \quad , \tag{3}$$

where W is the transition rate of the system from the ground state. Using perturbation theory with the interaction Hamiltonian

$$H'=\frac{ie\hbar}{mc}\vec{A}\cdot\vec{\nabla} \tag{4}$$

we get

$$W=\frac{2\pi}{\hbar^2}\sum_n|M_{n0}|^2\delta|(E_n-E_0-\hbar\omega) \quad , \tag{5}$$

where

$$M_{n0}=\left(\Phi_n\left|\sum_{k,k'}B_{k'k}C_{k'}^{\dagger}C_k\right|\Phi_0\right) \tag{6}$$

and

$$B_{k'k}=\int\psi_{k'}^{*}H'\psi_k d\tau \quad . \tag{7}$$

Here Φ_0 represents the ground state of the superconductor, Φ_n an excited state, E_0 and E_n the corresponding energies, whereas the ψ_k's are the electrons' eigenstates and the $C_k^{\dagger}$ and C_k the corresponding creation and annihilation operators.

If we take into account the relation $B_{k'k}=-B_{-k-k'}$, Eq. (6) becomes

$$\begin{aligned}M_{n0}&=B_{k'k}(\Phi_n|C_{k'}^{\dagger}C_k-C_{-k}^{\dagger}C_{-k'}|\Phi_0)\\&=B_{k'k}(V_kU_{k'}-U_kV_{k'})\end{aligned} \tag{8}$$

where

$$U_k^2=\frac{1}{2}\left(1-\frac{\epsilon_k}{E_k}\right) \text{ and } V_k^2+U_k^2=1 \quad . \tag{9}$$

In this expression ϵ_k is the energy of an electron measured relative to the Fermi surface,

$$\epsilon_k=\frac{\hbar^2k^2}{2m}-\frac{\hbar^2k_F^2}{2m} \tag{10}$$

and

$$E_k^2=\epsilon_k^2+\Delta^2 \quad , \tag{11}$$

Δ being the energy gap. Note that ϵ_k is positive if $k>k_F$ and negative if $k<k_F$.

Using Eqs. (3), (5), (6), (8), and (9), we obtain

$$\hbar\omega\sigma_1^s(\omega,q,l)=\frac{\pi c^2}{2|A_0|^2}\sum_{\vec{k},\vec{k}'}|B_{k'k}|^2\frac{E_kE_{k'}-\epsilon_k\epsilon_{k'}-\Delta^2}{E_kE_{k'}}\delta(E_k+E_{k'}-\hbar\omega) \quad , \tag{12}$$

with use of the fact that $E_n - E_0 = E_k + E_{k'}$.

If we let $\Delta \rightarrow 0$ in Eq. (12), and assume that the $B_{k'k}$ are the same in the normal state as in the superconducting state, we obtain the normal-state conductivity

$$\hbar\omega\sigma_1^N(\omega,q,l) = \frac{\pi c^2}{|A_0|^2} \sum_{\vec{k},\vec{k}'} |B_{k'k}|^2 \delta(|\epsilon_k| + |\epsilon_{k'}| - \hbar\omega) \ . \tag{13}$$

Let us now perform the summations in (12) and (13) in two steps. First we sum over the angles of $\vec{k}$ and $\vec{k}'$, and then over their absolute values. Only the $B_{kk'}$ are affected by the summations over the angles. Equation (12) can then be written as follows:

$$\hbar\omega\sigma_1^s(\omega,q,l) = \sum_{\epsilon_k,\epsilon_{k'}} \frac{E_k E_{k'} - \epsilon_k \epsilon_{k'} - \Delta^2}{E_k E_{k'}} F(\epsilon_k, \epsilon_{k'}) \delta(E_k + E_{k'} - \hbar\omega) \ , \tag{14}$$

where

$$F(\epsilon_k, \epsilon_{k'}) = \int |B_{\vec{k}',\vec{k}}|^2 \sin\theta \, d\phi \, d\theta \ ,$$

where θ and ϕ are the angles of $\vec{k}'$ relative to a system of reference for which the z axis is taken along the direction of $\vec{k}$. Let us replace the sums in (14) by integrals in ϵ, ϵ' and then change to the variables E and E':

$$\hbar\omega\sigma_1^s(\omega,q,l) = \int\int dE \, dE' \frac{EE' - \epsilon\epsilon' - \Delta^2}{|\epsilon\epsilon'|} F(\epsilon,\epsilon') \delta(E + E' - \hbar\omega) \ . \tag{15}$$

The integration in E' can be performed, and remembering that in the superconducting state $\epsilon\epsilon'$ can be both positive and negative, we obtain

$$\hbar\omega\sigma_1^s(\omega,q,l) = \int_\Delta^{\hbar\omega-\Delta} dE \, \{[g(E)+1]F(|\epsilon|,|\epsilon'|) + [g(E)-1]F(-|\epsilon|,|\epsilon'|)\} \ , \tag{16}$$

where

$$g(E) = \frac{EE'\Delta^2}{|\epsilon\epsilon'|}$$

and

$$E' = \hbar\omega - E, \quad \epsilon'^2 = E'^2 - \Delta^2 \ . \tag{17}$$

It is possible to obtain a similar expression for σ_1^N:

$$\hbar\omega\sigma_1^N(\omega,q,l) = 2\int_0^{\hbar\omega} dE \, F(|\epsilon|,|\epsilon'|) \ . \tag{18}$$

Now we use the property of the function $F(|\epsilon|,|\epsilon'|)$ to be a function of the difference $\epsilon' - \epsilon = |\epsilon'| + |\epsilon|$. This follows from the invariance of the physical properties of the system with respect to small changes in the Fermi sea. As in the normal state

$$|\epsilon'| + |\epsilon| = \hbar\omega \ ,$$

$F'(|\epsilon|,|\epsilon'|)$ can be taken outside the integral, to give

$$F(|\epsilon,|\epsilon'|) = \tfrac{1}{2}\sigma_1^N(|\epsilon'| + |\epsilon|,q,l) \ . \tag{19}$$

This expression for F can be substituted in (16):

$$\sigma_1(\omega,q,l) = \frac{1}{2\hbar\omega} \int_\Delta^{\hbar\omega-\Delta} dE \, \{[g(E)+1]\sigma_1^N(|\epsilon'| + |\epsilon|,l) + [g(E)-1]\sigma_1^N(|\epsilon'| - |\epsilon|,l)\} \ , \tag{20}$$

which is the expression for $\sigma_1^s(\omega,q,l)$ we were looking for.

$\sigma_2(\omega,q,l)$ can be obtained using the Kramer-Kronig relations[4]:

$$\sigma_2(\omega,q,l) = \frac{2A(q,l)}{\pi\omega} = \frac{2\omega}{\pi} \int_{0^+}^{\infty} \frac{\sigma_1^s(\omega',q,l)}{\omega^2 - \omega'^2} d\omega' \ , \tag{21}$$

where

$$A(q,l) = \int_0^\infty \sigma_1^N(\omega,q,l)\,d\omega - \int_0^\infty \sigma_1^s(\omega,q,l)\,d\omega \ . \tag{22}$$

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[3]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).

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