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Coherence length of a normal metal in a proximity system

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The Eilenberger equations are solved to find the correlation length in the normal (N) part of a proximity system at a temperature T which is well above the critical temperature T_{cN} and is not too far below T_{cS} ($T_{cN} < T \lesssim T_{cS}$). The result is valid for any concentration of nonmagnetic impurities.

I. INTRODUCTION

A number of situations of interest exist where a metal at a temperature T higher than its critical temperature for the normal-superconductor transition (let us call it N metal), supports a persistent current due to contact with a superconductor S ($T_{cN} < T < T_{cS}$). These are various kinds of *SN*, *SNS* "junctions" or "sandwiches" and "*in situ*" prepared compounds where S filaments are embedded in an N matrix.

The coherence length $\xi(T)$ is one of the basic characteristics of a superconductor in general and of a material upon which the superconductivity is imposed by proximity with a neighbor superconductor. Strictly speaking, this length should be found from the microscopic theory by solving the proper equations in the S and N parts of the system separately under the proper boundary conditions. This has been done for dirty materials by de Gennes.[1]

We shall show that in the temperature domain

$$T_{cN} < T \lesssim T_{cs} , \tag{1a}$$

the length $\xi(T)$ can be calculated by considering only the N part of the system for an arbitrary impurity concentration. Here $T_{cN} < T$ means "well above" or "out of the immediate vicinity of T_{CN}," whereas $T \lesssim T_{cS}$ implies "in the vicinity of T_{cS}." We shall use the Eilenberger formalism[2] or its simplified version for the dirty limit due to Usadel,[3] which seems to be the most convenient for our purpose.

Eilenberger[2] integrated the Gor'kov Green's functions [e.g., $F(\vec{r},\vec{r}\,',\omega)$ where $\hbar\omega = \pi T(2n+1)$, or, after a Fourier transform, $F(\vec{r},\vec{k},\omega)$] over the energy variable $\zeta(\vec{k})$ separating $\vec{k}$ into $\hat{k}=\vec{k}/k$ and k. For the functions $f(\vec{r},\hat{k},\omega)$, $f^\dagger(\vec{r},\hat{k},\omega)$, and $g(\vec{r},\hat{k},\omega)$ so obtained from the Gor'kov's F, $F^\dagger$, and G, respectively, he derived the set of equations:

$$(2\omega+\vec{v}\cdot\vec{\Pi})f = 2\Delta g/\hbar + \tau^{-1}(g\langle f\rangle - f\langle g\rangle) , \tag{1b}$$

$$\Delta \ln\frac{T_c}{T} = 2\pi T \sum_{\omega>0} \left[\frac{\Delta}{\hbar\omega} - \langle f\rangle\right] , \tag{1c}$$

$$f^\dagger(\vec{r},\vec{v},\omega) = f^*(\vec{r},-\vec{v},\omega), \quad g^2 = 1 - ff^\dagger . \tag{1d}$$

Here we restrict ourselves to the case of a spherical Fermi surface and use the Fermi velocity $\vec{v}$ instead of $\hat{k}$. The gauge-invariant gradient is $\vec{\Pi} = \vec{\nabla} - i(2e/\hbar c)\vec{A}$ and $\vec{A}$ is the vector potential. The gap function $\Delta(\vec{r})$ depends only on $\vec{r}$. The relaxation time for the impurity scattering is $\tau = l/v$, where l is the mean free path. The angle bracket sign $\langle \cdots \rangle$ means an average over the Fermi surface (or over all $\vec{v}$ directions). The temperature T is measured in energy units. In the term $\tau^{-1}(\cdots)$ of Eq. (1b), only the s scattering from nonmagnetic impurities is actually taken into account. The set (1b)–(1d) should be completed by adding a current equation and the Maxwell equations; we shall not use them in this paper.

The paper is organized as follows: First we recover the known result of the temperature dependence of the upper critical field $H_{c2}(T)$ (Refs. 4–6) using the Eilenberger formalism (Sec. II). A similar procedure will then be used in Sec. III and IV to calculate $\xi(T)$ in the temperature domain (1a). The treatment of the dirty limit is especially simple and is given separately in Sec. V. The important case $T_{cN} \simeq 0$ is considered in Sec. VI. In Sec. VII we compare the theory with other existing theories and with available experimental data.

II. UPPER CRITICAL FIELD

It is instructive for our purposes to see how $H_{c2}(T)$ can be obtained using Eqs. (1b)–(1d); we shall see that this is considerably simpler than the original approach.[4–6]

Let us begin with the clean limit where $\tau^{-1}=0$. Near the second-order phase transition at H_{c2}, both Δ/T and f are small, whereas $g\simeq 1$ in the linear approximation [see Eq. (1d)]. In this region the set (1b)–(1d) reduces to the self-consistency condition (1c) and the now *linear* equation

$$(2\omega+\vec{v}\cdot\vec{\Pi})f=2\Delta/\hbar\ . \tag{2}$$

The solution of this equation is

$$f=(2\omega+\vec{v}\cdot\vec{\Pi})^{-1}2\Delta/\hbar = \frac{2}{\hbar}\int_0^\infty d\rho\, e^{-\rho(2\omega+\vec{v}\cdot\vec{\Pi})}\Delta\ . \tag{3}$$

Substituting this in Eq. (1c) we obtain

$$-\frac{\Delta\ln t}{2\pi T}=\sum_{\omega>0}\left[\frac{\Delta}{\hbar\omega}-\frac{2}{\hbar}\int_0^\infty d\rho\, e^{-2\omega\rho}\langle e^{-\rho\vec{v}\cdot\vec{\Pi}}\Delta\rangle\right] \tag{4}$$

($t=T/T_c$). This is quite complicated linear homogeneous integral equation for $\Delta(\vec{r})$; in other words, $\Delta(\vec{r})$ is an eigenfunction of an operator defined by Eq. (4).

The solution of this equation is made possible by observation (see, e.g., Refs. 4–6, that the lowest eigenfunction of the equation

$$-\xi^2\Pi^2\Delta=\Delta \tag{5}$$

in a uniform magnetic field is a solution of our Eq. (4), if $\xi(T)$ is chosen properly. Physically, this means that at any temperature $T<T_c$ the structure of the mixed phase near H_{c2} is the same as that in the Ginzburg-Landau (GL) domain ($|T-T_c|/T_c\ll 1$) with a different ξ, however. Recall that near T_c, H_{c2} corresponds to the lowest eigenvalue of Eq. (5) and is given by

$$H_{c2}=\hbar c/2|e|\xi^2\ . \tag{6}$$

Another way of looking at the same problem is to consider Eq. (5) as an *ansatz* and then to try to satisfy Eilenberger's equations, or Eq. (4) in our case.

To proceed with the actual calculation we introduce the operators $\Pi^\pm=\Pi_x\pm i\Pi_y$, where z is chosen as the field direction; without any loss in generality we can put $\Pi_z=0$. The following relations are easily verified with the help of Eqs. (5) and (6);

$$\Pi^-\Delta=0,\quad \Pi^-\Pi^+\Delta=[\Pi^-,\Pi^+]\Delta=-2\Delta/\xi^2\ , \tag{7}$$

where [· · ·] stands for a commutator. We also $v^\pm=v_x\pm iv_y$ to obtain $\vec{v}\cdot\vec{\Pi}=(v^-\Pi^++v^+\Pi^-)/2$. Then using the operator identity

$$e^{P+Q}=e^Pe^Qe^{[Q,P]/2}\ , \tag{8}$$

which holds if $[Q,P]$ commutes with P and Q, one can evaluate $\exp(-\rho\vec{v}\cdot\vec{\Pi})$ in Eq. (4). Let

$$P=-\rho v^-\Pi^+/2,\quad Q=-\rho v^+\Pi^-/2\ , \tag{9}$$

then $(\exp[Q,P/2])\Delta=\Delta\exp(-\gamma)$, where $\gamma=(\rho v_\perp/2\xi)^2$, $v_\perp^2=v_x^2+v_y^2=v^-v^+$, and Eq. (7) has been used. Further, $e^Q\Delta=\Delta$ since $Q\Delta=0$. Therefore,

$$\langle e^{-\rho\vec{v}\cdot\vec{\Pi}}\Delta\rangle=\langle e^{-\gamma}e^P\Delta\rangle = \sum_{k=0}^\infty\frac{1}{k!}\left[-\frac{\rho}{2}\Pi^+\right]^k\Delta\langle e^{-\gamma}(v^-)^k\rangle$$

We now choose the spherical coordinates (θ,ϕ) on the Fermi sphere with the v_z axis parallel to the field: $v_\perp=v\sin\theta$, $v^\pm=v_\perp e^{\pm i\phi}$. Then the operation $\langle\cdots\rangle=\int_0^{2\pi}\int_0^\pi\sin\theta d\theta\, d\phi/4\pi\cdots$ excludes all terms in the sum over k with $k\neq 0$, i.e.,

$$\langle\exp(-\rho\vec{v}\cdot\vec{\Pi})\Delta\rangle=\Delta\langle\exp(-\gamma)\rangle$$

and

$$\langle f\rangle=\frac{2}{\hbar}\Delta\int_0^\infty d\rho\, e^{-2\omega\rho}\langle e^{-\gamma}\rangle\ . \tag{10}$$

Turning now to Eq. (4) we see that the solution of Eq. (5) at H_{c2} also satisfies Eq. (4), $\Delta(\vec{r})$ is just canceled. What remains gives an equation for $\xi(T)$:

$$-\frac{\ln t}{2\pi T}=\sum_{\omega>0}\left[\frac{1}{\hbar\omega}-\frac{2}{\hbar}\int_0^\infty d\rho\, e^{-2\omega\rho}\langle e^{-\gamma}\rangle\right]$$

or after some simple algebra:

$$\tfrac{1}{2}\ln t=\sum_{n=0}^\infty\left[\frac{1}{z}J(\alpha)-\frac{1}{2n+1}\right]\ , \tag{11}$$

where we introduced z by

$$\xi=\xi_N/z,\quad \xi_N=\hbar v/2\pi T \tag{12}$$

and

$$\alpha = z/(2n+1) = v/2\omega\xi \ . \tag{13a}$$

The function $J(\alpha)$ is given by

$$J(x) = 2^{1/2} \int_0^\infty d\eta\, e^{-\eta x} \times \int_0^{\pi/2} d\theta \sin\theta \exp(-\tfrac{1}{2}\eta^2 \sin^2\theta) \tag{13b}$$

where $\eta = \rho v/\xi 2^{1/2}$ and $x = 2^{1/2}/\alpha$. Another form of this function is given in Ref. 4:

$$J(\alpha) = 2\int_0^\infty dw \exp(-w^2)\tan^{-1} w\alpha$$

(a detailed discussion of its properties can be found in Ref. 6). Solving numerically Eqs. (11)–(13b) (Ref. 4) one finds $z(t)$ and then $H_{c2}(T)$ from Eqs. (6) and (12).

For an arbitrary impurity concentration the linearized Eq. (1b) reads

$$(2\omega' + \vec{v}\cdot\vec{\Pi})f = \frac{2}{\hbar}\left[\Delta + \frac{\hbar}{2\tau}\langle f\rangle\right], \tag{14}$$

where $\omega' = \omega + (2\tau)^{-1}$. This differs from the equation (2) of the clean case by replacing ω and Δ with ω' and $(\Delta + \hbar\langle f\rangle/2\tau)$, respectively. This substitution should also be done in Eq. (3) to obtain the solution of Eq. (14). We now notice that as is seen from the self-consistency relation (1c), the coordinate dependence of $\langle f\rangle$ is the same as that of $\Delta(\vec{r})$, i.e., $\langle f\rangle$ satisfies Eq. (5) along with Δ [apply Π^2 to Eq. (1c) and take Eq. (5) into account]. This means that the derivation for the clean case can be repeated to obtain

$$\langle f\rangle = \frac{2}{\hbar}\left[\Delta + \frac{\hbar}{2\tau}\langle f\rangle\right]\int_0^\infty d\rho\, e^{-2\omega'\rho}\langle e^{-\gamma}\rangle \tag{15}$$

instead of Eq. (10). This gives $\langle f\rangle$ which is then substituted into Eq. (1c) with the result:

$$\tfrac{1}{2}\ln t = \sum_{n=0}^{\infty}\left[\frac{J(\alpha')}{z - \lambda J(\alpha')/t} - \frac{1}{2n+1}\right], \tag{16}$$

where J is given in Eq. (13b),

$$\alpha' = \frac{z}{2n+1+\lambda/t}, \tag{17}$$

and the impurity parameter is

$$\lambda = \frac{\hbar}{2\pi T_c \tau} = \frac{\hbar v}{2\pi T_c l} \ . \tag{18}$$

For $\lambda = 0$ Eqs. (16) and (17) recover the formulas (11) and (13) of the clean limit.

III. CORRELATION LENGTH, $t > 1$

For the reduced temperature $t > 1$, the only uniform solution of the Eilenberger equations is trivial: $f = \Delta = 0$, $g = 1$ as it must be in the normal phase. However, if the N metal is a part of a proximity system, nontrivial solutions of the E equations are of a considerable interest (see, e.g., Ref. 7).

One can expect the E equations to simplify if the temperature is well above the critical temperature of the N metal ($T > T_{cN}$). The approximation $f \ll 1$, $\Delta/T \ll 1$, $g \simeq 1$ still holds here (as near H_{c2} for $0 < t < 1$) *almost* everywhere in N, except probably a layer of thickness ξ near the S-N boundary. In this layer f and Δ are lifted by the proximity with S. If, however, T is close to T_{cS}, $f \ll 1$ *everywhere* in N since this is true even deep in the S metal.

Thus, we consider the temperature domain (1a) where Eq. (1b) can be linearized. For simplicity we begin with the clean limit where Eq. (2) holds. The problem of finding $\xi(T)$ under this condition is similar to that of $H_{c2}(T)$ [or $\xi(T)$ for $0 < t < 1$]. We also assume that the magnetic field is absent or it is negligibly small; we shall discuss what this means later on. The solution (3) is valid here too.

The difference arises when we turn to Eq. (5). We had a good physical reason in the preceding section to try Eq. (5) as an ansatz. Here we have no such guiding idea. Nevertheless, we shall try the ansatz

$$\xi^2 \Pi^2 \Delta = \Delta \tag{19}$$

with the hope that its solutions will also satisfy the integral equation (4). The sign in (19) has been chosen to give the correct equation in the GL domain ($t \gtrsim 1$).

In the absence of a magnetic field $\vec{\Pi} = \vec{\nabla}$, i.e., all its components commute with each other. We now have from Eq. (19)

$$\Pi^+\Pi^-\Delta = \Pi^-\Pi^+\Delta = \Delta/\xi^2 \ , \tag{20}$$

instead of Eq. (7).

Applying Eq. (8) to $\exp(-\rho\vec{v}\cdot\vec{\Pi}) = \exp(P+Q)$ with P and Q given by Eq. (9), we have:

$$\langle f\rangle=\frac{2}{\hbar}\int_0^\infty d\rho\, e^{-2\omega\rho}\langle e^P e^Q\Delta\rangle=\frac{2}{\hbar}\int_0^\infty d\rho\, e^{-2\omega\rho}\sum_{l,m}\frac{(-\rho/2)^{l+m}}{l!m!}\langle v_\perp^{l+m}e^{i(m-l)\phi}\rangle(\Pi^+)^l(\Pi^-)^m\Delta$$

$$=\frac{2}{\hbar}\Delta\int_0^\infty d\rho\, e^{-2\omega\rho}\sum_{m=0}^{\infty}\left[\frac{\rho}{2\xi}\right]^{2m}\frac{\langle v_\perp^{2m}\rangle}{(m!)^2}\,, \tag{21}$$

where each step is obvious. We see that $\langle f\rangle\propto\Delta$, i.e., the ansatz (19) indeed solves the self-consistency equation (4). What remains from this equation after the $\Delta(\vec{r})$ is canceled out, gives an equation for $\xi(T)$.

Obviously, Eq. (19) describes an exponential attenuation of Δ in the N region with the decay length $k_N^{-1}=\xi$; for this reason ξ is usually called the pair penetration depth.

The integrations in Eq. (21) can be performed to give

$$\langle f\rangle=\frac{\Delta}{\hbar\omega}\sum_{m=0}^{\infty}\frac{\alpha^{2m}}{2m+1}=\frac{\Delta}{\pi Tz}\tanh^{-1}\alpha\,. \tag{22}$$

Substituting this in Eq. (1c) one obtains an implicit equation for $z(t)$ [or for $\xi(T)$]:

$$\tfrac{1}{2}\ln t=\sum_{n=0}^{\infty}\left[\frac{1}{z}\tanh^{-1}\alpha-\frac{1}{2n+1}\right]. \tag{23}$$

Note that the series (22) converges under the condition $\alpha<1$, which implies

$$z<1\,, \tag{24}$$

i.e., in the clean limit $\xi(T)>\xi_N=\hbar v/2\pi T$ at any temperature.

Equation (23) can be solved numerically; the result $z(T)$ is given in Fig. 1 for $\lambda=0$. However, one can further transform Eq. (23) to

$$\ln t=\psi\left[\frac{1}{2}\right]-\frac{1}{z}\ln\frac{\cos(\pi z/2)}{\pi}-\frac{2}{z}\ln\Gamma\left[\frac{1+z}{2}\right] \tag{25}$$

as is shown in Appendix A. Here Γ and ψ are the gamma and digamma functions.

It is easy to obtain the asymptotic behavior of $z(t)$:

$$z\simeq 1-0.28/t,\quad t\to\infty \tag{26}$$

$$z^2\simeq 12(t-1)/7\zeta(3),\quad t\to 1 \tag{27}$$

where $\zeta(3)=1.202$ and $0.28=2\exp\psi(\frac{1}{2})$. Equation (26) shows that with increasing t, the correlation length $\xi=\xi_N/z$ slowly decreases to ξ_N of Eq. (12); $\xi\simeq\xi_N$ within 1% at $t\gtrsim 30$. Formula (27) gives the GL result in the clean limit:

$$\xi_{\rm GL}=\frac{\hbar v}{2\pi T_c}\left[\frac{7\zeta(3)}{12(t-1)}\right]^{1/2}. \tag{28}$$

In the presence of a magnetic field the components of $\vec{\Pi}$ no longer commute: for any pair $\Pi_i\Pi_k$ $(i,k=x,y,z)$

$$[\Pi_i,\Pi_k]\Delta=-i\frac{2e}{c\hbar}e_{ikl}H_l\Delta=ie_{ikl}H_l\frac{2\pi\xi^2}{\phi_0}\frac{\Delta}{\xi^2}\,,$$

where ϕ_0 is the flux quantum and $H_i=\text{curl}_i\vec{A}$. Because $\Pi_i\Delta$ is of order Δ/ξ, the commutator $[\Pi_i,\Pi_k]$ is small, if

$$H\ll\phi_0/2\pi\xi^2\,. \tag{29}$$

With this restriction, the $\xi(T)$ found above, also holds in the presence of the magnetic field. It is worth noting, however, that in the dirty case (see Sec. V) the condition (29) can be relaxed.

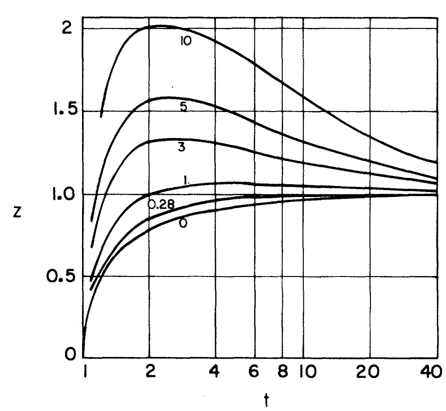

FIG. 1. Dependence $z(t)=\hbar v/2\pi T\xi$ obtained by solving Eq. (32) for the impurity parameter $\lambda=0$, 0.28, 1, 3, 5, and 10 (t is the reduced temperature).

IV. ARBITRARY IMPURITY CONCENTRATION

As in the corresponding case of $H_{c2}(T,\lambda)$, we begin here with Eq. (14). The same reasoning as in the preceding sections yields

$$\langle f\rangle=\frac{1}{\pi T z}\left[\Delta+\frac{\hbar}{2\tau}\langle f\rangle\right]\tanh^{-1}\alpha' . \quad (30)$$

The series leading to the $\tanh^{-1}\alpha'$, converges if

$$\alpha'=z/(2n+1+\lambda/t)<1$$

for any n, or if

$$z<1+\lambda/t . \quad (31)$$

This condition enters here instead of inequality (24) of the clean limit.

Extracting $\langle f\rangle$ from Eq. (30) and substituting it in Eq. (1c) we obtain

$$\frac{1}{2}\ln t=\sum_{n=0}^{\infty}\left[\frac{J_1(\alpha')}{z-\lambda J_1(\alpha')/t}-\frac{1}{2n+1}\right],$$
$$J_1(\alpha')=\tanh^{-1}\frac{z}{2n+1+\lambda/t} . \quad (32)$$

This is solved numerically with respect to $z(t,\lambda)$; the results are given in Fig. 1 for $\lambda=0$, 0.28, 1, 3, 5, and 10.

The clean-limit result (23) follows immediately from Eq. (32) with $\lambda=0$. In the dirty case $\lambda/t\to\infty$ and $\alpha'\simeq zt/\lambda$ can be considered as small. Expanding the $\tan^{-1}\alpha'$ to the third order in α', we obtain

$$\frac{1}{2}\ln t=\sum_{n=0}^{\infty}\left[\frac{1}{2n+1-z^2t/3\lambda}-\frac{1}{2n+1}\right] \quad (33)$$

or

$$\ln t=\psi\left[\frac{1}{2}\right]-\psi\left[\frac{1}{2}-\frac{z^2t}{6\lambda}\right] . \quad (34)$$

One should be careful when applying the dirty-limit formula: The actual parameter that should be large to obtain Eqs. (33) and (34) is

$$\frac{\lambda}{t}=\frac{\hbar v}{2\pi T l} \quad (35)$$

(not a bare $\lambda=\hbar v/2\pi T_c l$ as it was in the domain $0<t<1$). Note that for a metal with $T_c\to 0$, $\lambda\to\infty$ for any finite mean free path, whereas λ/t remains finite for $T\neq 0$.

The derivation which led to Eq. (32) for $\xi(T)$ is valid for temperatures well above T_{cN}. It is interesting to note, however, that Eq. (32) gives the exact GL result for ξ in the limit $t\to 1$ (Appendix B):

$$\xi_{\rm GL}^2=\frac{7\zeta(3)\hbar^2v^2}{48\pi^2T_c(T-T_c)}\chi(\lambda) . \quad (36)$$

Here $\chi(\lambda)$ is the Gor'kov function given in Appendix B. In the clean limit $\chi(0)=1$ and we recover Eq. (28); in the dirty case $\chi\simeq\pi^2/7\zeta(3)\lambda$ and

$$\xi_{\rm GL}^2=\frac{\hbar^2v^2}{48T_c\lambda}=\frac{\pi\hbar v l}{24(T-T_c)} . \quad (37)$$

The asymptotic behavior of $z(t)$ as $t\to\infty$ for an arbitrary but fixed λ is obtained in Appendix C and is given by

$$z\simeq 1+\frac{\lambda-0.28}{t} . \quad (38)$$

At $\lambda=0$ this reduces to the asymptotic expression (26) of the clean limit. We see that for $\lambda<0.28$, the curve $z(t)$ approaches 1 from below, whereas for $\lambda>0.28$ the curve descends to 1 from above. Since $z(1)=0$, all curves $z(t)$ for $\lambda>0.28$ have a maximum (see Fig. 1).

Dividing Eq. (38) by ξ_N we obtain the asymptotic formula for $\xi(T)$:

$$\frac{1}{\xi}=\frac{1}{\xi_N}+\frac{1}{l}-\frac{0.28}{\xi_N t} . \quad (39)$$

Here the last term cannot be removed; its presence assures convergence of $\tanh^{-1}z/(1+\lambda/t)$ that arises in Eq. (32). Note also that the asymptotic formulas (38) and (39) obtained for $\lambda/t\to 0$ are not applicable to the dirty limit where $\lambda/t\to\infty$.

V. DIRTY LIMIT

The result (34) for a dirty metal can be obtained in the most direct way if one begins with the E equations where the "dirtiness" is taken into account from the very start. A necessary modification of the E equations was done by Usadel,[3] who obtained for $F(\vec{r},\omega)\equiv\langle f\rangle$ and $G(\vec{r},\omega)\equiv\langle g\rangle$ the set of equations:

$$-D\vec{\Pi}(G\vec{\Pi}F-F\vec{\nabla}G)=2G\Delta/\hbar-2\omega F , \quad (40)$$

$$\Delta\ln\frac{T_c}{T}=2\pi T\sum_{\omega>0}\left[\frac{\Delta}{\hbar\omega}-F\right] , \quad (41)$$

$$G^2=1-|F|^2 . \quad (42)$$

Here $D=v^2\tau/3$ is the diffusion coefficient. Equations (40)–(42) are valid if[3]

$$G \gg 2\omega\tau, \quad F \gg 2\tau\Delta/\hbar \ . \tag{43}$$

In the temperature domain (1a), $F \ll 1$ and $G \simeq 1$ everywhere within the N metal as was argued above. Then the leading term in the left-hand side of Eq. (40) is $-D\Pi^2 F$; the neglected terms are of order F^3.

We now test the ansatz

$$F(\vec{r},\omega)=\Delta(\vec{r})/(\hbar\omega - a) \ , \tag{44}$$

where the temperature dependent $a(T)$ is to be found from the self-consistency relation (41):

$$\tfrac{1}{2}\ln = \sum_{n=0}^{\infty}\left[\frac{1}{2n+1-a/\pi T}-\frac{1}{2n+1}\right] . \tag{45}$$

We see that $a(T)$ is related to the $z(t)$ of the dirty limit given by Eq. (33): $a=\pi T t z^2/3\lambda$. We now substitute our ansatz (44) in the right-hand side of Eq. (40) to obtain $-2aF/\hbar$ in the linear approximation. Thus, we have

$$\xi_d^2 \Pi^2 F = F \ , \tag{46}$$

where the dirty-limit correlation length is $\xi_d=(\hbar D/2a)^{1/2}$ or in the notations of Ref. 1,

$$\xi_d^2=\frac{\hbar D}{2\pi T y(t)} \ . \tag{47}$$

The function $y(t)=a/\pi T=z^2 t/3\lambda$ is given by Eq. (45) or by

$$\ln t = \psi\left[\frac{1}{2}\right]-\psi\left[\frac{1}{2}-\frac{y}{2}\right] \tag{48}$$

(see Fig. 2), which has been considered by de Gennes.[1] It is easy to show that Eqs. (47) and (48) give the GL result (37) at $t \to 1$. For $t \gg 1$ we obtain

$$\xi_d^2=\frac{\hbar D}{2\pi T}\left[1-\frac{2}{\ln 4t}\right]^{-1} . \tag{49}$$

In the derivation of the $\xi_d(T)$ given here we did not use any assumption about the commutation properties of the $\vec{\Pi}$ components. This, along with the simplicity of the Usadel-based approach to the dirty case, makes it especially attractive. In particular, this implies that Eq. (46) with ξ_d of Eqs. (47) and (48), is applicable in any magnetic field within the temperature domain (1a).

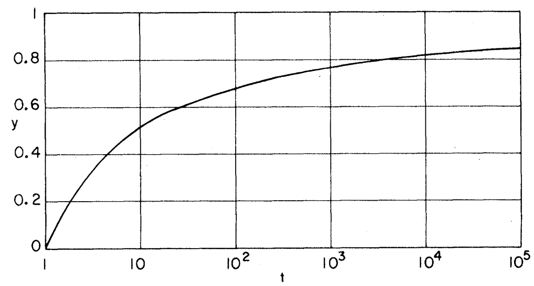

FIG. 2. Dependence $y(t)=\hbar D/2\pi T\xi_d^2$ obtained by solving Eq. (48) for the dirty limit (t is the reduced temperature).

VI. N METAL WITH ZERO OR NEGATIVE COUPLING

The general result (32) is of no use for a zero coupling constant where $T_{cN}=0$. One might have considered Eq. (32) for a small finite T_{cN} and then take the limit $T_{cN}\to 0$. Instead we go back to our starting point, namely, to the linear Eq. (14) and take the limit $\Delta \to 0$. Proceeding exactly as was done in Secs. III and IV, we obtain in this limit

$$\langle f\rangle = \frac{\hbar v}{2\pi T l z}\langle f\rangle \tanh^{-1}\alpha'$$

instead of Eq. (30), or

$$1-\frac{\lambda}{tz}\tanh^{-1}\frac{z}{2n+1+\lambda/t}=0 \ . \tag{50}$$

Here $z=\xi_N/\xi$ $\lambda/t=\xi_N/l$ depend only on the actual temperature T, ξ, and l. Therefore, Eq. (50) gives ξ as a function of T, l, and n. The n dependence of ξ is a specific feature of the case $T_{cN}=0$. This means that $\langle f(n,\vec{r})\rangle$ attenuates in the N metal at a distance which is different for different n's (recall $\langle f\rangle$ satisfies $\xi^2\Pi^2\langle f\rangle = \langle f\rangle$). Because of the exponential character of this decay, all $\langle f(n)\rangle$'s can be neglected with respect to the one with the largest $\xi(n)$ (or the smallest z). It is easy to see from Eq. (50) that the smallest root $z(n,\lambda/t)$ corresponds to $n=0$. We have, therefore, the equation for ξ_0:

$$\frac{l}{\xi_0}=\tanh^{-1}\frac{\xi_N l}{\xi_0(l+\xi_N)} \tag{51}$$

instead of Eq. (50) (the 0 subscript relates ξ_0 to the case $T_{cN}=0$).

In the dirty metal $\xi_N/l \gg 1$, and we easily obtain

$$\xi_0^2=\tfrac{1}{3}\xi_N l \ , \tag{52}$$

this is the dirty-limit result (49) where $t \to \infty$. In the case of large l's

$$\xi_N l / [\xi_0 (l+\xi_N)] = \tanh(l/\xi_0) \simeq 1 ,$$

or

$$\frac{1}{\xi_0} \simeq \frac{1}{\xi_N} + \frac{1}{l} . \tag{53}$$

This coincides with the asymptotic expression (39) for the clean case at $t=\infty$.

Let us now turn to the case of a negative coupling constant (the net electron repulsion) of the N metal in a proximity system. The Eilenberger equations and our result (32) for $\xi(T,l)$ are quite general and can also be applied to this situation. This was pointed out by Kupriyanov, Likharev, and Lukichev[8] who introduced a term "supernormal" for a metal with a repulsive electron interaction. The effective coupling constant $N(0)V$ enters the E equations in T_c only: $T_c \propto \exp(-1/N(0)V)$. Therefore, an N metal with $N(0)V<0$ can be described formally as having a large positive "critical" temperature T_c^*.

In order to see how a large T_{cN}^* arises in the formal structure of Eq. (32) for a supernormal metal, we first rewrite it as

$$T_c = T \exp[-2S(z,\lambda/t)] . \tag{54}$$

Here S stands for the $\sum_{n=0}^{\infty}$ of Eq. (32); the variables z and λ/t can be replaced by ξ, l, and T. When $T_c \to 0$, S diverges: According to Eq. (51), the denominator

$$1-\frac{\lambda}{tz}\tanh^{-1}\frac{z}{1+\lambda/t} , \tag{55}$$

in the term $n=0$ of $\sum_{n=0}^{\infty}$, is zero. Then, if ξ changes from $\xi_0+\delta\xi$ to $\xi_0-\delta\xi$ and $\delta\xi \to 0$, the denominator (55) changes from $+0$ to -0, i.e., the S jumps from $+\infty$ to $-\infty$ and T_c suffers a discontinuity from $+0$ to $+\infty$. This nonanalytic behavior of T_c certainly might have been expected from the very beginning: $T_c \propto \exp(-1/NV)$ is discontinuous when NV passes through zero.

There is a difference in the correlation lengths of a supernormal metal in the clean and dirty situations. To see this we first recall that the self-consistency relation cannot be satisfied if $z > 1+\lambda/t$ [see Eq. (31)]. In other words, the length ξ is always larger than

$$\xi_{\min} = \frac{\xi_N}{z_{\max}} = \frac{\xi_N}{1+\lambda/t} = \frac{\xi_N l}{l+\xi_N} . \tag{56}$$

In the absence of interaction ξ_0 is given by Eq. (53) in the clean case. The correction to this asymptotic formula is exponentially small:

$$\xi_0 = \frac{\xi_N l}{\xi_N + l}(1+2e^{-2l/\xi_N}) .$$

This actually coincides with the $\xi_{\min}$ of Eq. (56) because $l \gg \xi_N$. Thus in a clean supernormal metal the correlation length is actually the same as if the electron-electron interaction were absent.

In the dirty limit $\xi_{\min} \simeq l(1-l/\xi_N)$, whereas $\xi_0 = (\xi_N l/3)^{1/2}$. The ratio $\xi_0/\xi_{\min} \simeq (\xi_N/3l)^{1/2}$ is not small. In other words, there is substantially more room here for an effect of the repulsive interaction on the ξ value than in the clean case. We give some figures to illustrate this statement in the next section (see also Ref. 9).

VII. DISCUSSION

We now compare our result for $\xi(T)$ with other existing theories. The quantity of interest is usually the critical current j_c through the SNS system rather than $\xi(T)$. For thick N layers $j_c \propto \exp(-d/\xi)$, where the thickness d is considerably larger than ξ (see, e.g., Ref. 11). Kulik and Mitsay[9] (KM) solved this problem for a clean system using a model where $\Delta=0$ in the N part of the SNS junction. Their result for the correlation length is

$$\frac{1}{\xi} = \frac{1}{\xi_N} + \frac{1}{l} . \tag{57}$$

This is rather close to our asymptotic expression (39) for $t \gg 1$. Though the authors did not specify a temperature range where their result is valid, it is clear that $\Delta=0$ is a bad ansatz near T_{cN}. Therefore, the similarity between Eq. (57) and our result (39) at $t \gg 1$ is of no surprise. The curve $z=1+\lambda/t$ which corresponds to Eq. (57) is shown in Fig. 3 for $\lambda=3$ along with $z(t)$ found for the same λ from our Eq. (32).

Krahenbuhl and Watts-Tobin (KWT) proposed[10] an equation to describe superconductivity in an N metal, which can be obtained from the Usadel equation (40) by replacing the diffusivity $D=v^2\tau/3$ with $\tilde{D}=v^2\tau/3(2\omega\tau+1)$ (i.e., $\lambda/t \to 2n+1+\lambda/t$). When applied to Eq. (33) [which is equivalent to Eqs. (34) and (48) of the dirty limit] this replacement yields

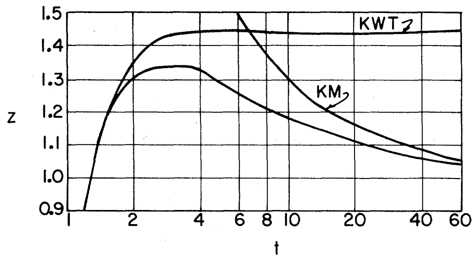

FIG. 3. Dependence $z(t)=\hbar v/2\pi T\xi$ obtained by solving Eq. (32). The KM curve corresponds to Eq. (57) obtained in Ref. 9. The KWT curve is obtained by solving Eq. (58) which corresponds to the theory of Ref. 10 (t is the reduced temperature). The impurity parameter $\lambda=3$ for all the curves.

$$\frac{1}{2}\ln t=\sum_{n=0}^{\infty}\left[\frac{1}{2n+1-z\alpha'/3}-\frac{1}{2n+1}\right]. \tag{58}$$

The solution $z_{\rm KWT}(t)$ of this equation for $\lambda=3$ is plotted in Fig. 3 along with $z(t)$ obtained from our Eq. (32). One can see that $z_{\rm KWT}(t)$ and $z(t)$ coincide in the GL domain; however, for t large, $z_{\rm KWT}$ exceeds our z by more than 40%. The coincidence of both z's in the GL region is not accidental: here $t\to 1, z\to 0$, and $\tanh^{-1}[z/(2n+1+\lambda/t)]$ of Eq. (32) can be expanded in a power series. Keeping only the first two terms in the expansion we obtain the KWT formula (58).

We believe that a mismatch out of the GL region can be understood as follows: To derive their equation the authors of Ref. 10 had to relax the first of two Usadel conditions (43), namely, $G\gg 2\omega\tau$. This, however, cannot be done at $t\gg 1$ without violating the second condition (43) or the self-consistency relation. Indeed, in this region $\Delta/T\ll 1$ and as is seen from the self-consistency equation (41), F is of order $\Delta/\hbar\omega$. Substituting the latter estimate in the second Usadel condition $F\gg 2\tau\Delta/\hbar$ we recover the first one: $1\gg 2\omega\tau$ ($G\simeq 1$); thus, $G\gg 2\omega\tau$ cannot be relaxed.

In the dirty limit our theory gives the same $\xi(T)$ as that of de Gennes[1] as well as KWT[10] [here the KWT equation (58) reduces to Eq. (33) because $\alpha'\simeq zt/\lambda$].

Most of the experiments on the thickness dependence of the critical current through *SNS* junctions with a thick N layer have been done with Cu, Ag, and their alloys as a material for the N film (see Refs. 11 and 12). The best collection of data relevant for our theory can be found in the paper by Niemeyer and Minnigerode[11] who measured ξ for a wide variety of mean free paths l. Unfortunately, direct comparison with the theory cannot be done because T_c for Cu and Ag are unknown. One can reverse the problem and try to calculate T_c for these metals from Eq. (32) using the measured values of ξ, l, and T (or z and λ/t). However, the actual numerical estimate of T_c turns out to be practically impossible: A spread in the experimentally determined ξ and l results in a huge uncertainty in T_c. Figure 4 demonstrates this quite clearly. Experimental points and the estimate of their accuracy are taken from Ref. 11. The dots represent ξ of CuGe alloys with different l's at $T=4.2$ K; the open circles show $\xi(l)$ for Ag and its alloys at the same T. The solid lines are calculated using Eq. (51), the upper line for Cu and the lower one for Ag, as if their T_c were zero (according to the estimate of Ref. 11, Fermi velocities for Cu and Ag are 1.56×10^6 and 1.38×10^6 m/sec, respectively). The dashed lines are calculated from Eq. (32) for Cu as if its T_c were 4.2×10^{-2} K (the upper dashed line) and $T_c^*=4.2\times 10^{+2}$ K (the lower dashed line), i.e., for $t=10^2$ and 10^{-2}, respectively. One can see that the data for ξ and l are far from being conclusive if we are interested in the determining T_c.

The nonanalyticity of $T_c(\xi)$ near $T_c=0$ which was discussed in the preceding section, makes the task of finding T_c for Cu and Ag even more difficult. To illustrate this we take as an example the data of Ref. 11 for CuGe (2.6 at. %): $\xi_N=450$ nm at $T=4.2$ K, $l=6.7$ nm so that the parameter $\lambda/t=\xi_N/l\simeq 67$. We first solve Eq. (51) for ξ_0 cor-

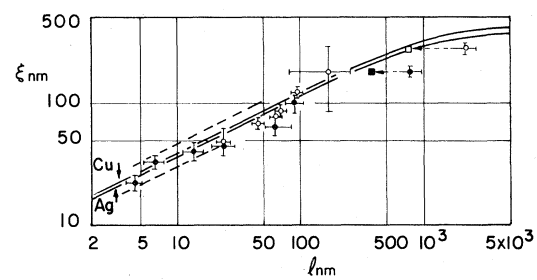

FIG. 4. Mean-free-path dependence of the correlation length $\xi(l)$. The experimental points are taken from Ref. 11. The black dots show $\xi(l)$ for Cu and its alloys; the open circles show $\xi(l)$ for Ag and its alloys. The solid curves are calculated from Eq. (51) for Cu and Ag as if their T_c were zero. The dashed lines are calculated from Eq. (32) for Cu as if its T_c were 4.2×10^{-2} K (the upper line) and $4.2\times 10^{+2}$ K (the lower line). The squares on the right are "the thin-film correction" to the l of the clean Cu and Ag as explained in the text.

responding to $T_c=0$; the result is $\xi_0=31.7$ nm. Note that the smallest possible ξ according to Eq. (56) is $\xi_{\min}\simeq 6.6$ nm. We now go back to Eq. (32) or better yet to its form (54) to estimate $T_c(\xi)$ for a series of points ξ from an interval which includes ξ_0:

$\xi_{nm}=$	34,	33,	32,	31,	30,	29;
$T_c(K)\simeq$	10^{-5},	10^{-10},	10^{-44},	10^{+20},	10^{9},	10^{6},

respectively (where ξ is expressed in nm and T_c in K). The divergence shows up between $\xi=31$ and 32 nm as it should. This example clearly indicates how futile the attempts were to determine T_c of Cu using the measured ξ (see Ref. 12 and references therein). Indeed, for this alloy the measured ξ is roughly between 29 and 39 nm according to Ref. 11.

The extremely low sensitivity of ξ to huge changes in T_c in the vicinity of $T_c=0$ (or to the value and sign of the coupling constant) has been pointed out in Ref. 8 in an analysis based on the Usadel equations, i.e., for the dirty case. The ξ is even less sensitive to T_c in clean Cu. For example, taking all of the same data for Cu ($\xi_N=450$ nm at 4.2 K) but $l=450$ nm so that the impurity parameter $\xi_N/l=1$, one obtains $\xi_0=235$ nm for $T_c=0$ [solve Eq. (51)]. Then Eq. (54) gives $T_c(\xi)$ in the vicinity of ξ_0:

$\xi_{(nm)}=$	242,	239,	236,	234,	231,	229;
$T_c(K)\simeq$	10^{-6},	10^{-10},	10^{-40},	10^{+41},	10^{10},	10^{6},

respectively (where ξ is expressed in nm and T_c in K). The situation here is qualitatively the same as in the preceding "dirty" example. However, regardless of the strength of the repulsion, it cannot reduce ξ under $\xi_{\min}=\xi_N/(1+\xi_N/l)=225$ nm which is much closer to ξ_0 than in the dirty case.

Still, as is seen from Fig. 4, the observed behavior of $\xi(l)$ and the theoretical curves are in reasonable agreement for a wide range of l's and ξ's. It is worth noting that the data are not inconsistent with the notion made in Ref. 8, that Cu may very well be a supernormal metal.

The points corresponding to the cleanest samples of Cu and Ag used in the experiment[11] are well off the theoretical curves. This is probably due to a somewhat overestimated l. Reference 11 gives $l=(770\pm200)$ nm for Cu. However, the experiment has been done for Cu films of thickness $500-1000$ nm. (This range for Ag was $500-2000$ nm whereas $l_{\rm Ag}=2200\pm500$nm.) Under these conditions an effective mean free path which takes into account the scattering at the *S-N* boundaries should be used rather than the bulk l. It is unclear whether or not one correctly includes the boundary effects in calculating l from the *bulk* formula[13] $l=\pi^2k_B^2/e^2\gamma v_F\rho$, even if the resistivity ρ is measured with a film sample.[11] Whatever the correct estimation procedure should be, $l_{\rm eff}$ should not substantially exceed the film thickness d. We estimated roughly $l_{\rm eff}$ using $l_{\rm eff}^{-1}=l^{-1}+d^{-1}$. Then the two points in question for the clean samples move to lower l's as is shown in Fig. 4 by closed and open squares for Cu and Ag, respectively.

It would be interesting to compare our theory with the data for ξ obtained from experiments with *SNS* junctions, where the N material has a known critical temperature. As we know, only one such an attempt has been reported by Hsiang and Finnemore.[14] They worked with Pb-Cd-Pb sandwiches (for Cd $T_c=0.52$ K) and were concerned mainly with magnetic field effects. Unfortunately, their data on $\xi(l,T)$ are not sufficient for a comprehensive comparison with the theory presented here.

Note added in proof. The result (51) has been given by M. Yu. Kupriyanov, Sov. J. Low Temp. Phys. 7, 342 (1941).

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APPENDIX A

Instead of summing up over m in Eq. (22) we substitute

$$\langle f\rangle=\frac{\Delta}{\hbar\omega}\sum_{m=0}^{\infty}\frac{\alpha^{2m}}{2m+1}=\frac{\Delta}{\hbar\omega}\sum_{m=0}^{\infty}\left[\frac{v}{2\omega\xi}\right]^{2m}\frac{1}{2m+1}$$

in the self-consistency relation (1c) and sum first over ω:

$$-\frac{1}{2\pi T}\ln t=\sum_{\omega>0}\left[\frac{1}{\hbar\omega}-\frac{1}{\hbar\omega}\sum_{m=0}^{\infty}\left[\frac{v}{2\omega\xi}\right]^{2m}\frac{1}{2m+1}\right]$$
$$=-\sum_{m=1}^{\infty}\left[\frac{\hbar v}{2\xi}\right]^{2m}\frac{1}{2m+1}\sum_{\omega>0}\frac{1}{(\hbar\omega)^{2m+1}}\ .$$

or

$$\tfrac{1}{2}\ln t=\sum_{m=1}^{\infty}z^{2m}\frac{\lambda(2m+1)}{2m+1}\ ,$$

where

$$\lambda(2m+1)\equiv(1-2^{-2m-1})\zeta(2m+1)=\sum_{n=0}^{\infty}(2n+1)^{-2m-1}\ . \qquad \text{(A1)}$$

With the help of the identity[15]

$$\sum_{m=1}^{\infty}\frac{x^{2m+1}}{2m+1}\zeta(2m+1)=\tfrac{1}{2}\ln\frac{\pi x}{\sin\pi x}-Cx-\ln\Gamma(1+x) \qquad \text{(A2)}$$

($C=0.5772$), that holds for $|x|<1$, we obtain the result (25).

APPENDIX B

In the GL domain $t\to 1$ and $\xi\to\infty$ (or $z\to 0$), i.e.,

$$\alpha'=z/(2n+1+\lambda/t)\ll 1\ ,$$

and

$$\tanh^{-1}\alpha'=\alpha'+\alpha'^3/3+\cdots\ .$$

Then Eq. (32) reads

$$\tfrac{1}{2}(t-1)=\frac{z^2}{3}\sum_{n=0}^{\infty}(2n+1)^{-2}(2n+1+\lambda)^{-1}\ .$$

In terms of the Gor'kov function

$$\chi(\lambda)=\frac{8}{7\zeta(3)}\sum_{n=0}^{\infty}(2n+1)^{-2}(2n+1+\lambda)^{-1}$$

we have

$$z^2=\frac{12(t-1)}{7\zeta(3)\chi(\lambda)}\ ,$$

which gives Eq. (36). In the clean limit $\lambda\to 0$ and

$$\sum_{n=0}^{\infty}(2n+1)^{-3}=\frac{7\zeta(3)}{8}\ ,$$

i.e., $\chi(0)=1$. If $\lambda\gg 1$,

$$\chi(\lambda)=\frac{8}{7\zeta(3)\lambda}\sum_{n=0}^{\infty}(2n+1)^{-2}=\frac{\pi^2}{7\zeta(3)\lambda}\ .$$

APPENDIX C

We look for a solution $z(t)$ of Eq. (32) as $t\to\infty$ in the form $z=1+\epsilon(t)$, where $\epsilon\to 0$. Then $\tanh^{-1}\alpha'$ behaves differently for $n=0$ and $n\neq 0$. It diverges if $n=0$,

$$\tanh^{-1}\frac{z}{1+\lambda/t}=\tanh^{-1}(1+\epsilon-\lambda/t)\simeq\frac{1}{2}\ln\frac{2}{\lambda/t-\epsilon}\ .$$

We keep here only the divergent and the constant terms. The remaining $\sum_{n=1}^{\infty}$ is convergent when both ϵ and λ/t go to zero. Therefore, it contributes only to the constant term of the divergent $\sum_{n=0}^{\infty}$. When evaluating this contribution we can put $\epsilon=\lambda/t=0$:

$$\sum_{n=1}^{\infty}\left[\tanh^{-1}\frac{1}{2n+1}-\frac{1}{2n+1}\right]=1-\ln 2-\frac{C}{2}$$

[use

$$\tanh^{-1}(2n+1)^{-1}=\sum_{k=0}^{\infty}(2n+1)^{-2k-1}(2k+1)^{-1}\ ,$$

sum up first over n and apply Eqs. (A1) and (A2)]. Finally, Eq. (32) gives $\epsilon=(\lambda-\tfrac{1}{2}e^{-C})/t$ where $\tfrac{1}{2}e^{-C}=0.2807$.

[1]P. G. de Gennes, Rev. Mod. Phys. 36, 225 (1964).
[2]G. Eilenberger, Z. Phys. 214, 195 (1968).
[3]K. Usadel, Phys. Rev. Lett. 25, 507 (1970).
[4]E. Helfand and N. R. Werthamer, Phys. Rev. 147, 288 (1966).
[5]G. Eilenberger, Z. Phys. 190, 142 (1966).
[6]G. Eilenberger, Phys. Rev. 153, 584 (1967).
[7]K. K. Likharev, Rev. Mod. Phys. 51, 101 (1979).
[8]M. Yu. Kupriyanov, K. K. Likharev, and V. F. Lukichev, Proceedings of the 16th International Conference on Low Temperature Physics [Physica (The Hague) 108B, 1001 (1981)].
[9]I. O. Kulik and Yu. N. Mitsai, Fiz. Nizk. Temp. 1, 906 (1975) [Sov. J. Low Temp. Phys. 1, 434 (1975)].
[10]Y. Krahenbuhl and R. J. Watts-Tobin, J. Low Temp. Phys. 35, 569 (1979).

[11]J. Niemeyer and G. von Minnigerode, Z. Phys. B 36, 57 (1979).
[12]J. Clarke, Proc. R. Soc. A308, 447 (1969).
[13]R. G. Chambers, Proc. R. Soc. London Ser. A 215, 481 (1952).
[14]T. Y. Hsiang and D. K. Finnemore, Phys. Rev. B 22, 154 (1979).
[15]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980). This is an error in formula (8.342.2); the correct form is given in Eq. (A2).