Two-dimensional XY magnets with random Dzyaloshinskii-Moriya interactions

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Two-dimensional XY ferromagnets with random Dzyaloshinskii-Moriya interactions are studied. Such systems can be mapped onto a Coulomb gas with a quenched random array of dipoles. For large amounts of randomness, the low-temperature phase of the XY model is destroyed entirely. For small amounts of randomness, the behavior with decreasing temperatures is first paramagnetic, then ferromagnetic, and finally becomes paramagnetic again via a second, reentrant phase transition. These phase transitions are driven by an unbinding of vortices, just as in pure XY models. In contrast to pure XY models, the exponent η and the spin-wave stiffness are nonuniversal at T_c. The reentrant phase transitions appear to persist when the model is continued to $2+\epsilon$ dimensions. Similar results should apply to spin-glass models with a small concentration of bonds with the wrong sign.

I. INTRODUCTION

A number of recent theoretical papers[1-5] have focused on two-dimensional XY spin models with quenched, random disorder. Pure systems of this kind are now well understood via a vortex unbinding picture proposed by Kosterlitz and Thouless.[6,7] In contrast to random-spin systems in higher dimensions,[8] one can easily determine the effects of disorder over a whole range of temperatures, from the critical point down to $T=0$. Here we study XY models with a quenched random Dzyaloshinskii-Moriya interaction.[9] Our starting point is the Hamiltonian

$$H=-J\sum_{\langle i,j\rangle}\vec{S}_i\cdot\vec{S}_j-\sum_{\langle i,j\rangle}J'_{ij}\hat{z}\cdot(\vec{S}_i\times\vec{S}_j)\ , \qquad (1.1)$$

where the sums run over nearest-neighbor classical spins $\vec{S}_i$ on a square lattice, and the uniform exchange coupling J is, for example, ferromagnetic ($J>0$). The spin vectors are of unit length and confined to the xy plane. The Dzyaloshinskii-Moriya interaction J'_{ij} varies randomly in sign and magnitude from bond to bond.

Recently, Fert and Levy[10] have shown that sizable random Dzyaloshinskii-Moriya interactions can occur in bulk spin-glasses, between transition-metal impurity spins in a (nonmagnetic) metallic host, mediated by additional nonmagnetic impurities. An analogous random interaction [like that displayed in Eq. (1.1)] would presumably occur between XY spins in the corresponding two-dimensional system. If the magnetic impurity spins are randomly distributed in space, one also expects an exchange coupling J which varies in sign, due to the usual oscillatory Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction.[11] To obtain the Hamiltonian (1.1), we imagine that the spacing between impurity spins is adjusted to make the RKKY interaction predominantly ferromagnetic. Small variations in the magnitude of J turn out to be irrelevant at long wavelengths. The long-range nature of the RKKY and Dzyaloshinskii-Moriya interactions will be ignored.

Our results are conveniently described in terms of a continuum reformulation of Eq. (1.1), namely (see Sec. II)

$$\tilde{H}/k_BT\equiv\tfrac{1}{2}K\int d^2r\,|\vec{\nabla}\theta(\vec{r})-\vec{q}(\vec{r})|^2\ , \qquad (1.2)$$

where K is an effective exchange coupling divided by k_BT, and $\theta(\vec{r})$ is the orientation of a spin at site $\vec{r}$ with respect to some reference axis. The quenched vector field $\vec{q}(\vec{r})$ describes the twist between neighboring spin directions induced by the random Dzyaloshinskii-Moriya interactions. We assume that the $\vec{q}(\vec{r})$ at different sites are uncorrelated, and that a given complexion occurs with probability

$$P(\vec{q}(\vec{r}))\propto\exp\left[-\frac{1}{2\sigma}\int d^2r\,|\vec{q}(\vec{r})|^2\right]. \qquad (1.3)$$

Deviations from this Gaussian distribution turn out to be unimportant at long wavelengths. This formulation of the model is similar to a description of spin-glasses suggested by Hertz and studied by him near four dimensions.[12] Hertz concluded that the kind of randomness summarized by Eq. (1.3) was irrelevant near the usual Wilson-Fisher fixed point. As we shall see, the situation is rather different near $d=2$.

Just as in nonrandom XY models,[6,7] it is important to allow for vortex configurations of spins, satisfying

$$\oint (\vec{\nabla}\theta)\cdot d\vec{l} = 2\pi m \ , \tag{1.4}$$

where m is an integer. When vortices are inserted in Eq. (1.2), one finds the usual neutral Coulomb gas of vortex charges[6,7] coupled to quenched distribution of dipoles, with moments

$$\vec{p}(\vec{r}) = \hat{z}\times\vec{q}(\vec{r}) \ . \tag{1.5}$$

The quantity σ can be viewed as a kind of frozen-in "temperature" parametrizing the distribution of dipole moments. Screening in this random Coulomb gas is closely related to the properties of the original spin system. The different phases we find are shown as a function of temperature and vortex number density y in Fig. 1. The behavior is metallic at high temperatures, and becomes insulating via a Kosterlitz-Thouless vortex pairing transition over an intermediate-temperature range at small y. In contrast to nonrandom scalar Coulomb gases,[7] we find a second, reentrant transition to a metallic phase at low temperatures. The second transition occurs (for small y) when the thermodynamic temperature T is of the order of the frozen-in temperature σ characterizing the quenched dipole array. Although the disordered dipole array does not contribute directly to the polarizability, it does make it easier for thermally excited vortex pairs to separate. At low temperatures, vortex pairs are ripped apart by the random potential generated by the quenched dipoles. At intermediate temperatures, there are enough thermal vortices to screen this potential, and prevent pairs from unbinding. We find that an intermediate insulating phase is only possible provided

$$0<\sigma<\pi/8 \ . \tag{1.6}$$

When σ exceeds $\pi/8$, the behavior is metallic at all

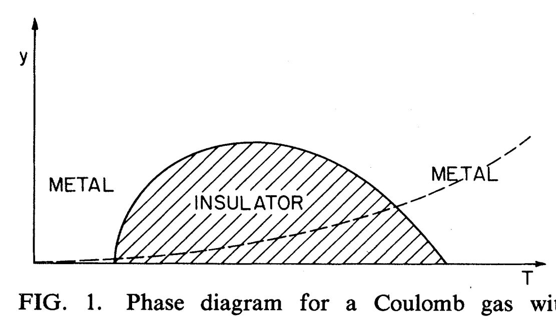

FIG. 1. Phase diagram for a Coulomb gas with quenched random dipoles, as a function of temperature T and vortex density y. The shaded insulating phase is absent entirely for sufficiently strong random potentials. The spin system moves along the dashed line as a function of temperature, and behaves paramagnetically in the regions marked "metal." There is quasi-long-range ferromagnetic order in the shaded region.

finite temperatures.

In the original XY spin model, the vortex charge density y is related to the core energy E_c by

$$y = e^{-E_c/k_BT} \ , \tag{1.7}$$

corresponding to the dashed locus in Fig. 1. As temperature decreases on this curve, the behavior is first paramagnetic, then "ordered," and finally paramagnetic again. The ordered phase is only ordered in the sense that the spin-spin correlation function decays *algebraically* to zero,

$$C(r) \equiv \langle \vec{S}(\vec{r})\cdot\vec{S}(\vec{0})\rangle \sim 1/r^{\eta(T,\sigma)} \ . \tag{1.8}$$

The exponent η depends both on the temperature and the strength of the randomness σ. Unbound thermal vortices produce exponential decay of $C(r)$ in the paramagnetic regions. There is no evidence for spin-glass behavior at any finite temperature, consistent with speculations[13] that the lower critical dimension for XY spin-glasses is well above $d=2$. In contrast to nonrandom XY spin models,[7] we find that η no longer assumes the universal values $\eta=\frac{1}{4}$ at T_c. The universal jump in the spin-wave stiffness[14] is also destroyed by the randomness, although this quantity still jumps discontinuously to zero across the ordered-to-paramagnetic phase boundary. Just as in pure systems,[7] there are only essential singularities in thermodynamic functions like the specific heat. The behavior near the high- and low-temperature phase transitions is qualitatively similar.

When vortices are neglected, we find that the randomness causes $C(r)$ to decay algebraically to zero even at $T=0$. In this sense, $d=2$ is the lower critical dimension for the destruction of XY long-range order by random Dzyaloshinskii-Moriya interactions. When the spin problem is continued into $2+\epsilon$ dimensions, we find that the randomness is unimportant (for small σ) both at $T=0$ and at a finite-temperature ferromagnetic fixed point with universal exponents. Reentrant phase transitions are still possible, however. Similar results apply to random Coulomb gases in $2-\epsilon$ dimensions.

It seems worth emphasizing that n-component ferromagnetic spins with random Dzyaloshinskii-Moriya interactions and $n\geq 3$ may behave rather differently. A natural generalization of the Hamiltonian (1.1) to the case $n=3$, for example, is

$$H = -J\sum_{\langle i,j\rangle}\vec{S}_i\cdot\vec{S}_j - \sum_{\langle i,j\rangle} J'_{ij}\hat{\omega}_{ij}\cdot(\vec{S}_i\times\vec{S}_j) \ , \tag{1.9}$$

where $\hat{\omega}_{ij}$ is a unit vector specifying a random rotation axis. In contrast to the XY case, Eq. (1.9) is not rotationally invariant for a fixed configuration of the $\{\hat{\omega}_{ij}\}$. One consequence is that standard renormalization-group procedures[15] generate random single-site anisotropies at long wavelengths.[16] The lower critical dimension for the destruction of ferromagnetic long-range order by such random "fields" is believed to be $d=4$.[17]

Reentrant phase transitions in two dimensions have been found previously in XY models with random p-fold symmetry-breaking fields.[3–5] In some sense, these models are more complicated than the Hamiltonian (1.1), because rotational invariance is broken explicitly. As shown in Appendix B via the replica trick, the renormalization-group equations for random Dzyaloshinskii-Moriya interactions are a special case of recursion relations obtained by Cardy and Ostlund for random symmetry-breaking fields.[4] The recursion relations needed in this paper are derived by a more physical technique in the main text, without recursion to replicas.

Our results are also related to work on ferromagnetic XY models with a small concentration of antiferromagnetic bonds.[1,2] As discussed by Villain,[1] XY models can adjust to an isolated exchange coupling with the wrong sign by trapping half-integer vortex dipoles or vortices with unit charge (and asymmetric cores) near an unfavorable cluster of bonds. At low temperatures the trapped integer vortices are energetically unfavorable, and will presumably be compensated if they occur at all by a nearby thermal vortex with opposite sign. The half-integer vortex dipoles form a source of quenched disorder like that studied in this paper. The concentration of "wrong" bonds, x, is related to the variance σ of our Gaussian distribution of dipole moments where σ is an increasing function of x. José[2] finds that algebraic decay of order-parameter correlations is preserved for small x, with, however, "some evidence" of a low-temperature instability. For $x\approx\frac{1}{2}$, he predicts exponential decay at all temperatures. These results agree qualitatively with our more detailed conclusions for random Dzyaloshinskii-Moriya interactions as a function of σ.

Experiments on Kosterlitz-Thouless transitions in helium films (which are mathematically quite similar to XY magnets) are often carried out on disordered substrates like glass or Mylar. Torsional oscillator experiments reveal some residual dissipation below T_c (Ref. 18) which may be attributable to configurations of vortices trapped by the disorder. If the trapped vortices form localized dipoles with frozen orientations on the time scale of the experiment, one might expect deviations from the predicted[14] universal jump in the superfluid density, and the destruction of superfluid order at very low temperatures. The degree of substrate disorder is easily controlled by changing the film thickness. Solids with quenched random impurities are also mathematically quite similar to the random magnets considered here. This analogy will be pursued elsewhere.[19]

In Sec. II we tabulate the properties of the model in the absence of thermally excited vortices, and describe how the Hamiltonian (1.1) is related to the Coulomb gas with a quenched array of random dipole moments. The properties of this random Coulomb gas are discussed in Sec. III. Renormalization-group recursion relations for this system, including an extension into $2+\epsilon$ dimensions, are derived in Sec. IV. Several technical calculations are summarized in the Appendixes.

II. THE MODEL

A. Spin-wave theory

Just as in treatments of nonrandom two-dimensional XY models,[6,7,20] it is useful to first study spin-wave fluctuations, before introducing vortices. To this end, we write the Hamiltonian (1.1) as

$$H=-\sum_{\langle i,j\rangle}[J\cos(\theta_i-\theta_j)+J'_{ij}\sin(\theta_i-\theta_j)]$$
$$=-\sum_{\langle i,j\rangle}\tilde{J}_{ij}\cos(\theta_i-\theta_j-q_{ij})\ , \tag{2.1}$$

where

$$\tilde{J}_{ij}=[J^2+(J'_{ij})^2]^{1/2}\ , \tag{2.2a}$$

$$q_{ij}=\arctan(J'_{ij}/J)\ . \tag{2.2b}$$

Because there is only a weak singularity in the specific heat of the pure XY model,[7] the Harris criterion[21] suggests that the randomness in the overall exchange constant $\tilde{J}_{ij}$ is unimportant (see Appendix B). Using the replica method discussed in Appendix B, one can, in fact, show that the spatial variation in $\tilde{J}_{ij}$ is irrelevant at long wavelengths for all temperatures up to and including T_c. Consequently, we replace $\tilde{J}_{ij}/k_BT$ by its mean value

$$\frac{\tilde{J}_{ij}}{k_BT}\to K\equiv[(J^2+J_{ij}^2)^{1/2}/k_BT]_d\ , \tag{2.3}$$

where the square brackets with subscript d (disorder) indicate an average over the randomness.

The spin-wave approximation[22] to H is obtained by expanding the cosine to second order in its argument, and neglecting higher-order terms. The integrations over the angle variables θ_i are extended to $\pm\infty$. In a convenient continuum notation, one finds an effective Hamiltonian (neglecting an unimportant constant),

$$\frac{H_{\rm sw}}{k_BT}\equiv\tfrac{1}{2}K\int d^2r\,|\vec{\nabla}\phi(\vec{r})-\vec{q}(\vec{r})|^2\ . \tag{2.4}$$

We have made the replacements

$$q_{ij}/a \to \vec{q}(\vec{r}) , \quad (2.5a)$$

where a is the lattice constant, and

$$\theta_i \to \phi(\vec{r}) , \quad (2.5b)$$

with the understanding that $\phi(\vec{r})$ is to be integrated over the range $[-\infty, +\infty]$. To completely determine the model, we need to specify the probability distribution governing the quenched local "wave vector" $\vec{q}(\vec{r})$. The simplest probability distribution consistent with Dzyaloshinskii-Moriya interactions J'_{ij} symmetric about zero is the Gaussian distribution (1.3). For small σ, we expect that deviations from this distribution are unimportant at long wavelengths.

It is straightforward to calculate the order-parameter correlation function

$$C(r) = [\langle \vec{S}(\vec{r}) \cdot \vec{S}(\vec{0}) \rangle]_d , \quad (2.6)$$

within the spin-wave approximation. Here the angular brackets indicate a thermodynamic average weighted by $\exp(-H_{sw}/k_B T)$ for a fixed configuration of the $\{\vec{q}(\vec{r})\}$, and the square brackets with subscript d (disorder) mean an average over the probability distribution (1.3). A simple calculation sketched in Appendix A gives

$$C(r) \sim 1/r^{\eta(K,\sigma)} , \quad (2.7a)$$

where

$$\eta(K,\sigma) = \frac{1}{2\pi}(K^{-1} + \sigma) . \quad (2.7b)$$

Note from Eq. (2.3) that K^{-1} is linear in the temperature and that, in contrast to pure spin systems, η remains finite even at $T=0$. One might expect effects of the randomness to be important at temperatures low enough so that

$$K^{-1} \lesssim \sigma . \quad (2.8)$$

As we shall see, the system is unstable to thermal vortex pairs outside a finite band of temperatures. Equation (2.7) is qualitatively incorrect at temperatures outside this range.

B. Vortices

Vortices can be added to the spin-wave description in the usual way.[7,20] Starting with the continuum Hamiltonian (1.2), we see that, for a fixed configuration $\{\vec{q}(\vec{r})\}$, extremal complexions of spin angles must satisfy

$$\nabla^2 \theta(\vec{r}) - \vec{\nabla} \cdot \vec{q}(\vec{r}) = 0 , \quad (2.9)$$

almost everywhere. Allowing for a set of quantized vortices satisfying Eq. (1.4), we find that the extremal solutions $\theta_{sing}(\vec{r})$ are given by

$$\partial_i \theta_{sing}(\vec{r}) = \int d^2r' q_j(\vec{r}') \partial_j \partial_i G(\vec{r} - \vec{r}') + 2\pi \epsilon_{ij} \int d^2r' m(\vec{r}') \partial_j G(\vec{r} - \vec{r}') . \quad (2.10)$$

Here ∂_i represents a partial derivative with respect to the ith component of the unprimed variable $\vec{r}$, $G(\vec{r})$ satisfies

$$\nabla^2 G(\vec{r}) = \delta(\vec{r}) , \quad (2.11)$$

and ϵ_{ij} is the antisymmetric 2×2 matrix, $\epsilon_{xy} = -\epsilon_{yx} = 1$. The quantity $m(\vec{r})$ is a vortex charge density, related to the charges m_k on a set of vortices at positions $\vec{r}_k$ by

$$m(\vec{r}) = \sum_k m_k \delta(\vec{r} - \vec{r}_k) . \quad (2.12)$$

For large r we have

$$G(r) \approx \frac{1}{2\pi} \ln(r/a) . \quad (2.13)$$

Upon decomposing $\theta(\vec{r})$ into a singular part and a smooth spin-wave part $\phi(\vec{r})$,

$$\theta(\vec{r}) = \theta_{sing}(\vec{r}) + \phi(\vec{r}) , \quad (2.14)$$

we can write the continuum Hamiltonian (1.2) as

$$\frac{\tilde{H}}{k_B T} = \frac{H_{sw}}{k_B T} + \frac{H_c}{k_B T} , \quad (2.15)$$

where $H_{sw}/k_B T$ is given by Eq. (2.4), and

$$\frac{H_c}{k_B T} = -\pi K \sum_{i \neq j} m_i m_j \ln\left[\frac{r_{ij}}{a}\right] + \left[\frac{E_c}{k_B T}\right] \sum m_i^2 + K \sum_i \int d^2r\, m_i \frac{\vec{p}(\vec{r}) \cdot (\vec{r} - \vec{r}_i)}{(\vec{r} - \vec{r}_i)^2} . \quad (2.16)$$

The quenched random vector field $\vec{p}(\vec{r})$ is related to $\vec{q}(\vec{r})$ via Eq. (1.5), and the vortex charges m_i satisfy

$$\sum_i m_i = 0 . \quad (2.17)$$

Evidently, we must deal with the usual scalar Coulomb gas[7] perturbed by a set of random dipole moments $\vec{p}(\vec{r})$.

III. PROPERTIES OF A COULOMB GAS WITH QUENCHED RANDOM DIPOLES

As shown in the preceding section, the Gaussian spin-wave excitations decouple from the vortex degrees of freedom. To proceed further one must understand the quenched random Coulomb gas (2.16). In this section we describe screening in this gas, and then show how the charge-charge correlation function renormalizes due to bound vortex pairs. In Sec. IV these results are used to derive renormalization-group recursions similar to those found by Kosterlitz[7] for the pure system, and to demonstrate exponential screening of the Coulomb interaction when the vortices are unbound.

A. Dielectric function

Screening in the Coulomb gas is conveniently described in terms of a dielectric function.[6,23] We imagine that a small perturbing potential $\phi_{\text{ext}}(\vec{r})$ is added to the system. Associated with this potential is a perturbing charge density $m_{\text{ext}}(\vec{r})$ determined by

$$-\nabla^2\phi_{\text{ext}}(\vec{r})\equiv 2\pi m_{\text{ext}}(\vec{r}) \;. \tag{3.1}$$

It is easy to see that the change in energy of the system is described by

$$H_c/k_BT \rightarrow H_c/k_BT+2\pi K\int d^2r\, m(\vec{r})\phi_{\text{ext}}(\vec{r}) \;, \tag{3.2}$$

where $m(\vec{r})$ is the vortex charge density (2.12). There is also a contribution from the interaction of the external charges with the random dipole array, but this drops out of the dielectric function calculated below. For a given distribution of quenched dipoles, we can determine a renormalized potential $\phi(\vec{r})$ in terms of $m_{\text{ext}}(\vec{r})$ and the thermodynamic average of $m(\vec{r})$,

$$-\nabla^2\phi(\vec{r})\equiv 2\pi[m_{\text{ext}}(\vec{r})+\langle m(\vec{r})\rangle] \;. \tag{3.3}$$

The average in Eq. (3.3) means a sum over configurations of thermal vortices

$$\langle m(\vec{r})\rangle \equiv \frac{\operatorname{Tr}'_{\{m_i\}}[m(\vec{r})e^{-H_c/k_BT}]}{\operatorname{Tr}'_{\{m_i\}} e^{-H_c/k_BT}} \;, \tag{3.4}$$

with the replacement (3.2). The prime on the traces means that we must respect the constraint (2.17) of charge neutrality. Upon averaging Eq. (3.3) over the random dipole distribution, we define a wave-vector–dependent dielectric function $\epsilon(\vec{q})$ in terms of the Fourier transforms $\phi_{\text{ext}}(\vec{q})$ and $[\phi(\vec{q})]_d$:

$$[\phi(\vec{q})]_d \equiv \frac{1}{\epsilon(\vec{q})}\phi_{\text{ext}}(\vec{q}) \;. \tag{3.5}$$

Expanding in the external potential, one finds from Eq. (3.4) that

$$\begin{aligned}\langle m(\vec{r})\rangle = \langle m(\vec{r})\rangle_0 & -2\pi K\int d^2r'\phi_{\text{ext}}(\vec{r}\,') \\ & \times[\langle m(\vec{r})m(\vec{r}\,')\rangle_0 - \langle m(\vec{r}\,')\rangle_0\langle m(\vec{r})\rangle_0] \;,\end{aligned} \tag{3.6}$$

where the angular brackets with subscript 0 mean a thermal average with no external potential. Upon averaging over the randomness, and passing to a Fourier-transform representation, we find

$$[\langle m(\vec{q})\rangle]_d = -2\pi K[\langle\,|\delta m(\vec{q})|^2\rangle_0]_d\phi_{\text{ext}}(\vec{q})/\Omega_0 \;, \tag{3.7}$$

where Ω_0 is the area of the system, and translational invariance and charge neutrality ensure that

$$[\langle m(\vec{q})\rangle_0]_d = 0 \tag{3.8}$$

for all $\vec{q}$, and

$$\delta m(\vec{q}) = m(\vec{q}) - \langle m(\vec{q})\rangle_0 \;. \tag{3.9}$$

Combining Eqs. (3.7) and (3.5) with the average of Eq. (3.3), we obtain finally

$$\frac{1}{\epsilon(\vec{q})} = 1-\frac{4\pi^2K}{q^2}[\langle\,|\delta m(\vec{q})|^2\rangle_0]_d \;. \tag{3.10}$$

This is, of course, just the standard dielectric formula,[23] modified to account for effects of the random medium.

Physically, $\epsilon(\vec{q})$ describes the screening of the potential between two widely separated vortex charges due to the remaining vortices in the medium of random dipoles. In an insulating phase, we expect that effective potential takes the form

$$\phi(\vec{q}) = 2\pi K_R/q^2 \;, \tag{3.11}$$

for small q. The "external" potential is just $2\pi K/q^2$ in this case. Equations (3.5) and (3.10) determine K_R,

$$\frac{K_R}{K} = 1-\lim_{q\to 0}\frac{4\pi^2K}{q^2}[\langle\,|\delta m(\vec{q})|^2\rangle_0]_d \;. \tag{3.12}$$

It is straightforward to evaluate this formula to leading order in the vortex fugacity

$$y = e^{-E_c/k_BT} \;. \tag{3.13}$$

Since $\langle m(\vec{q})\rangle_0$ is of $O(y^2)$, we have

$$[\langle\,|\delta m(\vec{q})|^2\rangle_0]_d = [\langle\,|m(\vec{q})|^2\rangle_0]_d + O(y^4) \;. \tag{3.14}$$

Upon expanding $[\langle |m(\vec{q})|^2\rangle_0]_d$ to second order in $\vec{q}$, and exploiting charging neutrality and isotropy of $[\langle m(\vec{r})m(\vec{0})\rangle_0]_d$, we obtain

$$[\langle |m(\vec{q})|^2\rangle_0]_d/\Omega_0 = -\tfrac{1}{4}q^2\int\frac{d^2r}{a^4}r^2[\langle m(\vec{r})m(\vec{0})\rangle_0]_d\ . \tag{3.15}$$

Collecting everything together, we see that

$$K_R = K+\pi^2K^2\int\frac{d^2r}{a^4}r^2[\langle m(\vec{r})m(\vec{0})\rangle_0]_d + O(y^4)\ . \tag{3.16}$$

The thermal average $\langle m(\vec{r})m(\vec{0})\rangle_0$ entering Eq. (3.16) follows immediately from the Coulomb-gas Hamiltonian (2.16),

$$\langle m(\vec{r})m(\vec{0})\rangle_0 = -2y^2\left[\frac{r}{a}\right]^{-2\pi K}\times\cosh[I(\vec{r},\vec{0})]+O(y^4)\ , \tag{3.17a}$$

where

$$\begin{aligned} I(\vec{r}_1,\vec{r}_2) &= K\int d^2r'\vec{p}(\vec{r}\,')\cdot\left[\frac{\vec{r}\,'-\vec{r}_1}{|\vec{r}\,'-\vec{r}_1|^2}-\frac{\vec{r}\,'-\vec{r}_2}{|\vec{r}\,'-\vec{r}_2|^2}\right] \\ &= 2\pi K\int d^2r'\vec{p}(\vec{r}\,')\cdot\vec{\nabla}' \times[G(\vec{r}\,'-\vec{r}_1)-G(\vec{r}\,'-\vec{r}_2)]\ . \end{aligned} \tag{3.17b}$$

Using the technique sketched in Appendix A, it is easy to evaluate $[\cosh(I(\vec{r},\vec{0}))]_d$ and find

$$[\langle m(\vec{r})m(\vec{0})\rangle_0]_d = -2y^2(r/a)^{-2\pi K(1-\sigma K)} + O(y^4)\ . \tag{3.18}$$

Our final result for K_R in the insulating phase is obtained by combining Eqs. (3.16) and (3.18), carrying out an angular average, and imposing the short-distance cutoff a,

$$K_R = K-4\pi^3y^3K^2\int_a^\infty\frac{dr}{a}\left[\frac{r}{a}\right]^{3-2\pi K(1-\sigma K)} + O(y^4)\ . \tag{3.19}$$

We see immediately that this perturbation expansion only makes sense over a finite range of temperatures, determined by

$$\pi K(1-\sigma K)>2\ . \tag{3.20}$$

For

$$\sigma>\pi/8\ , \tag{3.21}$$

it is impossible to satisfy this inequality at any finite temperature.

B. Charge-correlation function

It is straightforward to convert perturbation expansions such as (3.19) into renormalization-group recursions like those derived by Kosterlitz[7] for the pure Coulomb gas.[14,20] Because there are three important parameters (K, y, and σ) entering the random Coulomb gas, we need an additional equation to proceed further. Here we study the decay and the charge-correlation function, which we expect to be of the form

$$[\langle m(\vec{r})m(\vec{0})\rangle_0]_d \sim r^{-2\pi\hat{K}_R}\ , \tag{3.22}$$

in the insulating phase. Note from Eq. (3.18) that the $\hat{K}_R$ differs from K even to leading order in y^2. The method we use is adapted from a technique used for pure systems in an early study of two-dimensional melting.[24]

To calculate $\hat{K}_R$ to the next order in y^2, we write

$$[\langle m(\vec{r})m(\vec{0})\rangle_0]_d = \left[\frac{y^2\Gamma_2(\vec{r})+y^4\Gamma_4(\vec{r})+\cdots}{1+y^2Z_1+\cdots}\right]_d\ , \tag{3.23}$$

where the denominator is the expansion of the partition function for a fixed dipole configuration. The coefficient Z_1 is

$$Z_1 = \int\frac{d^2r_a}{a^2}\int\frac{d^2r_b}{a^2}\left[\frac{r_{ab}}{a}\right]^{-2\pi K}e^{I(\vec{r}_a,\vec{r}_b)}\ . \tag{3.24}$$

The numerator represents contributions of pairs, quadrupoles, etc., to the thermal average, just as in Ref. 24. To $O(y^4)$, we have

$$[\langle m(r)m(0)\rangle_0]_d = y^2[\Gamma_2(\vec{r})]_d+y^4[\Gamma_4(\vec{r})]_d - y^4[Z_1\Gamma_2(r)]_d\ . \tag{3.25}$$

From Eq. (3.18) we see immediately that

$$[\Gamma_2(\vec{r})]_d = -2\left[\frac{r}{a}\right]^{-2\pi\hat{K}}, \quad (3.26a)$$

where

$$\hat{K} = K(1-\sigma K). \quad (3.26b)$$

The second term in Eq. (3.25) may be evaluated straightforwardly as in Ref. 24, with the result

$$[\Gamma_4(\vec{r})]_d = -16\pi^4\left[\frac{r}{a}\right]^{-2\pi\hat{K}}\hat{K}^2\ln\left[\frac{r}{a}\right] \times \int_a^\infty \frac{dr'}{a}\left[\frac{r'}{a}\right]^{3-2\pi\hat{K}}. \quad (3.27)$$

The third term, which appears because we are carrying out a *quenched* rather than an annealed average, is given by

$$[Z_1\Gamma_2(\vec{r})]_d = -2\left[\frac{r}{a}\right]^{-2\pi K}\int\frac{d^2r_a}{a^2}\int\frac{d^2r_b}{a^2}\left[\frac{r_{ab}}{a}\right]^{-2\pi K}[\cosh(I(\vec{r},\vec{0}))e^{I(\vec{r}_a,\vec{r}_b)}]_d. \quad (3.28)$$

Terms proportional to the area of the system in Eqs. (3.27) and (3.28), and which cancel in Eq. (3.30), have been suppressed. Evaluating the quenched average by the method of Appendix A, and following the procedure of Ref. 24, we obtain

$$[Z_1\Gamma_2(\vec{r})]_d = -16\pi^4\left[\frac{r}{a}\right]^{-2\pi\hat{K}}K^4\sigma^2\ln\left[\frac{r_a}{a}\right]\int_a^\infty\frac{dr'}{a}\left[\frac{r'}{a}\right]^{3-2\pi\hat{K}}. \quad (3.29)$$

Collecting together the different contributions to Eq. (3.25), we have

$$[\langle m(\vec{r})m(\vec{0})\rangle_0]_d = -2y^2\left[\frac{r}{a}\right]^{-2\pi\hat{K}}\left\{1-\left[8\pi^4y^2(\hat{K}^2-K^4\sigma^2)\int_a^\infty\frac{dr'}{a}\left[\frac{r'}{a}\right]^{3-2\pi\hat{K}}\right]\ln(r/a)\right\}. \quad (3.30)$$

To lowest order in y^2, we see that the charge-correlation function may be written as in Eq. (3.22), with

$$\hat{K}_R \equiv K_R(1-\sigma_R K_R) = \hat{K} - 4\pi^3y^2(\hat{K}^2-K^4\sigma^2)\int_a^\infty\frac{dr'}{a}\left[\frac{r'}{a}\right]^{3-2\pi\hat{K}} + O(y^4). \quad (3.31)$$

When $\sigma=0$, the renormalizations (3.19) and (3.31) of K and $\hat{K}$ are identical. These are physically quite different quantities in the presence of disorder, however. Note that the same potentially divergent integral enters both Eqs. (3.19) and (3.31).

IV. RENORMALIZATION-GROUP RECURSION RELATIONS

A. Recursion relations in two dimensions

Difficulties in the perturbation expansions (3.19) and (3.31) can be studied using the method of José *et al.*[20] The potentially divergent integrations are split into two parts,

$$\int_a^\infty\frac{dr}{a} \to \int_a^{ae^l}\frac{dr}{a}+\int_{ae^l}^\infty\frac{dr}{a}, \quad (4.1)$$

and the small-r integrals are absorbed into effective couplings $K(l)$ and $\hat{K}(l)$. Upon rescaling the large-r integrations, we obtain a new perturbation series identical in form to (3.19) and (3.31), but with a new vortex fugacity $y(l)$. The differential recursion relations describing how $K(l)$, $\hat{K}(l)$, and $y(l)$ evolve under this procedure are

$$\frac{dK(l)}{dl} = -4\pi^3K^2(l)y^2(l), \quad (4.2)$$

$$\frac{d\hat{K}(l)}{dl} = \frac{d}{dl}[K(l)-\sigma(l)K^2(l)] = -4\pi^3[\hat{K}^2(l)-K^4(l)\sigma^2(l)]y^2(l), \quad (4.3)$$

and

$$\frac{dy(l)}{dl} = [2-\pi K(l)+\pi K^2(l)\sigma(l)]y(l). \quad (4.4)$$

Comparing Eqs. (4.2) and (4.3), one finds that they can only be consistent provided that

$$\frac{d\sigma(l)}{dl} = 0. \quad (4.5)$$

Recursion relations equivalent to (4.2), (4.4), and (4.5) are obtained via the replica trick in Appendix B. For $\sigma=0$, we recover the recursion relations of Kosterlitz.[7] Although $\sigma(l)$ is unrenormalized to this order in y, we would expect it to renormalize slightly when irrelevant variables [parametrizing, say, deviations from the purely Gaussian probability distribution (1.3)] are taken into account.

The Hamiltonian flows generated by Eqs. (4.2) and (4.4) in the (K^{-1},y) plane are shown in Fig. 2 for small σ. There are two special points along the fixed line $y=0$ where the eigenvalue of y vanishes, namely

$$K_{\mp}^{-1}(\sigma)=\frac{\pi}{4}[1\pm(1-8\sigma/\pi)^{1/2}] . \quad (4.6)$$

The trajectories are hyperbolic near K_{+}^{-1}, which marks the terminus of a locus of transition temperatures. The flows are elliptical near K_{-}^{-1}. The heavy line marks a special trajectory which leaves the fixed line at a finite temperature K_0^{-1} below K_{-}^{-1} and ends at K_{+}^{-1}. The vortex fugacity ultimately decays to zero in the region bounded by this trajectory. The random Coulomb gas is insulating in this region, and there is algebraic decay of correlations in the corresponding spin system. The initial increase in $y(l)$ at temperatures less than K_{-}^{-1} is caused by the random dipole potential. For

$$K_0^{-1}<K^{-1}<K_{+}^{-1} , \quad (4.7)$$

there are enough vortex dipoles to screen out this potential as long wavelengths. When K^{-1} exceeds K_{+}^{-1}, there is the usual Kosterlitz-Thouless vortex unbinding transition. When K^{-1} is below K_0^{-1}, a combination of the random potential and thermal effects causes the vortices to unbind. As σ approaches $\pi/8$, K_{+}^{-1} and K_{-}^{-1} merge, and the insulating phase shrinks to zero. The system is always unstable to vortices when σ exceeds $\pi/8$.

The dashed line of the initial Hamiltonian corresponding to Eq. (1.7) is also shown in Fig. 2. When this line crosses the heavy trajectory, transitions from algebraically decaying order to paramagnetism occur. The properties at both these transition temperatures are controlled by the Hamiltonian flows near K_{+}^{-1}. The exponent η entering parametrizing the algebraic decay of $G(r)$ approaches the value

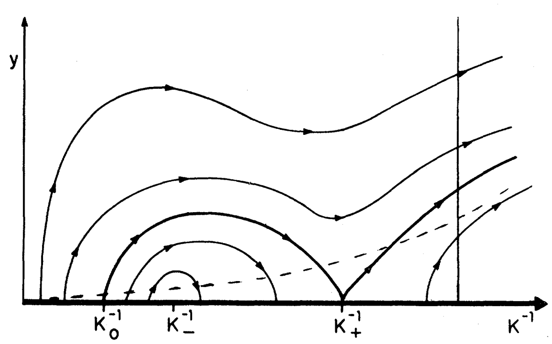

FIG. 2. Hamiltonian flows in the $(K^{-1}(l),y(l))$ plane. There is a line of fixed points at $y=0$. The heavy trajectory which starts at K_0^{-1} and ends at K_{+}^{-1} bounds a region of insulating behavior. The unstable trajectories outside this region can be integrated out to the vertical line, where the high-temperature Debye-Hückel approximation is appropriate.

$$\eta^*(\sigma)=\tfrac{1}{8}[1+(1-8\sigma/\pi)^{1/2}]+\frac{\sigma}{2\pi} , \quad (4.8)$$

at these transitions. This exponent decreases (quadratically in σ) from $\frac{1}{4}$ at $\sigma=0$,[7] to

$$\eta^*_{\min}=\tfrac{3}{16} , \quad (4.9)$$

just before the algebraically ordered phase vanishes at $\sigma=\pi/8$. The renormalized spin-wave stiffness constant K_R is given by[14]

$$K_R(K,\sigma,y)=\lim_{l\to\infty} K(l) . \quad (4.10)$$

As is evident from Fig. 2 this quantity approaches K_+ on the boundary of the ordered phase

$$\lim_{T\to T_c} K_R=K_+(\sigma) . \quad (4.11)$$

In contrast to pure systems,[14] this quantity is not universal.

In the paramagnetic region of Fig. 2, we expect that the spin-correlation function decays exponentially to zero,

$$G(r)\sim e^{-r/\xi} . \quad (4.12)$$

Taking over Kosterlitz's analysis[7] of the pure system, we find from the properties of the flows near K_+ that ξ diverges exponentially,

$$\xi\sim\exp(\text{const }|T-T_c|^{-1/2}) . \quad (4.13)$$

This same length controls screening in the metallic phase of the Coulomb gas. The singular part of the specific heat behaves like ξ^{-2} near the transition.

B. Screening in the metallic phase

Outside the insulating region of Fig. 2, the system is driven toward high temperatures and large y, and perturbation theory in the vortex fugacity breaks down. In this limit, we can show explicitly that the behavior is metallic. The general formula (3.10) for the dielectric constant may be written

$$\frac{1}{\epsilon(\vec{q})}=1-\frac{4\pi^2K}{q^2}\{[\langle|m(\vec{q})|^2\rangle_0]_d - [|\langle m(\vec{q})\rangle_0|^2]_d\} . \quad (4.14)$$

When unbound vortices are present, the averages, in Eq. (4.14) can be evaluated by integrating, rather than summing, over the vortex degrees of freedom in H_c. This "Debye-Hückel" approximation[25] should be valid for small q, since many free vortices are contained in a region of size q^{-1}.

When rewritten in terms of the Fourier-transformed charge distribution $m(\vec{q})$, the Coulomb-gas Hamiltonian becomes

$$\frac{H_c}{k_BT}=\frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\left[\frac{4\pi^2K}{q^2}+\frac{E_ca^2}{k_BT}\right]|m(\vec{q})|^2$$
$$+2\pi K\int\frac{d^2q}{(2\pi)^2}m(\vec{q})\frac{i\vec{p}(\vec{q})\cdot\vec{q}}{q^2}\ , \tag{4.15}$$

where a is the lattice spacing implicit in Eq. (1.1). By treating $m(\vec{q})$ as the Fourier transform of a continuous vector field, it is easy to show that

$$[\,|\langle m(\vec{q})\rangle_0|^2]_d=\frac{4\pi^2K^2/q^2}{\left[\dfrac{4\pi^2K}{q^2}+\dfrac{E_ca^2}{k_BT}\right]^2}\ , \tag{4.16}$$

$$[\langle\,|m(q)|^2\rangle_0]_d=\frac{1}{\dfrac{4\pi^2K}{q^2}+\dfrac{E_cq^2}{k_BT}}+\frac{4\pi^2K^2/q^2}{\left[\dfrac{4\pi^2K^2}{q^2}+\dfrac{a^2E_c}{k_BT}\right]^2}\ . \tag{4.17}$$

Inserting these results into Eq. (4.14) we find the wave-vector–dependent dielectric constant characteristic of a metal:

$$\epsilon(\vec{q})=1+\frac{1}{(q\xi)^2}\ , \tag{4.18}$$

where

$$\xi^2=E_ca^2/4\pi k_BTK\ . \tag{4.19}$$

The resulting interaction between widely separated vortices dies off exponentially at large distances. Using the renormalization group, one can scale onto this Debye-Hückel calculation everywhere outside the insulating region of Fig. 2. The recursion relations can be integrated until the trajectories intersect the vertical line in Fig. 2, where the high-temperature Debye-Hückel theory is a good approximation. This sort of scaling analysis shows that ξ diverges as in Eq. (4.13) near the metal-insulator transition.

C. Continuation into $2+\epsilon$ dimensions

It is interesting to see if random Dzyaloshinskii-Moriya interactions cause reentrant phase transitions in $2+\epsilon$ dimensions. Although the precise meaning of "vortices" in $2+\epsilon$ dimensions is debatable, one may be able to think of them as ϵ-dimensional singularities which become singular lines in three dimensions. Just as in pure systems,[26] results in $2+\epsilon$ dimensions can be obtained by accounting for the effect of length rescalings on quantities like K, which are dimensionless in $d=2$. Terms in the recursion relations proportional to the vortex fugacity can be evaluated in two dimensions, to lowest order in ϵ.[26] The resulting renormalization-group equations for XY magnets with random Dzyaloshinskii-Moriya interactions are

$$\frac{dK^{-1}}{dl}=-\epsilon K^{-1}+4\pi^3y^2\ , \tag{4.20a}$$

$$\frac{dy}{dl}=(2+\tfrac{1}{2}\epsilon-\pi K+\pi K^2\sigma)y\ , \tag{4.20b}$$

$$\frac{d\sigma}{dl}=-\epsilon\sigma\ . \tag{4.20c}$$

The same recursion relations apply to a d-dimensional Coulomb gas with random dipoles in $2-\epsilon$ dimensions.[27] Note that the randomness decays to zero, and is ultimately unimportant at long wavelengths at all temperatures. The $T=0$ and finite-temperature fixed points which occur for $\sigma=0$ were discussed in Ref. 26.

For fixed σ, reentrant phase transitions are still possible as a function of temperature. Figure 3 shows qualitatively the result of numerically integrating the recursion relations for $\sigma=\pi/16$, $d=2.1$, and a range of initial values in the (K^{-1},y) plane. The shaded region exhibits long-range ferromagnetic order, and corresponds to the set of initial conditions which are attracted to the $T=0$ fixed point. The shaded region is bounded by a line of ferromagnet-paramagnet phase transitions with universal critical exponents. The correlation length, for example, diverges like

$$\xi\sim\frac{1}{|T-T_c|^{\nu}}\ , \tag{4.21a}$$

with[26]

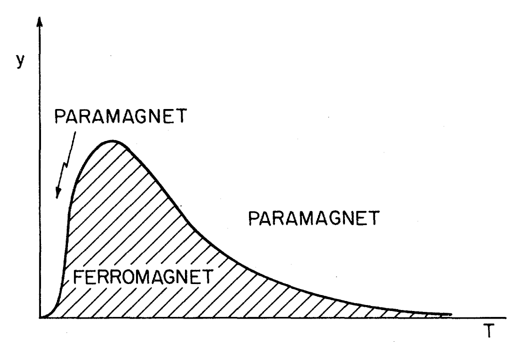

FIG. 3. Schematic phase diagram for XY spins with a random Dzyaloshinskii-Moriya interaction in $2+\epsilon$ dimensions.

$$\nu \approx \tfrac{1}{2}\sqrt{\epsilon} \ . \tag{4.21b}$$

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APPENDIX A: SPIN-WAVE THEORY

In this Appendix we determine the decay of order-parameter correlations in the absence of vortices. The remaining Gaussian excitations are usually called "spin waves."[20] Starting with the definition (2.6), we can write

$$C(r) = \mathrm{Re}[\langle e^{i\phi(\vec{r})-i\phi(\vec{0})}\rangle]_d \ . \tag{A1}$$

The thermal average takes the form

$$\langle \exp[i\theta(\vec{r})-i\theta(\vec{0})]\rangle = \frac{\left\langle \exp\left[i\phi(\vec{r})-i\phi(\vec{0})-K\int d^2r'\vec{\nabla}\phi\cdot\vec{q}\right]\right\rangle_0}{\left\langle \exp\left[-K\int d^2r'\vec{\nabla}\phi\cdot\vec{q}\right]\right\rangle_0} , \tag{A2}$$

where the angular brackets with subscript 0 mean an average over an ensemble specified by the Hamiltonian

$$H_0/k_BT = \tfrac{1}{2}K\int d^2r\,|\vec{\nabla}\phi|^2 \ . \tag{A3}$$

(The same symbol is used in a different context in Sec. III.) Because the average is over a Gaussian ensemble, the numerator of (A2) may be written,

$$\begin{aligned}\langle \exp[i\phi(\vec{r})-i\phi(\vec{0})-K\int d^2r'\vec{\nabla}\phi\cdot\vec{q}]\rangle_0 &= \exp\{-\tfrac{1}{2}\langle[\phi(\vec{r})-\phi(\vec{0})]^2\rangle_0\}\\ &\quad\times\langle\exp(-K\int d^2r'\vec{\nabla}\phi\cdot\vec{q})\rangle_0\\ &\quad\times\exp[-iI(\vec{r},\vec{0})] ,\end{aligned} \tag{A4}$$

where

$$\begin{aligned}I(\vec{r}_1,\vec{r}_2) = K\int d^2r'\vec{q}(\vec{r}\,')\cdot\vec{\nabla}\,'\\ \times\langle[\phi(\vec{r}_1)-\phi(\vec{r}_2)]\phi(\vec{r}\,')\rangle_0 \ .\end{aligned} \tag{A5}$$

The first term in Eq. (A4) is easily evaluated for large r by passing to a Fourier representation,

$$\begin{aligned}\langle[\phi(\vec{r})-\phi(\vec{0})]^2\rangle_0 &= 2\int\frac{d^2q}{(2\pi)^2}\frac{1}{Kq^2}\\ &\quad\times(1-e^{i\vec{q}\cdot\vec{r}})\\ &\approx\frac{1}{\pi K}\ln(r/a) \ .\end{aligned} \tag{A6}$$

After applying similar manipulations to $I(\vec{r}_1,\vec{r}_2)$ we find that

$$\begin{aligned}I(\vec{r}_1,\vec{r}_2) = \int d^2r'\vec{q}\,'(\vec{r}\,')\cdot\vec{\nabla}\,'\\ \times[G(\vec{r}\,'-\vec{r}_1)-G(\vec{r}\,'-\vec{r}_2)] ,\end{aligned} \tag{A7}$$

where $G(\vec{r})$ is given by (2.13) for large r, and satisfies the relation (2.11).

Equations (A2), (A4), and (A6) can be combined into an expression for $C(r)$, namely

$$C(r) \approx \left[\frac{a}{r}\right]^{\eta_T}(e^{-iI(\vec{r},\vec{0})})_d , \tag{A8}$$

where

$$\eta_T = \frac{1}{2\pi K} \ . \tag{A9}$$

Since $I(\vec{r},\vec{0})$ is a linear functional of the Gaussian variable $\vec{q}(\vec{r})$, we have

$$[\exp(-iI(\vec{r},\vec{0}))]_d = \exp\{-\tfrac{1}{2}[(I(\vec{r},\vec{0}))^2]_d\} , \tag{A10}$$

where

$$\begin{aligned}[(I(\vec{r},\vec{0}))^2]_d = \sigma\int d^2r'\\ \times|\vec{\nabla}\,'[G(\vec{r}\,'-\vec{r})-G(\vec{r}\,')]|^2 \ .\end{aligned} \tag{A11}$$

The integral over $\vec{r}\,'$ is readily evaluated by integrating by parts, and using the relation (2.11). The resulting expression for $C(r)$ is

$$C(r) \sim \frac{1}{r^{\eta_T+\eta_\sigma}} , \tag{A12}$$

where η_T was given in Eq. (A9), and

$$\eta_\sigma = \sigma/2\pi \ . \tag{A13}$$

APPENDIX B: RECURSION RELATIONS VIA REPLICAS

A useful check on the recursion relations for the random Coulomb gas derived in Sec. IV is provided by the replica trick.[28] With a fixed configuration of dipoles, the free energy associated with the Coulomb-gas Hamiltonian (2.16) is given by

$$e^{-F/k_BT}=Z\equiv \mathop{\mathrm{Tr}'}_{\{m_i\}} e^{-H_c/k_BT} . \tag{B1}$$

We can average this free energy over the randomness via the relation

$$(F/k_BT)_d=\left[\lim_{n\to 0}\frac{1-Z^n}{n}\right]_d . \tag{B2}$$

Assuming that the limit process $n\to 0$ and the random average commute, we first integrate over the randomness, and take the limit $n\to 0$ at the end of the calculation. Proceeding in this way, we find a "replicated" Coulomb-gas Hamiltonian, namely

$$\begin{aligned}\frac{H_c}{k_BT}= &-\pi K_1\sum_{i\neq j}\sum_{\alpha} m_i^\alpha m_j^\alpha \ln(r_{ij}/a)\\ &-\pi K_2\sum_{i,j}\sum_{\alpha\neq\beta} m_i^\alpha m_j^\beta \ln(r_{ij}/a)\\ &+\frac{E_c}{k_BT}\sum_i\sum_\alpha (m_i^\alpha)^2 ,\end{aligned} \tag{B3}$$

where α is a replica index running from 1 to n, and

$$K_1=K-\sigma K^2,\ \ K_2=-\sigma K^2 . \tag{B4}$$

Equation (B3) is a special case of a more general Hamiltonian in replica space considered by Cardy and Ostlund.[4] Although the coupling between replicas K_2 is not present in their initial Hamiltonian, it is generated by their random p-fold symmetry-breaking field. Specializing the Cardy-Ostlund results to the Hamiltonian (B3), we find (via the same technique used originally by Kosterlitz[7] for nonrandom Coulomb gases) the recursion relations

$$\frac{dK_1}{dl}=-4\pi^3y^2[K_1^2+(n-1)K_2^2] , \tag{B5a}$$

$$\frac{dK_2}{dl}=-4\pi^3y^2[2K_1K_2+(n-2)K_2^2] , \tag{B5b}$$

$$\frac{dy}{dl}=(2-\pi K_1)y . \tag{B5c}$$

Taking limit $n\to 0$, we obtain results equivalent to the recursion relations (4.2), (4.4), and (4.5) derived without replicas in Sec. IV.

One can also use the replica technique to check that a small amount of randomness in the coupling $\tilde{J}_{ij}$ in Eq. (2.2a) is irrelevant along the stable portion of the xy fixed line. We apply the replica trick to a long-wavelength version of Eq. (2.1), namely

$$\begin{aligned}\frac{H'}{k_BT}= &\tfrac{1}{2}K\int d^2r\,|\vec\nabla\theta|^2+\tfrac{1}{2}\int \delta K(\vec r)\,|\vec\nabla\theta|^2\\ &-K\int d^2r\,\vec q(\vec r)\cdot(\vec\nabla\theta) ,\end{aligned} \tag{B6}$$

where

$$\delta K(\vec r)=\delta J(\vec r)/k_BT \tag{B7}$$

represents a local variation in the coupling strength. The remaining terms in Eq. (B6) are the same as those in Eq. (1.2), with the contribution proportional to $|\vec q(\vec r)|^2$ suppressed. The variable $\vec q(\vec r)$ has a quenched probability distribution given by (1.3), and we assume that the distribution for the quenched variable $\delta J(\vec r)$ is

$$P'(\delta J(\vec r))\propto \exp\left[-\frac{1}{2\Delta}\int d^2r\,|\delta J(\vec r)|^2\right] . \tag{B8}$$

Upon applying the replica trick[28] to (B6), we find the replicated Hamiltonian

$$\begin{aligned}\frac{H_c'}{k_BT}= &\int d^2r\left[\tfrac{1}{2}K_1\sum_\alpha|\vec\nabla\theta_\alpha|^2+\tfrac{1}{2}K_2\sum_{\alpha\neq\beta}\vec\nabla\theta_\alpha\cdot\vec\nabla\theta_\beta\right]\\ &-\tfrac{1}{8}\Delta\int d^2r\,|\vec\nabla\theta_\alpha|^2\,|\vec\nabla\theta_\beta|^2 ,\end{aligned} \tag{B9}$$

where α and β are replica indices, and K_1 and K_2 are given by (B4). Upon inserting vortices into the first two terms of (B9), we obtain the Coulomb-gas Hamiltonian (B3) considered above. When vortices can be neglected, the last term in (B9) is clearly irrelevant along the corresponding Gaussian fixed line because it involves four powers of the gradient operator.[29] Since Δ iterates to zero at long wavelengths, we are justified in making the replacement (2.3).

It is interesting to see how the Harris criterion[21] leads to the same conclusion. The Harris criterion[30] is applicable in the following sense: A spatially varying coupling $J(\vec r)\equiv J+\delta J(\vec r)$ causes local variations in T_c relative to a system without this sort of randomness. What is important is the shift in the critical temperature $\delta T_c(\vec r)$ averaged over a coherence area ξ^2,

$$\langle\delta T_c(\vec r)\rangle\sim \int_{\text{coh. area}} d^2r\,\delta J(\vec r)\Big/\xi^2 . \tag{B10}$$

Just above the Kosterlitz-Thouless critical temperature T_c, we can calculate the mean-square fluctuation of this quantity, averaging over the probability distribution (B8). The result is

$$[\langle\delta T_c(\vec r)\rangle^2]_d\sim\frac{\Delta}{\xi^2} . \tag{B11}$$

These fluctuations must be negligible relative to $(T-T_c)^2$ for randomness in J to be unimportant near the critical point,

$$\frac{\Delta}{\xi^2}\ll(T-T_c)^2 . \tag{B12}$$

Since (see Ref. 7) ξ diverges strongly,

$$\xi \sim \exp(\text{const}\,|T-T_c|^{-1/2}) , \tag{B13}$$

this criterion is more than satisfied for two-dimensional *XY* models. Equation (B12) reduces to the usual Harris criterion when $\xi(T)$ diverges as a power law $\xi \sim |T-T_c|^{-\nu}$.

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