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Block-spin renormalization group in the large-n limit

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We present a real-space renormalization-group (RG) study of the n-component classical Heisenberg model in the large-n limit. We obtain exact expressions for correlation functions of block spins, given those of the original spins. We show how to extract the critical behavior of the model from such information. The model provides a good testing ground for evaluating the effectiveness of Monte Carlo RG techniques. We study numerically the implementation of the real-space RG transformation on finite lattices. In two dimensions, we obtain the β function as a function of temperature. In three dimensions, we calculate the critical temperature and the thermal exponent. We discuss the effects of the use of finite lattices and of the use of different forms for the initial action on the accuracy of the results.

I. INTRODUCTION

Real-space renormalization-group (RG) methods[1] provide a conceptually appealing and potentially powerful way of studying critical phenomena. When coupled with Monte Carlo techniques these block-spin RG's can be implemented without restriction to a perturbative regime, and without the use of uncontrolled approximations.[2,3] It is important to understand the strengths and weaknesses of these methods and the approximations inherent in them in order to have confidence when applying them to new systems. To this end we would like to have some system where a real-space RG can be implemented exactly and its working followed in detail. Bell and Wilson[4] have done just this with the Gaussian model using a linear RG. Their work is very instructive and we will have more to say about it later. Hilhorst *et al.*[5] have solved the two-dimensional Ising model on a triangular lattice exactly with the use of an ingenious procedure based upon the star-triangle transformation. This work, although very interesting, seems difficult to generalize.

In this paper we study via block-spin techniques the n-component classical Heisenberg model in the large-n limit, an exactly solvable model equivalent to the spherical model. This system displays interesting critical behavior: asymptotic freedom and dynamical mass generation in two dimensions (2D), and non-Gaussian critical point in three dimensions (3D).

Ma[6] has already discussed an exact momentum shell RG for this model, illuminating the workings of such RG's. We focus here on a lattice version of the model and on a block-spin RG. In particular, we show how to implement some aspects of such an RG exactly. We utilize a formulation of the large-n limit developed by a number of authors[7] in which one studies spin-rotationally invariant quantities only. Instead of using $\vec{S}_i$ as a variable we use $F_{ij} \equiv \vec{S}_i \cdot \vec{S}_j$. All correlations as $n \to \infty$ are determined by the saddle-point value of F_{ij}, the so-called master field.[7] We show how to deduce the master field of the block spins from the original one. We do not know how to determine the block H but we can get all correlations of block spins. Monte Carlo renormalization-group (MCRG) techniques are designed to use only correlations, so we can follow their workings exactly.

II. ANALYTIC DEVELOPMENTS

The classical Heisenberg model with an $O(n)$ symmetric interaction is defined by the Hamiltonian

$$H = -\sum_{i,j} \rho_{ij} \vec{S}_i \cdot \vec{S}_j \ , \tag{2.1}$$

where the $\vec{S}_i$ are n-component unit vectors defined on the sites of a d-dimensional lattice. ρ_{ij} is a short-ranged and translationally invariant interaction. The partition function for the model is given by

$$Z = \int D^n S_i \prod_i \delta(S_i^2 - 1) e^{-H/T} \ . \tag{2.2}$$

We divide the lattice into cubes of two sites per side (2^d-site blocks) and define block-spin variables by the relation

$$\vec{t}_l=\frac{\sum_a \vec{S}_{l,a}}{\left|\left|\sum_a \vec{S}_{l,a}\right|\right|} . \tag{2.3}$$

Here, l labels a block and a runs through all the sites in the block. Note that the new block-spin variables are again of unit length. The interactions between the block-spin variables are determined by the renormalized Hamiltonian $H'(\vec{t}\,')$ defined by

$$\exp\left[-\frac{1}{T'}H'(\vec{t}\,')\right]=\int \prod_{l,a}[D\vec{S}_{l,a}\delta(\vec{S}_{l,a}^2-1)]\prod_i \delta\left[\vec{t}_l-\frac{\sum_a \vec{S}_{l,a}}{\left|\left|\sum_a S_{l,a}\right|\right|}\right]\exp\left[-\frac{1}{T}H(\vec{S})\right] . \tag{2.4}$$

From (2.3) and (2.4) we can obtain a relation for the two-point correlation functions of the block-spin variables:

$$\langle \vec{t}_l\cdot\vec{t}_m\rangle_{H'}=\left\langle \frac{\sum_{a,a'}\vec{S}_{l,a}\cdot\vec{S}_{m,a'}}{\left|\left|\sum_a \vec{S}_{l,a}\right|\right|\left|\left|\sum_{a'}\vec{S}_{m,a'}\right|\right|}\right\rangle_H , \tag{2.5}$$

where the angle brackets stand for the averages

$$\langle A\rangle_H=\frac{\int DS_i\prod_i \delta(S_i^2-1)e^{-H/T}A}{Z} . \tag{2.6}$$

The relation (2.5) simplifies in the large-n limit. To see this, it is convenient to rewrite the partition function (2.2) in terms of the spin-rotationally invariant quantities[7]:

$$F_{ij}=\vec{S}_i\cdot\vec{S}_j . \tag{2.7}$$

The Jacobian for this transformation is given by

$$J(F_{ij})=\int D\vec{S}_i\prod_{i,j}\delta(F_{ij}-\vec{S}_i\cdot\vec{S}_j) . \tag{2.8}$$

After implementing the δ-function constraints by using Lagrange multipliers α_{ij} and integrating over the spin variables, one obtains

$$J(F_{ij})=\int\prod_{i,j}D\alpha_{ij}\exp\left[i\sum_{i,j}\alpha_{ij}F_{ij}-\frac{n}{2}\mathrm{tr}\ln(i\alpha_{ij})\right] . \tag{2.9}$$

In the large-n limit the integral (2.9) can be obtained from the saddle point and gives

$$J(F_{ij})=\text{const}\exp\left[\frac{n}{2}\mathrm{tr}\ln F_{ij}\right] , \tag{2.10}$$

so that the partition function (2.2) can be rewritten as

$$Z=\int\prod_{i,j}DF_{ij}\prod_i\delta(F_{ii}-1)\times\exp\left[\frac{1}{T}\sum_{i,j}\rho_{ij}F_{ij}+\frac{n}{2}\mathrm{tr}\ln F_{ij}\right] . \tag{2.11}$$

It is convenient to implement the fixed-length constraint by introducing an additional integration variable, yielding

$$Z=\int dF_{ij}D\lambda_i\exp\left[\frac{n}{T}\left[\sum_{i,j}\rho_{ij}F_{ij}+\frac{T}{2}\mathrm{tr}\ln F_{ij}+\sum_i\lambda_i(F_{ii}-1)\right]\right] . \tag{2.12}$$

We have also rescaled the interactions ρ_{ij} and λ_i by a factor n. The average of any function of the invariants F_{ij} is now obtained by evaluating the function at the saddle-point field of (2.12), i.e., for a given function f

$$\langle f(\vec{S}_i\cdot\vec{S}_j)\rangle_H=f(\tilde{F}_{ij}) , \tag{2.13}$$

where the saddle-point fields $\tilde{F}_{ij}$ are given by the equation (assuming a translational invariant saddle point)

$$\tilde{F}(k)=\frac{1}{2}\frac{T}{\rho(k)+\lambda} , \tag{2.14}$$

with λ determined by the fixed-length condition $\tilde{F}_{ii}=1$, i.e.,

$$\frac{1}{2N}\sum_k\frac{T}{\rho(k)+\lambda}=1 . \tag{2.15}$$

Note that λ acts like a mass squared in Eq. (2.14).

In two dimensions Eq. (2.15) has a nonzero solution for λ for arbitrary T, which for small T behaves as

$$\lambda \sim \exp(-4\pi/T) . \tag{2.16}$$

This behavior is usually called asymptotic freedom. For $d>2$ there is a critical point at a finite temperature T_c determined by

$$\frac{1}{2N}\sum_k \frac{1}{\rho(k)} = \frac{1}{T_c} , \tag{2.17}$$

and λ has the behavior

$$\lambda \sim e^{2/(d-2)} , \tag{2.18}$$

with $\epsilon = |T-T_c|$. The thermal exponent follows from Eq. (2.18):

$$\nu = 1/(d-2) , \tag{2.19}$$

i.e., it is non-Gaussian. At the critical point, however, λ is zero and the model behaves as a Gaussian model so that the exponent $\eta = 0$.

Returning now to the block-spin correlation function (2.5), we see from Eq. (2.13) that we can factor the right-hand side to get

$$F'_{l,m} = \frac{1}{c}\sum_{a,a'} F_{la,ma'} , \tag{2.20a}$$

$$c = \sum_{aa'} F_{la,la'} , \tag{2.20b}$$

where the prime denotes correlation functions of block spins.

Equation (2.20) is the basic equation of our renormalization-group transformation. Since in the large-n limit higher-order correlation functions factor onto products of two-point correlation functions, Eq. (2.20) completely defines arbitrary correlation functions of the block-spin system. Unfortunately, it does not seem possible to obtain a simple closed form for the Hamiltonian $H'(t)$ describing the interactions between the block-spin variables. From Eq. (2.4) we obtain

$$\exp\left[-\frac{1}{T'}H'(F')\right]\delta(F'_{ll}-1) = \int D\tilde{F}_{la,ma'} \exp\left[\frac{1}{T}H(\tilde{F}_{ij}) + \frac{n}{2}\operatorname{tr}\ln\tilde{F}_{ij} - \frac{n}{2}\operatorname{tr}\ln F'_{lm}\right] , \tag{2.21}$$

where the integral is restricted to the fields $\tilde{F}_{la,ma'}$ satisfying the equation

$$F'_{l,m} = \frac{1}{c}\sum_{aa'} \tilde{F}_{la,ma'} , \quad c = \sum_{aa'} \tilde{F}_{la,la'} , \tag{2.22}$$

for given block-spin correlations $F'_{l,m}$. We have not been able to obtain a closed expression for the renormalized Hamiltonian $H'(F')$. The renormalized Hamiltonian $H'(F')$ contains more information than all the correlation functions in this model, since the correlation functions are determined only by the saddle-point fields of H' in the large-n limit. Thus different H''s can give rise to the same correlation functions. One would need to compute finite-n corrections to the $n=\infty$ correlation functions to obtain from them the Hamiltonian H'.

Here we will focus on the recursion relation for the correlation functions, Eq. (2.20). To put the discussion in the same framework as usual renormalization-group transformations, we will write a corresponding Hamiltonian for the block-spin variables assuming it is *bilinear* in the spins, as the original one is. In that case the relation between correlation functions and the underlying Hamiltonian is given by Eqs. (2.14) and (2.15). We will call this the effective Hamiltonian for the block spins.

Fixed points and perturbations

The recursion relation for the correlation functions (2.20) can be written symbolically as

$$F'(\vec{y}) = \frac{DF(2\vec{y})}{c} , \tag{2.23}$$

with the operator D denoting the block-sum of Eq. (2.20), and

$$c = DF(0) . \tag{2.24}$$

In Fourier space this equation becomes

$$F'(q) = \frac{1}{c}\frac{1}{2^d}\sum_{\vec{l}=0,1} F\left(\frac{\vec{q}}{2}+\pi\vec{l}\right)\left|u_2\left(\frac{\vec{q}}{2}+\pi\vec{l}\right)\right|^2 , \tag{2.25}$$

with

$$u_2^2(q) = \prod_{i=1}^{d}\frac{\sin^2 q_i}{\sin^2(q_i/2)} , \tag{2.26}$$

$$c = \frac{1}{N^d}\sum_{\vec{q}}\sum_{\vec{l}=0,1} F\left(\frac{\vec{q}}{2}+\pi\vec{l}\right)\left|u_2^2\left(\frac{q}{2}+\pi\vec{l}\right)\right| . \tag{2.27}$$

Equation (2.23) looks like a linear recursion relation; in fact, it is just the recursion relation for the Gaussian model[4] if c is a constant. The constraint on c embodied in Eq. (2.24), however, makes it a non-

linear transformation. The corresponding effective Hamiltonian $\rho'(q)$ satisfies the relation

$$F'(q)=\frac{1}{2}\frac{T'}{\rho'(q)+\lambda'}\ , \tag{2.28}$$

with $F'(q)$ given by (2.22) and λ' determined by

$$\frac{2^d}{2N}\sum_q \frac{T'}{\rho'(q)+\lambda'}=1\ , \tag{2.29}$$

and T and T' determined by some normalization condition on the interaction $\rho(q)$, for example, $\lim_{q\to 0}\rho(q)/q^2=1$.

The fixed-point Hamiltonian is obtained from iterating Eq. (2.23), as[4]

$$\rho^*(q)=\left[\sum_{\vec{l}=0}^{\infty}\frac{1}{|\tilde{q}+2\pi\vec{l}\,|^2}\times\sum_{i=1}^{d}\frac{\sin^2(q_i/2)}{|(q_i/2)+\pi\vec{l}\,|^2}\right]^{-1}\ , \tag{2.30}$$

which is the same as in the Gaussian model. At the fixed point one has also $c^*=2^{d+2}$ and $\lambda^*=0$. Under iterations, T_c is also changed and tends to its fixed-point value T_c^* given by

$$T_c^*=\prod_n \frac{c^*}{c(n)}T_c\ , \tag{2.31}$$

where $c(n)$ denotes c at the nth iteration. Note the analyticity of $\rho^*(q)$ at $q=0$, implying locality in real space. This leads us to expect that finite-lattice approximations may work.

It is simple to obtain the spectrum of irrelevant perturbations. They are of the form

$$\delta\rho_i(q)=\frac{2}{T_c^*}\rho^{*2}(q)f_i(q)-\rho^*(q)\ , \tag{2.32}$$

where f_i satisfies

$$\frac{1}{N}\sum_q f_i(q)=1\ , \tag{2.33a}$$

$$\frac{Df_i(2\vec{y})}{c^*}=x_i f_i(\vec{y})\ . \tag{2.33b}$$

It is easy to verify that a perturbation of the form (2.32) leaves the system at the critical point, since to first order in ϵ

$$\frac{1}{\rho^*(q)+\epsilon\delta\rho(q)}=\frac{1}{\rho^*(q)}-\epsilon\left[\frac{2}{T_c^*}f_i(q)-\frac{1}{\rho^*(q)}\right]\ , \tag{2.34}$$

so that Eq. (2.15), with $\lambda=0$, is still satisfied due to the condition (2.33a). To see that (2.32) is indeed an eigenfunction, we write

$$F(q)=\frac{T_c^*}{2}\frac{1}{\rho^*(q)+\epsilon\delta\rho}=F^*-\epsilon(f_i-F^*)\ . \tag{2.35}$$

Under the transformation (2.23) we obtain, to lowest order in ϵ and using (2.33b),

$$F'=\frac{c^*}{c}F^*-\epsilon(x_iF_i-F^*)\ . \tag{2.36}$$

From (2.24) we obtain for c

$$c=c^*[1-\epsilon(x_i-1)]\ , \tag{2.37}$$

and replacing in (2.36) we obtain

$$F'=F^*-x_i\epsilon(f_i-F^*)\ , \tag{2.38}$$

which implies

$$\delta\rho_i'(q)=x_i\delta\rho_i(q)\ . \tag{2.39}$$

To find the eigenvalues and eigenfunctions we iterate the relation (2.33b). The eigenvalues that correspond to short-ranged interactions are of the form

$$x_i=1/2^{2i}\ , \tag{2.40}$$

with $i\geq 1$ an integer, and the corresponding rotationally invariant[8] eigenfunctions are

$$f_i(q)=\sum_{\vec{l}=0}^{\infty}\frac{1}{|\vec{q}+2\pi\vec{l}\,|^{2-2i}}\prod_{j=1}^{d}\frac{\sin^2(q_j/2)}{|(q_j/2)+\pi l_j|^2}\ . \tag{2.41}$$

The fact that i is an integer guarantees that $\delta\rho_i(q)$ is analytic at the origin so that the interactions in real space are short ranged. Though the irrelevant eigenvalues are the same as in the Gaussian model,[4] the corresponding eigenfunctions (2.32) are somewhat different in form. One important difference with the Gaussian case is the absence of a marginal operator. Here, the constraint of fixed-length spins produces a nonlinear RG with no marginality. In his analysis of the soft-spin large-n model, Ma[6] finds additional irrelevant operators. We presumably avoid these by our restriction to quadratic initial actions.

The relevant eigenfunction is found to be

$$\delta\rho_0(q)=\frac{1}{F^*(q)}\ , \tag{2.42}$$

and from (2.25) and (2.28) one obtains the recursion relations

$$\epsilon'=2^{d-2}\epsilon\ , \tag{2.43a}$$

$$\lambda'=2^2\lambda\ , \tag{2.43b}$$

with $\epsilon=T-T_c$. Equation (2.43a) implies that the

thermal eigenvalue is $\nu=1/(d-2)$ while (2.43b) simply says that the correlation length in the block-spin system is half the one in the original system.[9] For the case of $d=2$ one has to go to higher order in the recursion relation (2.43a) and gets

$$\epsilon'=\epsilon+\frac{\ln 2}{2\pi}\epsilon^2 , \tag{2.44}$$

indicating marginal behavior, from which the behavior (2.16) follows. The parameter c close to the fixed point is given by

$$\frac{c}{c^*}=1+(1-d^{d-2})\epsilon . \tag{2.45}$$

III. NUMERICAL STUDIES ON FINITE LATTICES

We have seen in the last section how to calculate expectation values of block spins from those of original spins. In this section we discuss techniques for doing RG analysis with the use of the block-spin correlations. In 2D, where the ($n \geq 3$) Heisenberg model has no phase transition, a particularly simple scheme has been applied by Shenker and Tobochnik (ST).[3] Imagine that the flows are like those in Fig. 1. Pick some path in H space, such as the pure nearest-neighbor line. Iterate point D until it is very close (point F) to the one-dimensional unstable manifold of the zero-temperature fixed point (the renormalized trajectory). Now search the path for some point E such that E iterated one fewer time than D reaches F. One verifies this by comparing a

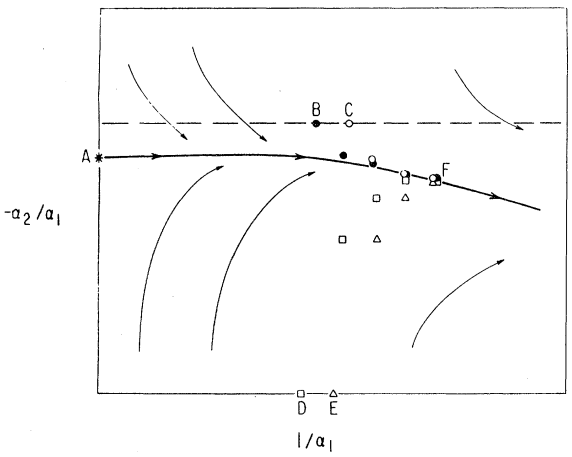

FIG. 1. Schematic diagram of renormalization-group flows. α_1 and α_2 are coupling constants in the Hamiltonian which are proportional to $1/T$. Point A represents the low-temperature fixed point. B, C, D, and E represent possible starting points for the iteration of the RG procedure.

variety of block expectation values of the two H's. If expectation values match, we say that the two H's are the same. To eliminate some finite-lattice effects we require the expectation values to be computed on the same size lattice. A 32×32 lattice iterated twice would yield an 8×8 lattice of block spins. This could be compared to a 16×16 lattice blocked once, again giving an 8×8 block lattice. Since the correlation length is reduced by a factor of 2 at each iteration and since both H's at point F have (almost) the same ξ, it follows that ξ at point D is twice ξ at point E. In this way we can map out the ξ behavior along the path.

This method can break down in a variety of ways. The H's on the path chosen could be too far away from the renormalized trajectory to reach it in a reasonable number of iterations. (Remember, we plan to iterate by blocking spins on some finite-size lattice. The number of iterations we can perform is limited by the size of the lattice we start with.) Choosing a better path, such as the upper one in Fig. 1, can help this. At sufficiently high temperature we expect the renormalized trajectory to stop being attracting altogether, indicating the lack of universality in systems where correlation lengths are short.

We implemented this technique in the large-n limit. Nearest-neighbor (NN), next-nearest-neighbor (NNN), and third-nearest-neighbor (3NN) correlation functions of block spin were computed on finite lattices by iterating Eq. (2.25) and Fourier-transforming at the end to obtain the correlation functions in real space. We determined matching by requiring the NN expectation value on the largest scale permitted by the lattice size to agree. Table I displays an example of the procedure used: The NN correlation function of 32×32 blocks on a 64×64 lattice at coupling contant $K=0.80$ matches the same correlation function of 16×16 blocks on a 32×32 lattice at coupling constant $K'=0.690\,55$. Note that when matching is imposed on the NN correlation function, the other correlation functions also approximately agree, indicating that the Hamiltonians describing both lattices are approximately the same, as we would expect for such low temperatures. Note also that matching also occurs approximately at one lower level of blocking, indicating that here one is already very close to the renormalized trajectory.

In Fig. 2 we show results for $\Delta K=K'-K$, obtained with the use of nearest-neighbor action, starting from 32×32 and 64×64 lattices. The exact results for ΔK are obtained by computing the value of K' for which the correlation length ξ obeys

$$\xi(K')=\tfrac{1}{2}\xi(K) . \tag{3.1}$$

TABLE I. An example of the matching for the two-dimensional lattice using 64×64 and 32×32 lattices: $K=0.8$, 64×64 lattice; $K'=0.69055$, 32×32 lattice.

Block length for	NN		NNN		3NN	
64×64 lattice	64×64	32×32	64×64	32×32	64×64	32×32
1	0.68802		0.60305		0.54741	
2	0.72869	0.64003	0.63571	0.54269	0.54875	0.47999
4	0.70788	0.67340	0.59666	0.56393	0.48824	0.46478
8	0.65059	0.63977	0.51952	0.51056	0.40181	0.39628
16	0.58490	0.58266	0.44947	0.44833	0.40017	0.39988
32	0.68512	0.68512	0.56997	0.57063		

The exact correlation length is computed along a lattice axis (such as the x axis) by finding the pole of Eq. (2.14) closest to the real axis with $k_y=0$. It is shown on the upper horizontal line of Fig. 2. At low temperatures the correlation length behaves as

$$\xi(K)=\exp(2\pi K)\ , \tag{3.2}$$

and the β function for this model is

$$\beta(T)=\frac{dT}{d\ln b}=\frac{1}{2\pi}T^2\ , \tag{3.3}$$

so that our answers should approach the asymptotic value

$$\Delta K(K)=\frac{1}{2\pi}\ln 2 \tag{3.4}$$

since we are changing length scales by a factor of 2. Notice in Fig. 2 the increased accuracy in going to larger lattices, as the effect of irrelevant operators are iterated away. At very large K (low T, large ξ) the amplitude of the marginally relevant operator is extremely small and its effect on expectation values is swamped by that of irrelevant operators, even after many blockings. Thus accuracy is degraded at very large K. At very small K the renormalized trajectory ceases to be attracting and the results also start to deviate markedly from the expected ones.

We have repeated the calculation using another trajectory that is closer to the renormalized trajectory, a path specified by

$$\rho(k)=4-2\cos k_x-2\cos k_y +0.2(4-2\cos k_x-2\cos k_y)^2\ , \tag{3.5}$$

which was used in Ref. 3. On the scale of Fig. 2, the results are almost indistinguishable from the exact answers for the whole range of temperatures for both the 32×32 and 64×64 lattices, and are therefore not shown in that form. In Fig. 3 we show the fractional error of the renormalization-group calculation for this trajectory, as a function of the correlation length ξ. The results remain very good even at high temperatures (correlation length $\xi\sim1.5$), suggesting that we are very close to the renormalized trajectory.

We also display in Fig. 3 results obtained by matching at one lower level (16×16 blocks on the 64×64 lattice match 8×8 blocks on the 32×32 lat-

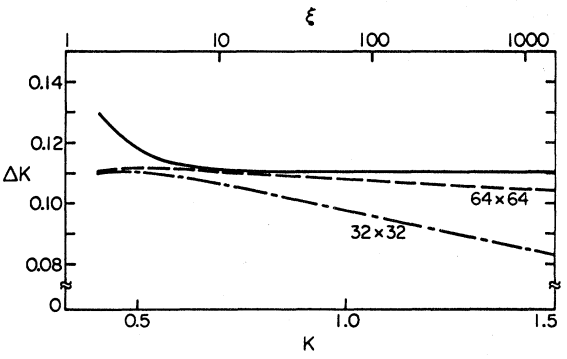

FIG. 2. ΔK vs K for the nearest-neighbor action. ΔK is the change in K needed to go from a lattice with correlation length ξ (at K) to one of $\xi/2$. The solid line is exact results for the infinite lattice. The dashed and dash-dotted lines are block-spin RG results for 64×64 and 32×32 lattices, respectively.

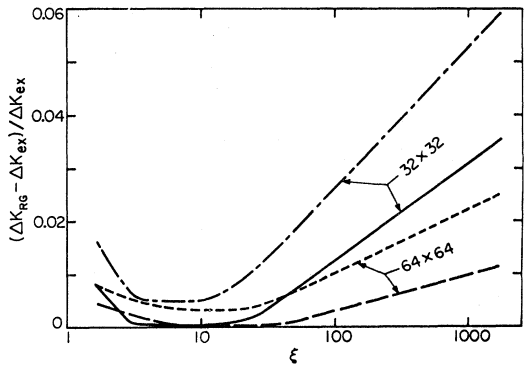

FIG. 3. Fractional error in ΔK vs K for the trajectory equation (3.5). The solid and long-dashed line are RG results in 32×32 and 64×64 lattices matching on the largest scale permitted by the lattice size. The dash-dotted and short-dashed lines are obtained by matching at one level less in the iteration.

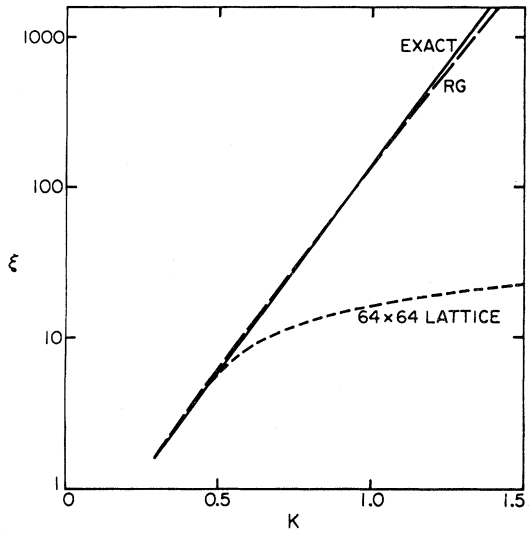

FIG. 4. Correlation length vs K for the trajectory equation (3.5). Solid line: exact results. Long-dashed line: RG results from 64×64 onto 32×32 lattices. Short-dashed line: correlation length for the 64×64 lattice.

tice). Such a procedure was used by ST due to the large statistical errors involved in the larger blocks. As can be seen, this increases the error substantially. Nevertheless, for the case studied by ST [32×32 lattice matches with 8×8 blocks, trajectory Equation (3.5)] Fig. 3 suggests that the error was less than 5% for ξ from approximately 1 to 1000.

Finally, we show in Fig. 4 the correlation length ξ as a function of K for the action equation (3.5). The RG results were obtained by iterating the results for ΔK vs K up to high temperatures ($K\sim0.25$), and using the value of ξ for the finite lattice at these temperatures. That information can be obtained in an actual calculation from a high-temperature expansion or by straight Monte Carlo techniques. We also show in Fig. 4 the correlation length for the finite 64×64 lattice. Note how rapidly the finite-lattice correlation length starts to deviate from the exact answer. In contrast, the RG results, which also used a 64×64 lattice, are in remarkable agreement with the exact solution even for correlation lengths much larger than the lattice used in the calculation.

3D

For $n\to\infty$ in 3D we have a conventional phase transition presumably described by a standard fixed point. Methods to deal with such a situation have been discussed by Swendsen[2] and Wilson.[10] First, one wants to locate the critical temperature. For simplicity, imagine starting with an H of the form

$$H=H^*+a_R O_R+a_I O_I \tag{3.6}$$

(O_R and O_I are relevant and irrelevant operators with eigenvalues λ_R and λ_I, respectively, and a_R and a_I are small enough that a linear analysis can be applied). After n iterations one has for the block H

$$H^{(n)}=H^*+a_R\lambda_R^n O_R+a_I\lambda_I^n O_I\ . \tag{3.7}$$

The effect of the relevant perturbation grows at large block sizes. If $a_R=0$ (one is at the critical point), the block H's converge to H^* and the block expectation values converge to their fixed-point values. So to find K_c one searches for the value of K where the nth and $(n+1)$th expectation values agree most closely. This becomes more and more accurate as n grows. Errors in the procedure due to the irrelevant operators are nominally of order $(\lambda_I/\lambda_R)^n$, the leading irrelevant operators dominating eventually. Table II shows the matching obtained comparing blocks on a 64^3 lattice and a 32^3 lattice. (Again we arbitrarily chose to match the NN correlations at the lowest level as closely as possible and use different size lattices to cancel finite-lattice effects.) In Table III we display results for K_c obtained in this manner for different size lattices. Note the convergence to the correct answer, although it is difficult to find evidence for the naive $(\lambda_I/\lambda_R)^2\sim(\frac{1}{8})^2$ convergence rate.

To actually compute the relevant eigenvalue we

TABLE II. Matchings for the 3D lattice using 64^3 and 32^3 lattices for $K=0.252\,65$. The value of K was determined by requiring the NN correlations at the lowest level to coincide.

Block length for 64^3 lattice	NN		NNN		3NN	
	64^3	32^3	64^3	32^3	64^3	32^3
1	0.340 49		0.218 38		0.172 46	
2	0.441 98	0.340 96	0.311 71	0.219 07	0.253 76	0.173 32
4	0.497 64	0.444 33	0.360 05	0.315 23	0.295 77	0.258 13
8	0.525 39	0.507 51	0.389 08	0.375 16	0.326 17	0.314 69
16	0.569 76	0.566 09	0.452 60	0.450 30	0.402 17	0.400 51
32	0.765 15	0.765 15	0.683 94	0.684 56	0.643 00	0.643 91

TABLE III. Critical temperature of the 3D lattice from different lattice sizes (near-neighbor action).

Size	K_c
8^3	0.231 14
16^3	0.248 25
32^3	0.252 06
64^3	0.252 65
∞	0.252 73

TABLE IV. Thermal exponent of the 3D lattice from different lattice sizes (near-neighbor action).

Size	$1/\nu$
8^3	1.221
16^3	1.126
32^3	1.039
64^3	1.0129
∞	1

could follow the method of Swendsen, picking a finite set of operators and computing a truncated version of the linearized RG. (Here we would not be able to compute the derivatives by correlations, but would have to do it numerically.) Instead we choose to demonstrate a method proposed by Wilson[10] that does not depend on a truncation. Imagine again an H as in Eq. (3.6). If we consider the derivative of a block correlation function with respect to a_R, we have, after n iterations,

$$\left[\frac{d\langle SS\rangle_n}{da_R}\right]_{a_R=0}=\lambda_R^n\langle SSO_R\rangle_c\ , \tag{3.8a}$$

$$\left[\frac{d\langle SS\rangle_{n+1}}{da_R}\right]_{a_R=0}=\lambda_R^{n+1}\langle SSO_R\rangle_c\ . \tag{3.8b}$$

If we divide the second equation by the first, we see that the right-hand side is just λ_R (again we use different size initial lattices so our block lattices are of the same size to cancel finite-lattice effects). In a real MCRG calculation one would compute the right-hand sides of Eqs. (3.8). For our case, however, these correlation functions vanish in the large-n limit by factorization, and we should therefore compute these correlation functions to higher order in $1/n$. However, it is simpler to compute the left-hand sides of Eqs. (3.8) by numerical differentiation. Note that a higher-order term in the recursion relations such as

$$a_R'=\lambda_R a_R+a_R a_I \tag{3.9}$$

causes an error of order λ_I^n, so we expect slower convergence here than for T_c.

In Table IV we display results for ν obtained using Eq. (3.8), with $\langle SS\rangle$ being the nearest-neighbor correlation function. Other operators give similar results. Although the answers systematically improve, we do not observe quite the expected rate.

In order to improve convergence we attempted to reduce the amplitude of the leading irrelevant operator. Proceeding in an *ad hoc* fashion we added terms to the action so that the small-k behavior of $\rho(k)$ reproduced that of $\rho^*(k)$ to order k^4. The action is given by

$$\begin{aligned}\rho(k)=&\,6-2\cos k_x-2\cos k_y-2\cos k_z+\tfrac{1}{12}(6-2\cos k_x-2\cos k_y-2\cos k_z)^2\\&+\tfrac{1}{12}[(2-2\cos k_x)^2+(2-2\cos k_y)^2+(2-2\cos k_z)^2]\ .\end{aligned} \tag{3.10}$$

Results for this H are shown in Table V. Note the improved numbers, but also note the nonuniform convergence. Interaction of errors in the T_c determination with the ν determination are a possible explanation.

IV. DISCUSSION

The analytic results of Sec. II confirm the existence of eigenvalues and eigenvectors necessary to produce the correct critical behavior of the model. (Again note the absence of certain irrelevant operators when we restrict ourselves to quadratic actions.) The operators are local in real space and hence we expect finite-lattice approximations to be effective. The numerical results of Sec. IV confirm this. In 2D along a well-chosen line of H's we can get an accuracy of about 1% for β-function numbers in regimes where the correlation length is far larger than the lattice. In 3D with a reasonable size lattice we can get T_c to a few parts in 10 000 and ν to $\sim 1\%$.

Although these 3D numbers are good, the approach to these numbers does not seem to follow the

TABLE V. Critical temperature and thermal exponents for the 3D lattice and the action given by Eq. (4.5).

Size	K_c	$1/\nu$
8^3	0.167 07	1.066
16^3	0.176 39	0.998
32^3	0.177 96	0.990
64^3	0.178 21	0.990
∞	0.178 27	1

naive convergence rates predicted by the linearized analysis. More subtle effects that we do not completely understand are contributing to the errors. These need to be further elucidated. In connection with this, techniques for picking out the leading irrelevant operators should be implemented on this model.

Other extensions that seem worth doing include adding in magnetic fields. The $n \to \infty$ model has a first-order transition in H below T_c controlled presumably by a discontinuity fixed point. It is straightforward to see that the magnetic field scales appropriately at $T=0$, and so the situation seems amenable to analysis.

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[8]There are also other eigenfunctions with the same eigenvalues that are only cubic invariant.

[9]If one extends this perturbative analysis around the fixed point to higher than linear order, one finds nonanalytic recursion relations for noninteger d. We believe that this is an artifact of our use of an effective quadratic action.

[10]K. G. Wilson (unpublished).