PHYSICAL REVIEW B VOLUME 26, NUMBER 2 15 JULY 1982

Conditions for charge fractionalization

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(Received 5 October 1981; revised manuscript received 26 January 1982)

We prove a theorem which gives necessary and sufficient conditions under which charge will be fractional, in a certain well-defined sense. The model of Su, Schrieffer, and Heeger, which gives fractional charge, is completed in several ways which illustrate the theorem and the postulates made to prove it. The theorem applies to a chain of arbitrary complexity which consists of charged regions separated by long stretches of electrically neutral and electrically insulating chain. The theorem states that the integral of the charge density over a charged region will be a noninteger multiple of the charge on the proton if and only if the electrical polarity of the neutral chain is different on the two sides of the charged region. In short, fractional charges in the sense here discussed exist only on ferroelectric domain walls. Further, these charges will not be simple fractions, but will be irrational multiples of the protonic charge. A brief discussion is given of how fractional charges may exist in other senses with an experimental meaning.

I. INTRODUCTION

Considerable interest has been aroused by the possibility that certain topological solitons in physically realizable quasi-one-dimensional systems may carry fractional charge. In particular, Su, Schrieffer, and Heeger[1] have produced a model, applicable to polyacetylene, for which the solitons have half integral electronic charge for each spin. Since there are two spins, however, the total charge is integral. This model is a close analog to an earlier field theoretical model of Jackiw and Rebbi,[2] which has been discussed recently by Goldstone and Wilczek.[3] It has been generalized by Su and Schrieffer[4] to one which displays charge $1/n$. (We speak of all models in this class as SSH models.) They suggest that such a system with $n=3$ may be realized in tetrathiafulvalene-tetracyanoquinodimethane.[4] Other cases are $NiSe_3$ (Ref. 5) and TaS_3 (Ref. 6), which have parallel chains of charge-density waves with commensuration four (or close to it). The discommensurations are topological solitons which carry $\frac{1}{4}$ charge per spin, or charge $\frac{1}{2}$ all told. Bak[7] has interpreted existing data[5] as verifying this prediction.

Several meanings can be attached to the term "fractional charge." A fundamental particle, e.g., a quark, may be fractionally charged. Even if only integer charged particles are admitted, fractional charge may occur as an average. The average may be quantum, it may be a time average, or it may be an average over many species of charged entities. Generally, if an average is involved, fluctuations in principle exist. However, Kivelson and Schrieffer[8] (KS) have introduced an operator which, in their model, has as its expectation value the charge that couples to a slowly varying electric field and which has arbitrarily small fluctuations. This operator is exhibited in the Appendix. It is not an electron "number" operator, not having integral eigenvalues. The number operator, as standardly defined, continues to have this property in the finite systems under consideration, of course.

There is, in fact, one type of average which is known to exhibit nonfluctuating fractional, indeed, irrational, charge. That is the average made in the macroscopic theory of electromagnetism.[9] That the charge may be nonintegral is illustrated by the fact that the total charge within a Gaussian surface surrounding a test charge q is q/ϵ where ϵ is the dielectric constant of the medium within which the test charge and Gaussian surface are embedded. The fractional part of the charge is due to the polarization, and the polarization charge density is $\rho_p=-\nabla\cdot\vec{P}$ where $\vec{P}$ is the electrical polarization, the dipole moment per unit volume, of the medium. There are charges on the surface of any polarized material which must also be taken into account. The fluctuations are not identically zero, but are "macroscopically small," that is, small insofar as the averaging volume contains many elementary charges. The classical theory is indepen-

dent of the quantization of the elementary charges, of course. It is this type of average which gives fractional charge in the SSH models. The total macroscopically averaged charge density is $\rho=\rho_F+\rho_P$ where ρ_F is the density of "free" charges.

We are interested in an electrically insulating quasi-one-dimensional system, which we call a chain, whose length is great compared with its transverse dimensions. (A conducting chain would screen any charge, so this case is of no interest here.) We further assume that ρ is negiligible (and has negligible integral) except in a finite region U, that is, the chain is electrically neutral except for U. The total charge Q in U is $Q=\int \rho\, dV$ where the integral is over any region containing U. [We do not need to use KS's definition (although that would not be incorrect) since ρ is already macroscopically averaged and vanishes outside U.] A test charge q located a long distance R from U will experience a force of magnitude qQ/R^2. (Since classical electromagnetic theory is consistent, any other method of measuring Q based on that theory would give the same result.) We may also have other charged regions on the chain lying at great distances from U, with corresponding changes in the definitions.

We speak of U and Q as being *electrically isolated.* This is distinguished from a charge on matter which is *physically isolated,* by which we mean that the matter is far from any other matter. (We consider electrical isolation only in the quasi-one-dimensional sense and do not deal with charges embedded in surfaces or volumes of material.)

We now state three postulates on which we base our theorem. The first two of these are very general and are believed to hold for any physical system, or acceptable model of a physical system. The third is also quite general and is true in the models we are discussing.

The first we call the postulate of locality. This postulate, whose physical content has been clear since Einstein, states that matter in one locality is neither directly influenced by nor directly influences matter in a distant region, but rather the matter in a specific locality is influenced only by the local electromagnetic (and gravitational) fields, and of course by the short-range interactions with matter in neighboring regions. In particular, quantities such as charge are *locally* conserved, and net charge cannot be created in a particular region even if compensated by charge of the opposite sign created in a distant region.

The second we call the oil-drop postulate. This states that the charge on matter which is physically isolated must be an integral multiple of the fundamental charge. In other words, if the charge as measured in a Millikan oil-drop experiment is fractional, say $e/3$, it must be concluded that an isolated quark or some new particle with fundamental charge $e/3$ is present on the drop and not that the ordinary matter of the oil drop has taken up some unusual configuration.

The third is the postulate of quasistatic metastability. Here we assume that the matter in the chain is in a state sufficiently slowly varying in time and is in a sufficiently stable condition, that the local electric properties of the material in the chain cannot be altered by, e.g., weak electromagnetic signals or sound waves propagating down the chain from distant events. (This postulate may not be strictly necessary if relativity is invoked.)

II. THEOREM

We prove first the following lemma.

Lemma. All neutral oriented chains may be classified by a parameter p, where $-\frac{1}{2}<p\leq\frac{1}{2}$. A chain is oriented by giving a coordinate x which increases in the positive direction of the chain.

Proof. Consider cutting a neutral chain, say with a spark cutter or laser, at points y,z with $y\ll z$. By the postulate of quasistatic metastability, this will affect only a finite region near the cuts, and by the postulate of locality, the results of the two cuts are independent of one another. There may be a nonvanishing total charge on the newly cut ends. We enumerate the charges on the end of the chain with $x<y$ which result from each possible cut. These charges $\alpha_1,\alpha_2,\ldots,$ are the elements of a set we call $E^<(y)$. Similarly, the other end near y will have possible charges $\beta_1,\beta_2,\ldots,$ forming a set $E^>(y)$. (The fact that the α's and β's may be correlated is of no interest.) There are also sets $E^<(z)$ and $E^>(z)$, which have no correlation with the sets associated with y. Since the chain between y and z is now physically isolated, and since the only charge on the chain is near its ends, we have by the oil-drop postulate that $\alpha+\beta=ke$ for all α in $E^<(z)$ and all β in $E^>(y)$ where k is an integer. This can only happen if every α is of the form $ne+pe$ and every β is of the form $me-pe$ where n and m are integer and $-\frac{1}{2}<p\leq\frac{1}{2}$. By considering other hypothetical cuts it is easily proven that all charges in the sets $E^<(x)$ have the form $ne+pe$ and charges in the sets $E^>(x)$ have the form $me-pe$, with p indepen-

dent of x. The value of p is thus uniform along a neutral chain, and characterizes the chain.

The quantity $P=pe$ has an easily deduced meaning. It is the noninteger part of the total electric polarization along the chain. Since, in one dimension, the polarization charge density $\rho_P(x) = -dP/dx$, and since by assumption dP/dx vanishes except near the ends of the chain, and P must vanish outside the chain, it follows that a chain will have charges P and $-P$ at its two ends. A chain with $p=0$ is an ordinary dielectric chain. A chain with $p=\frac{1}{2}$ is exceptional. It may be reflection symmetric, but will be a (one-dimensional) polar crystal. Other chains will have a preferred direction and will be ferroelectric. (These cases are illustrated in Fig. 3.) We say, for short, that the value of p (or P) is the polarization characterizing the ferroelectric state of the chain. We are now in a position to prove the main theorem.

Theorem. An electrically isolated charge Q will be nonintegral if and only if the charged region is connected to chains in different ferroelectric states. If the chain to the negative side of Q has polarization P, and to the positive side has polarization P', then $Q=P-P'+ke$ with k integer.

Proof. We proceed as before, with the region bearing the charge Q now somewhere between the cuts at y and z, and far from either of them. The value of Q cannot be affected by the cuts nor can it affect the results of the cutting. From the lemma the chain on the two sides of Q must be characterized by polarizations P and P', which possibly are different. The total charge which is physically isolated after the cuts is $Q+P'-P+ke$ from which the result follows. We are thus entitled to assert that any electrically isolated nonintegral charge must be at a ferroelectric domain wall.

III. SSH MODEL

The theorem applies to models of chains which obey or can be extended so as to obey the postulates. We shall examine the class of models of Refs. 1, 2, and 4, completed in ways which illustrate aspects of the theorem. We first describe the SSH model and derive the fractional charge in a way equivalent to that of the originators of the model.

The SSH model speaks explicitly of a linear chain of sites, with elastic forces of general character between them. The electronic Hamiltonian is

$$H_e=-\sum_i t(u_i-u_{i+1})(a_i^\dagger a_{i+1}+a_{i+1}^\dagger a_i)\,, \qquad (1)$$

with a_i the electron-annihilation operator on site i. The hopping matrix element t is a function of the displacements u_j of the sites from the nominal site position at, for example, jd. The number of electrons (of a given spin) on the chain in its ground-state configuration is taken to be one for every n sites. The cases $n=2$ and 3 have been considered in detail.[1,4] The metallic ground state is unstable, the Peierls instability, and the lattice n-merizes so that the lattice constant is nd.

SSH consider the case of solitons in this chain. We may reproduce their results by considering a very long closed loop consisting of N n-mers and N electrons. Imagine removing an n-mer with its electron, and letting the chain heal. The electron is then taken from the removed n-mer and discarded, the n-mer is split up into its constituent sites, and each of these is inserted interstitially at widely different points of the loop. There are again nN sites, but there is now a deficit of one electron. Finally, the lattice positions are again allowed to relax. The configuration near each of the inserted "interstitials" cannot relax back to perfect n-merization, since there will always be an extra site which cannot be accommodated into an n-mer. The result may be termed a topological soliton, because the lattice configuration interpolates between regions of essentially perfect n-merization, or in the nomenclature of Jackiw and Rebbi[2] the configuration interpolates between equivalent "vacua." After relaxation, each disturbed region approaches the perfectly n-merized configuration in an exponential way and is in this sense confined to a finite length of chain. This process is illustrated in Fig. 1 for $n=2$.

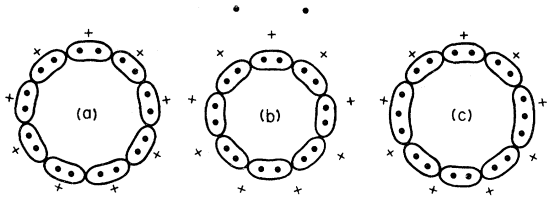

FIG. 1. Schematic construction of solitons in an n-merized loop with $n=2$ and spinless electrons. The solid circles represent sites. Electrons can hop to nearest-neighbor sites. The lines delineate regions containing one electron on the average. (a) is perfectly dimerized, (b) has one dimer removed, and (c) shows the solitons, centered to left and right. A possible set of positive charges giving neutrality is shown by the pluses.

Each soliton is completely equivalent, and therefore must share the electron deficit equally between them. Therefore, each has a deficit of $1/n$ and the charge on each soliton is $+e/n$. Further, this charge is localized exponentially on the soliton and there is no net charge density between solitons. (We stress this because there seems to be a common intuition that fractional charge can be "explained" by supposing that a complementary fraction is spread thinly out along the perfect chain. This can indeed happen in the model of a metal formed of noninteracting electrons. However, interacting electrons in a metal will always screen a static charge so as to maintain strict charge neutrality. The noninteracting electron model is not a satisfactory one unless it is complemented by a Friedel sum rule which ensures charge neutrality.) This argument, based on symmetry, is simple and apparently compelling, but we shall show below that it is *not applicable* when the theorem applies.

For completeness, we consider the energy bands associated with the chain. Before n-merization there is a single band containing nN electron states. After n-merization, but before introducing the solitons, there are n bands each containing N states of which the lowest is filled. The solitons introduce midgap states, each of degeneracy n. There is one midgap state (in each gap) associated with each soliton. These states are localized (exponentially) on each soliton. The total number of states in the lowest band is $N-1$. In the situation as we have envisaged it, the lowest band is full, and all midgap states are empty. Thus the electrons are in a nondegenerate ground state, separated from excited electronic states by a finite gap. The deficit electron, or hole, *cannot* then be shared quantum mechanically by the n solitons. If that were so the state would be nearly degenerate, since the relative phase of the hole on widely separated solitons is energetically irrelevant. This rules out a second intuitive "explanation" for fractional charge.

We give in Fig. 2 the electronic charge density near a soliton in an illustrative case for $n=2$. Only two values of the hopping elements t are involved, which we denote by t_1 and t_2 with $(t_2/t_1)^2 \equiv \eta < 1$. The figure gives the bond sequence and site labels. In this case, the midgap state has a wave function odd about the center of the soliton. The expected electronic number density on site j, is $\frac{1}{2}$ for j even, and is $\frac{1}{2}-\frac{1}{4}\eta^l(1-\eta)$ for $j=2l+1>0$. Fig. 2 shows the case $\eta=\frac{1}{2}$. Note that the expected electron number on a large number of sites containing the soliton and including only complete n-mers is integral. (A direct evaluation of the charge which avoids the concept of cutting the chain can be made in simple enough models, such as this one.)

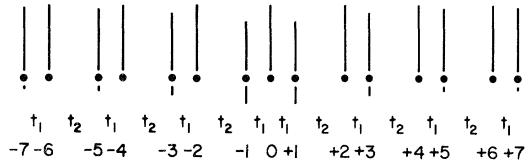

FIG. 2. Electronic number density near a soliton which has charge $\frac{1}{2}$ in the SSH model. Solid circles represent sites, labeled $\ldots -7, \ldots +7, \ldots$. The bond alternation is also shown. Vertical bars on each site represent expected electron density (above) and density deficit (below) which approach $\frac{1}{2}(0)$ far from the soliton. The figure assumes $(t_2/t_1)^2=\frac{1}{2}$. A total of $7-\frac{1}{64}$ electrons occupy the sites pictured.

There are other topological distinct solitons, and midgap states may or may not be filled. A handy mnemonic method for computing the charge and seeing the general features of the midgap states and bands is to consider the case in which the electron transfer between n-mers is very small, so that the n-mers approximate independent linear n-sited molecules. The soliton we have considered is then approximated by a molecule with $n+1$ sites and one electron. This preserves all the topological features of the soliton. (Other solitons may be formed by "molecules" with m sites and other numbers of electrons, where $m \bmod n \neq 0$.) The soliton has charge $+e/n$ because it has the negative charge of one electron but a positive charge of $e(1+1/n)$ since the charge on each site is $+e/n$, as required by charge neutrality of the n-merized chain.

IV. EXTENDED SSH MODELS

In the original SSH model $n=2$, but the spin of the electron was accounted for so the total charge of each site was $+e$. In that model there is no need to introduce charges other than on the sites of the model and on the electrons. Indeed the sites physically represent carbon ions. In the extended versions of the model it is necessary to introduce other sites or introduce some other device to avoid endowing the sites with fractional charge. This was not spelled out in Ref. 4 but KS as well as Schrieffer[10] have made clear the situation that was in mind.

The situation envisaged is that the chain explicitly studied in the SSH model is complemented by parallel chain or chains of net positive charge which is electronically inert. It is further approximated that the positively charged chain does not move appreciably from its original configuration when the solitons are created. (Some minor movements may also be contemplated.) In this case, the positive-charged chain drops out of the problem of finding the charge on the soliton, since when one considers the macroscopically averaged charge density of the positive chain it will be constant, and just such as to cancel the charges of the n-merized chain on the average. It is then possible to compare the *electronic* charge with and without solitons in order to find the charge on any one soliton. When the SSH model correctly describes the electronic charge, then under these assumptions the charge will be nonintegral. We shall see that there is a very simple way of calculating the charge in this model, which gives exactly the results found by the methods of Ref. 8.

In order for the SSH models to fit within the framework of the theorem, the positive charges must be explicitly placed. We have already ruled out that only the sites actually appearing in the model are charged, if they are all equivalent. In versions of the SSH model making contact with field theories, a continuum limit is usually taken, which amounts to treating the positive charge as jellium. This version of the model does not satisfy the oil-drop postulate which clearly depends on the explicit quantization of all charges. [In the Jackiw-Rebbi model,[2] the positive background never appears (except perhaps as a renormalization constant). The oil-drop postulate *is* satisfied but in the sense that insofar as the model is correct, it predicts a particle with fundamental charge a fraction of the protonic charge.]

We therefore place the positive charges on a parallel chain, which we imagine has no dynamic coupling to the SSH chain. One possibility is to have one unit positive charge for each n-mer. Several configurations are illustrated in Fig. 3 for $n=2$. Figure 3(a) is a ferroelectric domain wall of the usual variety. The values of p and p' in this case are $-\frac{1}{4}$ and $\frac{1}{4}$, respectively. In Fig. 3(b), the state of the chain to the left of the soliton has $p=\frac{1}{2}$, while to the right $p'=0$.

Next we imagine introducing a parallel chain consisting of n positive ions of charge $+e$ for each n-mer of the SSH chain, and which will contain $n-1$ electrons for each n ions. If this chain has

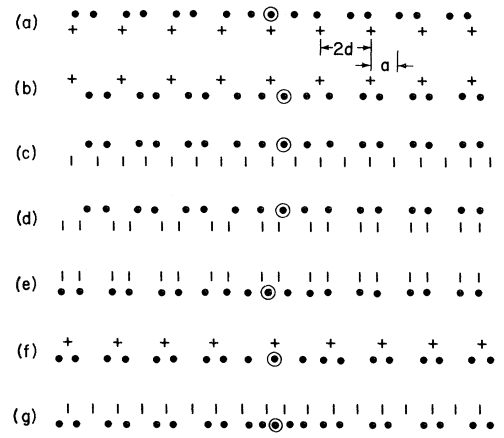

FIG. 3. Configurations of solitons and positively charged sites needed for charge neutrality which illustrate the theorem of the text. Solid circles are uncharged sites of the soliton-bearing chain. The center of the soliton is at the circled site. Pluses are positions of charges $+e$. The vertical bars represent either charges $+e/2$ or charges $+e$ which have associated an electron for every two such sites. (a) has net charge $Q=+e/2$, polarization to the left $P=-e/4$, to the right $P'=e/4$; (b) has $Q=+e/2$, $P=e/2$, $P'=0$; (c) would have $Q=e/2$, except the positively charged chain is metallic, and would either dimerize or screen out the charge; (d) has $Q=+e/2$, $P=e/2$, $P'=0$; (e) has $Q=P+P'=0$; (f) has $Q=P=P'=0$. By the definition of Ref. 9, the electronic charge of the soliton-bearing chain is $+e/3$, the (protonic) charge of the positive chain $-e/3$; (g) would have $Q=-e/2$. The charge of the soliton chain is $-2e/3$, the charge on the positive chain is $+e/6$. The positive chain, however, is metallic.

periodicity d, there will be a charge of precisely $1/n$ near each ion, and is illustrated in Fig. 3(c). This model fails to satisfy the postulates because it is metallic. If the positive chain is also allowed to n-merize, or if the hopping between positively charged sites is arbitrarily changed to prevent metallic behavior, then the kind of situation envisaged in Figs. 3(a) and 3(b) is recovered, e.g., as in Fig. 3(d).

An easy way to find the fractional part of the charge consists of first finding the net (positive or negative) displacement a of the perfect n-mers on the positive side of the soliton, relative to their positions in the absence of the soliton. The nonintegral part of the charge is then ea/nd. Note that, in the absence of additional constraints, the charge need not be rational, that is, a may not equal d. Thus the method of KS is in principle different

from the arguments which are based on the symmetry of the solitons and their charge deficit.

We now relax the condition that the parallel chain of positive charge be independent of the solitons. We give two configurations which satisfy the postulates of the theorem and give integral charge. In Fig. 3(e) the parallel chain is the same as without the soliton but the forces between it and the original chain are imagined to compress the soliton-bearing chain in such a way as to make $p=p'=0$. The effect of this compression is to restore integral charge to the soliton. In Fig. 3(f) we imagine that the parallel chain is distorted by the existence of the soliton in such a way that the same result is achieved. This of course means that its contribution to the total charge becomes nontrivial. In this case, each chain has fractional, indeed irrational charge, if calculated by KS, but the net charge of the two chains together is integral. Finally, in Fig. 3(g) we modify 3(c) by allowing the positive charges to move slightly so as to make $p=p'=0$ even though $a\neq d$. This gives charge e/n but the positive chain is again conducting so the charge would be screened out if the Coulomb interaction were taken into account. We thus see that the SSH model can be readily extended to satisfy the postulates and will indeed give nonintegral charge if and only if the soliton interpolates between chains in different ferroelectric states. Subbaswamy and Brill[11] have also produced examples emphasizing the importance of explicit consideration of the positive charges.

V. DISCUSSION

At this point we reconsider the argument based on the symmetry of the solitons. This argument is somewhat misleading in the sense that the lemma implies that it is *impossible* to have symmetric solitons, that is, a chain with multiple solitons, all of which are equivalent, cannot exist, unless the charge on each soliton is integral or half integral. The half-integral case is rather special, as we shall see.

We dispense with the proof of this assertion, and simply note that it is impossible to assign polarizations to the neutral chains separating the solitons, so as to have nonintegral charge for each soliton, and each soliton equivalent to all others. The exceptional case is an alternation of polarizations 0 and $\frac{1}{2}$. This case is special because these two polarizations do not determine a preferred direction along the chain. Figure 1(c) illustrates this case (approximately). Assuming symmetric solitons therefore tacitly implies a breakdown of the postulates, e.g., a use of fractionally charged sites.

Without symmetry, there is no reason to expect *rational* charge. The polarizations will be determined by energetic, not topological, factors and the charge at ferroelectric domain walls will be irrational.

Although fractional charge is a polarization charge, we note that the polarization *induced* by either the test charge or the charge on the soliton plays no role, since its effect can be made arbitrarily small by taking the region U and/or R large enough. If the medium between the test charge and the chain is not vacuum, as we have assumed, then it is necessary to include the dielectric constant ϵ in the definition of the force on the test charge.

For completeness, we briefly discuss models of chain cutting which consist of making particular bonds become longer and longer with the concomitant vanishing of the corresponding hopping matrix elements. If a hopping which was originally a relatively weak one connecting two n-mers is allowed to vanish it turns out that no charge is induced on either of the freshly cut ends. If a relatively strong bond internal to an n-mer is cut there appear midgap "end" states which become degenerate in the limit of complete cutting. The electron or electrons which occupy these states must then choose one end or the other as the ends become well separated.

We have thus seen a number of ways in which simple models can illustrate the theorem and the postulates. We have not, however, discussed them all. The reader is invited to consider the situation in Fig. 1(c) and to reconcile the symmetry prediction $e/2$ with the result of direct calculation ea/d.

A final remark is that the chief idealization of the theorem is that it is assumed that between the charged regions there lie arbitrarily long stretches of neutral chain. In practice, for the kind of systems physically envisaged by SSH, there will be many solitons in parallel strings of atoms which are extremely close to one another. The chains may be rather short as well. The positive charges may be associated with more than one soliton bearing string. With many solitons present, it could well be that, on the average, the positive charges do not respond, and thus, on the average, the solitons would carry the charge predicted by the SSH theory. This kind of argument is at the basis of the discussion of Bak.[7] A more delicate and

specific discussion is needed to decide what charge will be observed in an experiment measuring the "shot noise" due to a current of charged solitons. If the solitons are well isolated, and there is no ferroelectricity, then the result must be integral charge, but if they are not, fractional, or multiple charge could well result, depending on the correlations between solitons.

We have shown that electrically isolated nonintegral charge can indeed arise, in fact by a known effect, that of the ferroelectric domain wall. There are specific situations, however, where fractional charge which is not electrically isolated can be a useful concept in the absence of ferroelectricity. Each case must be considered separately, but it seems likely that the concept of fractional charge introduced by SSH will be useful.

ACKNOWLEDGMENTS

We thank J. R. Schrieffer, W. S. Su, J. Kivelson, S. das Sarma, V. Korenman, J. Bhattacharjee, and D. Thouless for discussions, and K. R. Subbaswamy for a copy of his work in advance of publication. The work was supported in part by NSF Grant No. DMR 7908819.

APPENDIX

For completeness, we give the definitaion of charge used by Kivelson and Schrieffer[8] and compare it to the definition used in standard classical electromagnetic theory. Let $\rho_m(x)$ be the microscopic charge density. Then define macroscopic density

$$\rho(x)=\int f(x-x')\rho_m(x')dx' ,$$

where $f(x)$ is some function with unit integral, which is relatively large over a region containing macroscopic numbers of charges, and drops off smoothly to zero. It is known that the results are independent of the detailed choice of $f(x)$. According to our hypothesis, $\rho(x)$ vanishes except in a region U. Thus $\int \rho(x)dx$ is well defined, and equals $\int \rho_m(x)dx$ whenever the latter is defined. Kivelson and Schrieffer introduce

$$Q_F=\int \rho_e(x)F(x)dx ,$$

where $F(x)$ is a function essentially unity for x in U, and which drops off smoothly, for example, exponentially, outside U. The integral is over the electronic charge density ρ_e, however, not the total charge. The charge on the soliton is given by $Q_S=Q_F-\langle Q_F\rangle_0$, where $\langle Q_V\rangle_0$ is the value of the Q_F in the situation with no soliton. Clearly, $-\langle Q_F\rangle_0$ is the positive charge in the neutral, no-soliton situation, so if the positive charge is assumed not to change upon introduction of the soliton, Q_S will give the total charge. If $F(x)$ is chosen as

$$F(x)=\int f(x-x')dx' ,$$

where the integral is over some region containing U and sufficiently larger such that F is effectively unity for x in U, we have an exact correspondence between the two methods. Q_F is not a number operator. The electron number operator associated with some volume V is

$$N_V=\int V\tilde{\rho}_e(x)dx/(-e) ,$$

where the integral is that of the electron charge-density operator over a specified volume V. N_V has integral eigenvalues, and its expectation value is the average number of electrons to be found in the volume in the given state.

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