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Theoretical interpretation of resistive transition data from arrays of superconducting weak links

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The effects of a vortex-unbinding phase transition on the properties of a square lattice of superconducting weak links are discussed. Formulas relating bulk static quantities such as the penetration depth and transition temperature to single-junction parameters are given, as well as formulas for the current- and temperature-dependent resistance. Except for temperatures very near the transition temperature, these results can differ significantly from the continuum results of Halperin and Nelson, and thus may be more appropriate for description of junction arrays and some granular films.

I. INTRODUCTION

A large number of recent experimental and theoretical papers have been devoted to the superconducting-to-normal phase transition in two dimensions.[1–12] This work is largely in response to theoretical work[1,2] which noted that the Kosterlitz-Thouless[13,14] vortex-unbinding picture of the transition, which was originally applied to experiments on neutral superfluids, should apply to two-dimensional superconductors as well, provided that the perpendicular penetration depth $\lambda_\perp$ is larger than the sample width and length.[1]

Most of the experimental work to date has been on thin high-resistance films,[3–12] which were taken to be analogous to the neutral superfluid. All of the workers have noted difficulties which are presumably due to sample inhomogeneity. In spite of a large amount of care in both measurement and sample preparation, it is not certain that the vortex-unbinding picture completely describes the transition in the samples measured to date.[3,6]

Although the problems inherent to the thin-film experiments will probably be overcome, a number of workers have made large two-dimensional (2D) arrays of superconducting weak links[15–19] to study the problem in its lattice version. This method depends on making large numbers of weak links with nearly identical characteristics, which may be easier than making a uniform thin film. Voss and Webb[17] have made arrays of tunnel junctions, taking advantage of the large IBM effort to make identical junctions for a superconducting computer.[20] Resnick *et al.*[18] and Abraham *et al.*[19] are using S-S'-S and S-N-S junctions which, because of their relatively large characteristic dimensions (micrometers), can be made with good reproducibility.

In view of the importance of these measurements and the necessity for distinguishing between vortex unbinding and other effects in the interpretation of results, we have worked through the vortex-unbinding theory of the resistive transition in arrays of weak links. This makes it possible to compare experiment directly with an appropriate theory, rather than with an empirically modified continuum theory. In addition, most thin films which have been studied have been granular, and are perhaps better described by a lattice than a continuum.

In Sec. II we discuss static properties such as the transition temperature, penetration depth, and coherence length. Although many of the results of this section have been derived elsewhere, a unified treatment which clarifies what is exact and what is approximate is worthwhile. We also point out that some of these results have *no* adjustable parameters and are thus excellent tests of the theory.

In Sec. III the temperature and current dependences of the resistance near T_c are calculated. We find that, although the results are similar to the continuum case arbitrarily close to T_c, some of the formulas appropriate to the lattice are significantly different from the corresponding continuum formulas over the temperature range usually studied in experiments. (In this paper we will refer to equations which do not have a lattice spacing built in and which assume a Ginzburg-Landau temperature dependence for the superfluid density as continuum formulas.) In addition, we are able to place bounds on some of the coefficients, which should help to rule out fortuitous agreement between theory and experiment which can occur when unreasonable values of the coefficients are used.

II. STATIC PROPERTIES

Consider two superconducting islands connected by a single weak link which has a superconducting interaction energy[21]

$$E=E_J(T)[1-\cos(\phi_2-\phi_1)]\ , \qquad (1)$$

where $\phi_2-\phi_1$ is the difference in the phase of the order parameter on the two islands. Associated with this coupling energy there is a supercurrent[22,21]

$$i=i_c(T)\sin(\phi_2-\phi_1)\ , \qquad (2)$$

where $i_c(T)=e^*E_J(T)/\hbar$, with $e^*=2e$. In the absence of thermal fluctuations the parameter i_c would have the physical significance of a critical current, that is, a maximum current without voltage. Strictly speaking, however, for any $T>0$, there is always a nonzero voltage for any applied current because of thermally activated phase-slip processes, so the *observed* "i_c" will depend on sensitivity and will not coincide exactly with the parameter i_c which characterizes the coupling strength.

In a square array of superconducting islands connected by weak links, Eq. (1) can be summed over the islands to give a Hamiltonian

$$H=\sum_{\langle i,j\rangle} E_J(T)[(1-\cos(\phi_i-\phi_j)]\ , \qquad (3)$$

where the sum is taken over nearest-neighbor pairs with each pair counted once. Equation (3) shows that an array of weak links is isomorphic to the XY model, with a temperature-dependent coupling energy, and that the partition function for the system depends on the dimensionless parameter $T'\equiv kT/E_J(T)$.

To compare the lattice model to the more familiar continuum model in the long-wavelength limit, we require that the characteristic energies of vortices in the two systems be the same. First consider a vortex in a uniform film. The superfluid velocity at a distance r, far from the core, is given by $v_s=\hbar/m^*r$ for particles of mass m^*, so that the total kinetic energy in a region bounded by r_1 and r_2 is given by

$$U=\int_{r_1}^{r_2}(\tfrac{1}{2}m^*v_s^2)n_s^*(T)2\pi r\,dr$$
$$=\pi n_s^*(T)\frac{\hbar^2}{m^*}\ln(r_2/r_1)\ , \qquad (4)$$

where $n_s^*(T)$ is the areal superfluid pair density.

The energy of a vortex on a square lattice can be calculated in a similar way. At distance r, far from the vortex center, $\phi_i=\tan^{-1}(y_i/x_i)$ and, from (1),

$$E_{ij}\cong(E_J/2)(\phi_i-\phi_j)^2\ .$$

To calculate the coupling energy per island, we look at the interaction with the islands to the right and above; this prescription avoids double counting. Then

$$\phi_{ix}=\tan^{-1}[y_i/(x_i+s)]$$

and

$$\phi_{iy}=\tan^{-1}[(y_i+s)/x_i]\ ,$$

where s is the lattice spacing. For $x_i\gg s$ and $y_i\gg s$ (which will be true for most islands as $r\to\infty$),

$$\phi_{ix}-\phi_i=sy_i/(x_i^2+y_i^2)$$

and

$$\phi_{iy}-\phi_i=sx_i/(x_i^2+y_i^2)$$

so that

$$E_i\cong(E_J/2)[(\phi_{ix}-\phi_i)^2+(\phi_{iy}-\phi_i)^2]$$
$$=(E_J/2)(s^2/r^2)$$

is the energy per island. Thus

$$U=\int_{r_1}^{r_2}\frac{E_J}{2}\frac{s^2}{r^2}\frac{1}{s^2}2\pi r\,dr=\pi E_J\ln(r_2/r_1)\ . \qquad (5)$$

An important distinction must be made before we proceed. While (4) is general, (5) ignores the effects of the fluctuations associated with the Kosterlitz-Thouless transition; in the language of the renormalization group, $E_J(T)$ [or $i_c(T)$] is a "bare" or unrenormalized quantity. The corresponding unrenormalized quantity in the continuum model is called $n_0^*(T)$. Roughly speaking, $n_0^*(T)$ is the superfluid density which would exist without the vortex fluctuations and can be identified with the Ginzburg-Landau estimate of the pair density. The Kosterlitz renormalization procedure[14] shows how to obtain $n_s^*(T)$ from $n_0^*(T)$; that is, it "renormalizes" $n_0^*(T)$ into $n_s^*(T)$.

The proper procedure, then, is to define an effective bare superfluid density $n_0^*(T)$ for a lattice of weak links by equating the result (4) [with $n_s^*(T)$ replaced by $n_0^*(T)$] with (5):

$$n_0^*(T)=m^*E_J(T)/\hbar^2=m^*i_c(T)/\hbar e^*\ . \qquad (6)$$

Equation (6) is the first example of the difference between a lattice model and a Ginzburg-Landau continuum model. In the latter the superfluid density is assumed to vary linearly with T_{c0}-T, where T_{c0} is the Ginzburg-Landau transition temperature, while (6) shows that the effective unrenormalized superfluid density in a lattice can have a different temperature dependence, depending on how i_c varies with T. The result (6) is essentially equivalent to the result Berezinskii[23,24] obtained by comparing a Bose lattice gas to a magnetic XY model. Equations (5) and (6) are changed by a multiplicative constant if the lattice is not square.[25]

Renormalized quantities are the macroscopic quantities measured in experiments. The appropriate measurable quantity for an array is the kinetic inductance per square $L_{K\square}(T)$, which measures the superfluid density. Its definition allows us to define an effective (renormalized) superfluid density $n_s^*(T)$ for an array by

$$n_s^*(T)=m^*/L_{K\square}(T)e^{*2} . \tag{7}$$

[This result and others in this paper involving macroscopic measured quantities (such as $L_{K\square}$) are also true for thin films.] At sufficiently low temperatures $n_s^*(T)=n_0^*(T)$ which, with (6) and (7), produces the correct low-temperature result

$$L_{K\square}(T)=\hbar/e^*i_c(T)\equiv L_J(T) ;$$

the kinetic inductance per square equals the inductance of a single junction $L_J(T)$ (Ref. 21) at low temperatures in a square lattice.

The last two equations allow us to calculate static quantities such as the transition temperature and penetration depth using continuum formulas from the literature. Since the penetration depth for perpendicular fields is given by[26]

$$\lambda_\perp(T)=m^*c^2/4\pi n_s^*e^{*2} ,$$

we obtain

$$\lambda_\perp(T)=L_{K\square}(T)c^2/4\pi\cong\Phi_0c/8\pi^2i_c(T) , \tag{8}$$

where $\Phi_0\equiv hc/e^*$ and in the last part of (8) it is assumed that $n_s^*(T)\cong n_0^*(T)$. (This is not a bad approximation for $T<T_c$.[27]) The last result was obtained previously by Hebard.[28]

Two properties of (8) are worth noting. First, the temperature dependence of $\lambda_\perp(T)$ is determined by $i_c(T)$ and thus can differ from the standard thin-film dirty-limit form. For example, if T_{cs} is the transition temperature of the islands and the weak links are *S-N-S* junctions[29] with length d and normal-metal coherence length $\xi_N(T)$,

$$i_c(T)\propto(1-T/T_{cs})^2e^{-d/\xi_N(T)}$$

near T_{cs} so that $\lambda_\perp$ has a temperature dependence which is quite different from the conventional $(1-T/T_{c0})^{-1}$ thin-film dependence. Second, if the critical current $i_c(T)$ and normal-state resistance r_n of a junction are related by

$$i_c(T)r_n=\frac{\pi\Delta(T)}{e^*}\tanh\frac{\Delta(T)}{2kT}$$

(as for ideal tunnel junctions[30]) we obtain the standard dirty-limit result[21,1]

$$\lambda_\perp(T)=\frac{\Phi_0^2e^{*2}r_n}{16\pi^4\hbar\Delta(0)}\left[\frac{\Delta(T)}{\Delta(0)}\tanh\frac{\Delta(T)}{2kT}\right]^{-1} . \tag{9}$$

An array of ideal tunnel junctions has the same penetration depth $\lambda_\perp(T)$ as a dirty film with the same energy gap $\Delta(T)$ and with $R_\square=r_n$. We note that (8) is the general result; (9) is a special case.

In the continuum the transition temperature T_c is implicitly determined by[13,14,27,31,32]

$$kT_c=\pi\hbar^2n_s^*(T_c)/2m^* \tag{10a}$$

or

$$n_s^*(T_c)/m^*T_c=7.9029\times10^{37} , \tag{10b}$$

in units of $(\mathrm{g\,cm^2\,K})^{-1}$, which, combined with (6) and (7), give

$$kT_c=\frac{\pi\hbar^2}{2e^{*2}}L_{K\square}^{-1}(T_c)\cong\frac{\pi\hbar}{2e^*}i_c(T_c) \tag{11a}$$

or

$$L_{K\square}(T_c)T_c=1.2323\times10^{-8} \tag{11b}$$

in units of H K and

$$i_c(T_c)/T_c\cong26.706 \tag{11c}$$

in units of nA/K, where we have again set $n_0^*(T)\cong n_s^*(T)$ in (11c) and in the last equation of (11a). Equations (11) have been obtained previously.[17] It appears that (11c) may overestimate T_c, when compared to experiment.[19]

The last equations are important from an experimental standpoint. Equation (11b) predicts a discontinuity in the kinetic inductance; when $L_{K\square}(T)$ reaches the value given by (11b), it jumps to infinity. Unfortunately, $L_{K\square}(T)$ cannot be measured directly, but must be unfolded from measurements of the sample's total impedance, which is determined by the kinetic inductance, plus geometrical inductance, normal-state resistance, junction capacitance, etc. Measurements of $L_{K\square}(T)$ are further complicated by nonlinear effects due to nonzero measuring currents.[19] Hebard and Fiory's $L_K(T)$ data[4] have a sharp change near T_c which could be a smeared-out discontinuity. We would estimate from their graph that

$$0.7\times10^{-8}\ \mathrm{H\text{-}K}<L_{K\square}(T_c)T_c<1.4\times10^{-8}\ \mathrm{H\text{-}K} ,$$

which is consistent with (11b), but further measurements are needed to provide a more accurate test of this result.

The approximate result (11c) is useful because it may not differ substantially from the more rigorous result (11b) (Ref. 27) and because $i_c(T)$ is simpler to measure than $L_{K\square}(T)$. Nonetheless, we note that since $i_c(T)$ is the critical current of a single junction in the absence of fluctuations, it cannot be directly measured. It must be inferred from the *I-V* curves for single junctions, measured at low temperatures and extrapolated to higher temperatures, extracted from the low-temperature critical-current data of an array,[19] or calculated theoretically.

From the definition of $\lambda_\perp(T)$ and (10) we obtain

$$kT_c = \frac{\Phi_0^2}{32\pi^2}\frac{1}{\lambda_\perp(T_c)} \qquad (12a)$$

or

$$\lambda_\perp(T_c)T_c = 0.9807 \qquad (12b)$$

in units of cm K. These are identical to the dirty thin-film results,[1] even though we have *not* assumed that the weak links are ideal tunnel junctions, demonstrating the universality of relations based on macroscopic measured properties of the superconducting state.

The correlation length for an array of junctions is similar to the correlation length for the lattice XY model[14,33]; it is given by

$$\xi_+(T) = c_1 s \exp\{[c_2/(T' - T_c')]^{1/2}\} , \qquad (13)$$

where c_1 and c_2 are of order unity[14,33] and we have again defined a dimensionless temperature $T' \equiv e^* kT/\hbar i_c(T)$, as in the discussion following Eq. (3). The correlation length is the size of fluctuations associated with the phase transition; that is, it is the average distance between unbound vortices. The prefactor, which is called ξ_c in Ref. 2, is proportional to the lattice spacing s here, as would be expected in a lattice model.

III. RESISTIVE BEHAVIOR

Consider a square lattice of junctions with lattice spacing s. When a current flowing along the length L of the sample causes free vortices to move across the width W of the sample, there will be an end-to-end voltage difference given by

$$V = \frac{\hbar}{e^*}\frac{d\theta}{dt} , \qquad (14)$$

where the phase difference θ along the length increases by 2π each time a vortex crosses the width of the sample. If the areal density of free vortices is denoted by n_f, then

$$\left\langle \frac{d\theta}{dt} \right\rangle = 2\pi L v_d n_f , \qquad (15)$$

where the angular brackets denote time average and v_d is the average vortex drift velocity across the sample. This drift velocity is given by s/τ, where τ is the average time for a vortex to cross one junction. If i is the current through one junction and I is the total current, then $I/W = i/s$ so that (14) and (15) lead to

$$R_\square = \frac{V}{I}\frac{W}{L} = \frac{\hbar}{e^*} 2\pi (s^2 n_f)\frac{1}{i\tau} . \qquad (16)$$

To use (16) we need to determine τ and n_f. We show in the Appendix that, to a good approximation,

$$\frac{1}{i\tau} = \frac{e^*}{2\pi\hbar} r_n [I_0(E_J(T)/10kT)]^{-2} , \qquad (17)$$

where $I_0(x)$ is the hyperbolic Bessel function of order zero and r_n is the normal-state resistance of one of the junctions in the array. Equation (17) assumes that the junctions are small compared to the Josephson penetration depth and the normal-metal coherence length, but makes no assumption about the relation between r_n and E_J.

The density of free vortices is proportional to ξ_+^{-2}, so that

$$n_f = b_1 s^{-2} \exp\{-[b_2/(T' - T_c')]^{1/2}\} \qquad (18)$$

where b_1 is of order unity,[2,33] $b_2 \equiv 4c_2$, and $T' \equiv e^* kT/\hbar i_c(T)$ is the temperature measured in the appropriate reduced units, as in (13). Combining (16)–(18) we get the final result

$$R_\square = b_1 [I_0(E_J(T)/10kT)]^{-2} \times \exp\{-[b_2/(T' - T_c')]^{1/2}\} r_n . \qquad (19)$$

Sufficiently close to T_c [where $E_J(T) \cong E_J(T_c) = \text{const}$], this has the same form as the Halperin-Nelson result[2]

$$R_\square = b_3 \exp\{-[b_4(T_{c0} - T_c)/(T - T_c)]^{1/2}\} R_{\square n} , \qquad (20)$$

where T_{c0} is the BCS transition temperature of the film and b_3 and b_4 are of order unity[2]; but there are significant differences between (19) and (20), in both the prefactor and the exponential factor.

Consider first the prefactor: In contrast to the constant b_3 in (20), the prefactor $b_1[I_0(E_J/10kT)]^{-2}$ is temperature dependent. If, however, (11) holds, then the argument of the Bessel function ranges only from 0 to $2/10\pi$ for $T > T_c$, which causes the prefactor to vary by less than 1%. The smallness of this effect at first appears surprising, since in a single isolated junction the analogous factor gives the exponential disappearance of the resistance[34] $r = r_n[I_0(E_J/kT)]^{-2}$ which dominates the resistive transition. If one applied this latter formula to a single junction embedded in an array, one would find that its resistance would have dropped much more, to $0.82r_n$, as T dropped down to the T_c of the array. The much smaller effect at the same temperature in the array can be traced back to the smaller activation energy in the lattice, as found in the Appendix. The effect of the other junctions in the lattice, then, is to suppress the single-junction temperature dependence of the resistance in favor of the exponential factor in (19) (which reflects the variation of the free vortex density with temperature). In particular, it makes the zero-current resistance strictly zero (as opposed to exponentially small) for $T < T_c$, where the number of free vortices vanishes. In an array, resistance results from overcoming *two* barriers—the first to nucleation of a vortex, which is infinite for $T < T_c$, and the second to vortex motion, which is relatively small in an array. By contrast, a single junction has only the barrier opposing phase slip, with no cost for nucleation.

Turning now to the exponential factor, here there is a more significant difference between (19) and (20). It arises from the distinction between T' and T, since $i_c(T)$ can vary appreciably with temperature. [The appropriate dimensionless temperature for the continuum case, close to T_{c0}, is $T/(T_{c0}-T_c)$, which is proportional to T.] For example, when the data in Ref. 19 are plotted according to (19) and (20), the best fit to (19) gives $b_1 \cong 3$, while the best fit to the form (20) gives $b_3 \cong 100$. Since both of these factors are supposed to be of order unity, the fit to (19) is clearly more reasonable. From this perspective it is interesting that the published fitting parameters of Wolf *et al.*[7] can be combined to give $b_3 \cong 1000$, Bancel and Gray's[6] parameters yield $b_3 \cong 220$, and the data plotted in Resnick *et al.*[18] give, according to our rough estimates, $b_3 \cong 440$ and 150 for their samples 5 and 6. The value that we infer from the data and fits of Voss and Webb[17] is $b_3 \cong 2.5-14$, depending on what value is used for $R_{\square n}$. It is interesting that in the last case, where b_3 is of order unity, $T_c \ll T_{c0}$ so that $E_J(T) \cong E_J(0) = \text{const}$, which makes (19) and (20) equivalent for fitting purposes. The difference between (19) and (20) might explain the large values of b_3 which characterize many samples, but we could not test this conjecture by calculating b_1 for these samples because data on $i_c(T)$ were not available. [It is worth noting that, while the theoretical results should apply only in a narrow temperature region close to T_c, numerical simulations[33] and experiments seem to fit the asymptotic-limit formulas (13) and (19) reasonably well over a wide range.]

By comparing the continuum and lattice versions of formulas, we obtain the correspondences listed in Table I. This table allows us to "translate" additional formulas from the continuum to the lattice language. With the use of Eqs. (36)–(38) in Ref. 2, we can, for example, obtain the current-voltage characteristics below T_c for a lattice of weak links. This procedure gives

$$\frac{V}{I} = x r_n [I_0(E_J(T)/10kT)]^{-2} \, (i/i_c)^{2+(x/2)} , \tag{21}$$

where we have dropped a multiplicative factor of order unity and the value of x depends on whether the vortex density is dominated by thermal or finite-current effects:

$$x = \max(x_T, x_I) \tag{22}$$

with

$$x_T = 4\left[\frac{L_{K\square}(T_c)T_c}{L_{K\square}(T)T} - 1\right] \cong 4\left[\frac{i_c(T)T_c}{i_c(T_c)T} - 1\right] , \tag{23a}$$

$$x_T \cong 4\pi\left[\frac{T_c' - T'}{b_2}\right]^{1/2} , \tag{23b}$$

$$x_I = 1/\ln[i_c(T)/i] . \tag{23c}$$

TABLE I. Guide to the equivalence between a uniform 2D superconducting film and a square array of superconducting weak links. Symbols are defined in the text.

Uniform film	Array of weak links
"Bare" superfluid density $n_0^*(T)$	$\dfrac{m^* i_c(T)}{\hbar e^*} \equiv \dfrac{m^* E_J(T)}{\hbar^2}$
Renormalized superfluid density $n_s^*(T)$	$\dfrac{m^*}{e^*} L_{K\square}^{-1}(T)$
Vortex core size $\xi_c \cong \xi_{GL}(T_c)$	Lattice spacing s
Normal-state sheet resistance $R_\square$	$r_n \left[I_0 \left[\dfrac{\hbar i_c(T)}{10 e^* kT} \right] \right]^{-2} \cong r_n$
$\dfrac{T}{T_{c0} - T_c}$	$\dfrac{e^* kT}{\hbar i_c(T)}$

The second half of (23a) is approximate because it assumes that $n_0(T) \cong n_s(T)$, while (23b) holds very close to T_c; note that the same constant b_2 appears in (19) and (23b).[24]

For small I, $V \propto I^{a(T)}$. Equations (19) and (21)–(23) imply that $a(T_{c-})=3$ and $a(T_{c+})=1$; that is, there is a *jump* in $a(T)$ at T_c, just as in the continuum case.[2] This discontinuity will be smeared by a nonzero current, but the experimental effort to find it would be worthwhile in order to test the theory. Empirically, any real superconductor will have $V \propto I^{a(T)}$ with $a(T)=1$ somewhat above the nominal T_c and $a(T) \gg 1$ somewhat below the nominal T_c, and will thus pass through $a(T)=3$ (or 2 or 4 or 5.7, for example) at a temperature not very far from the nominal T_c. It is *detailed* agreement with (19) and (21) that identifies the Kosterlitz-Thouless transition.

IV. CONCLUSIONS

We have discussed the vortex-unbinding transition in a discrete lattice system. Predictions for the continuum case have been "translated" to the lattice language to provide expressions which can be compared directly to data from 2D arrays of weak links. For example, the association between n_s and $L_{K\square}$ gives an exact relation between $L_{K\square}$ and T_c as a direct consequence of the Nelson and Kosterlitz universal jump in the superfluid density at T_c. By examining vortex motion in a lattice of junctions, we have shown how the interaction between junctions suppresses the Ambegaokar-Halperin temperature-dependent resistance of an isolated junction, leaving the free vortex density as the dominant temperature-dependent factor in the expression for the array's resistance. Perhaps the most important result for the evaluation of experimental data is the need to adopt an appropriate dimensionless temperature parameter, $T' \equiv e^* kT/\hbar i_c(T)$, which differs from the one $[T/(T_{c0}-T_c)]$ used in the continuum case, because $i_c(T)$ can have a strong temperature dependence near T_c.

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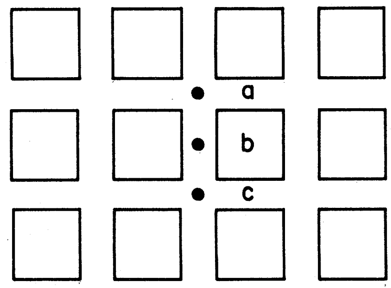

FIG. 1. Three positions for a vortex center in an array. Points a and c are equivalent; both have minimum energies. To get from a to c, point b must be crossed, which is a position of maximum energy.

APPENDIX

In order for a vortex center to move from point a to point c in a square array (see Fig. 1), it must cross a potential barrier of height E_{HA} at point b. In an isolated zero-dimensional junction the barrier height against the analogous process, phase slip, is just twice the coupling energy

$$E_H = 2E_J(T) \equiv 2\hbar i_c(T)/e^* \ .$$

In that case, the average time for a phase slip is given by[34]

$$\tau = 2\pi \frac{\hbar}{e^*} \frac{1}{i r_n} I_0^2(E_H/2kT) \ , \qquad (A1)$$

where $I_0(x)$ is the hyperbolic Bessel function of order zero. We use (A1) to *estimate* the time τ for a vortex to move past one junction in an array by replacing E_H with the (as yet unknown) barrier height E_{HA} for an array.

Figure 2 shows two phase configurations corresponding to vortex centers located at points a and b in Fig. 1. Configuration a corresponds to a local minimum in energy, while b corresponds to a maximum, so that $E_{HA} = E_b - E_a$.

In both configurations the phase at very large r is known by symmetry: $\phi(r \gg s) = \tan^{-1}(y/x)$, which determines one boundary condition. For configuration a symmetry requires that the phase vectors on the four islands nearest the center point radially outward, as shown in Fig. 2(a). For configuration b the phase angle is 0 for islands on the line to the right of the center and π for those to the left, as shown in Fig. 2(b).

To find the phase at all the other islands (2) is minimized, which leads to

$$\sum_j \sin(\phi_i - \phi_j) = 0 \qquad (A2)$$

for each island i, where j runs over the nearest neighbors to i; that is, the net current flowing into

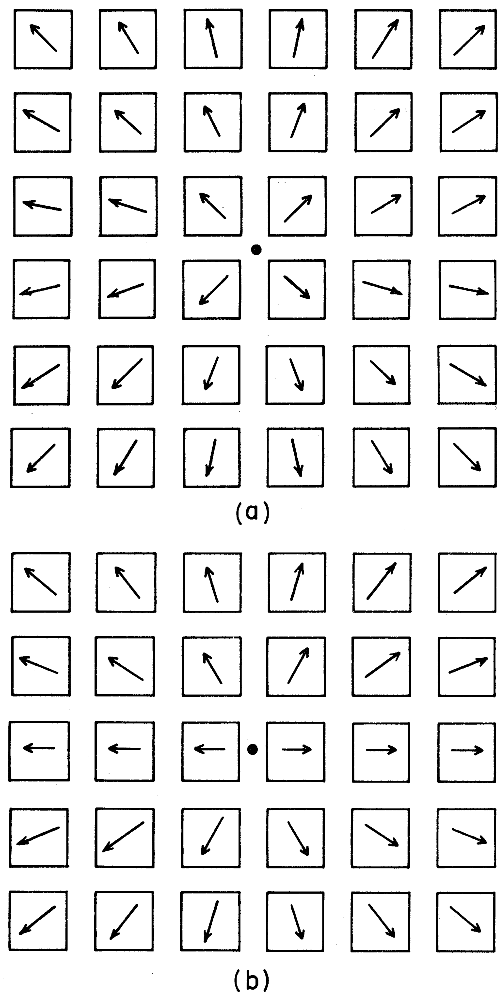

FIG. 2. Phase-angle configurations for two positions of the geometrical center of the vortex. (a) is a low-energy configuration, (b) is a high-energy configuration.

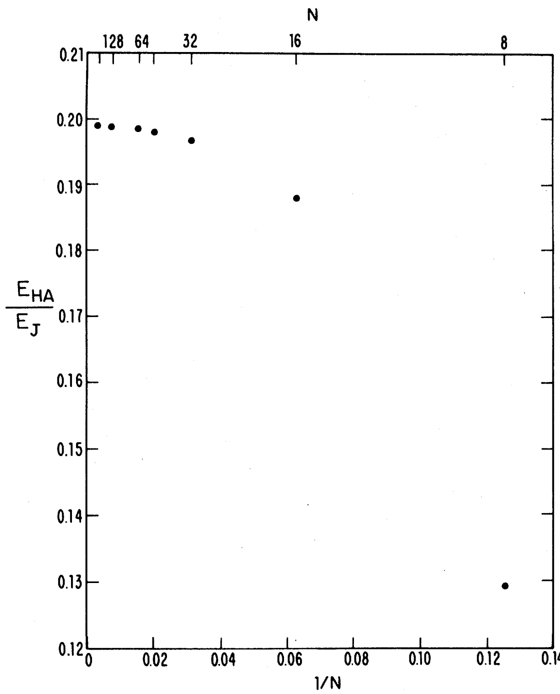

FIG. 3. Plot of the barrier height E_{HA}/E_J against inverse sample size $1/N$. When extrapolated to $N=\infty$, these data yield $E_{HA}/E_J \cong 0.199$.

each island is zero [see (2)]. To solve (A2), it is rewritten as

$$\tan\phi_i = \sum_j \sin\phi_j \Big/ \sum_j \cos\phi_j \ . \tag{A3}$$

These equations can be solved iteratively on a computer. At the n^{th} iteration we set

$$\tan\phi_i^{(n)} = \sum_j \sin\phi_j^{(m)} \Big/ \sum_j \cos\phi_j^{(m)} \ , \tag{A4}$$

where $\phi_j^{(m)}$ is the latest value of ϕ_j available (i.e., $m=n$ or $n-1$). When the ϕ's are known, E_a and E_b can be calculated using (2), and thus the barrier height $E_{HA}=E_b-E_a$ is determined.

This procedure was followed on square lattices containing N^2 islands. For $N=8$, 16, 32, 50, 64, 128, and 256, $E_{HA}/E_J \cong 0.131\,90$, 0.188 00, 0.196 54, 0.198 00, 0.198 40, 0.198 70, and 0.198 80. Extrapolating this to $N=\infty$ by plotting against $1/N$, we obtain the estimate $E_{HA}/E_J \cong 0.199 \cong \frac{1}{5}$ (see Fig. 3).

Adopting the Ambegaokar-Halperin result[34] (A1), but using this numerically determined barrier height, we obtain

$$\tau = 2\pi \frac{\hbar}{e^*} \frac{1}{i r_n} I_0^2(E_J(T)/10kT) \ , \tag{A5}$$

which is the desired relation (17). Details of this calculation change when it is done for a triangular lattice. We have determined that the barrier height for a triangular lattice is $\sim 0.043 E_J$, which changes the 10 in (A5) to ~ 47. This does not alter the physics of (19).

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