Conductivity of a random directed-diode network near the percolation threshold

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We have investigated the conductivity in a random directed network of "ohmic" diodes, elements which exhibit ohmic response under a forward-bias voltage. A numerical method is developed to identify first the correct "state" of the network, the distribution of forward- and back-biased diodes within the directed backbone, and then calculate the conductivity near the percolation threshold. Extrapolation of Monte Carlo data on the square lattice by *anisotropic* finite-size scaling yields a directed conductivity exponent of $t_+ \cong 0.60$.

Directed percolation is a relatively simple generalization of pure percolation in which a lattice is randomly occupied by *directed* bonds (diodes) which allow connectivity to "flow" in only one direction. This model has attracted considerable attention because it displays novel critical behavior characterized by two independent diverging lengths to describe cluster shapes near the percolation threshold.[1] Further interest stems from the connections that directed percolation has with Reggeon field theories,[2] branching Markov processes,[3] and strong-field hopping conductivity.[4]

In this Communication, we study the conductivity in directed percolation using Monte Carlo simulations in conjunction with an *anisotropic* finite-size scaling method. Our motivation for this investigation is the understanding of how the anisotropic character of directed percolation and the directionality constraints of the diodes manifest themselves in the behavior of the conductivity. The anisotropy of the phase transition requires a finite-size scaling that is anisotropic in nature. For the numerical values of the correlation length exponents of directed percolation, if the linear dimension of the lattice perpendicular to the anisotropy doubles, the linear dimension parallel to the anisotropy must approximately triple. This novel feature should arise in finite-size scaling for any model with more than one critical correlation length. The Lifshitz point problem,[5] and the nematic–to–smectic-A transition in liquid crystals[6] are other examples where this behavior occurs.

The directionality constraints give rise to an interesting problem of predicting the "state" of the system—the distribution of forward- and back-biased diodes in a random network. Since the determination of the network state requires the final distribution of voltages in the lattice, the problem is inherentaly nonlinear. We have developed a numerical relaxation method to find the state and simultaneously calculate the conductivity of a random network. We have applied this method to directed percolation, where almost all diodes are forward biased, and we hope to apply the method to systems with a larger fraction of back-biased diodes such as networks with randomly oriented diodes.

We consider a square lattice which is randomly occupied, with probability p_+, by "ohmic" diodes constrained to conduct only to the upper-right or the upper-left (Fig. 1). Each such diode is defined to allow a current flow which is proportional to the voltage under forward-biased conditions, and no current flow otherwise. Thus a positively biased ohmic diode behaves like an ideal resistor while a negatively biased ohmic diode behaves like an open circuit. For simplicity, we consider only the "directed" conductivity, obtained by applying the field along the anisotropy axis of directed percolation (Fig. 1).

To calculate the directed conductivity of a given configuration, we first obtain the underlying directed "backbone" by stripping away all dangling branches and isolated clusters. All series configurations and all nonnested parallel configurations are then reduced in order to save time in the subsequent numerical relaxation calculation for the conductivity. In order to introduce our approach for the directed diode network, we briefly outline relaxation for the random resistor network.[7] Starting with an initial guess for the voltages at each node of the network, local current conservation is used to update the node voltages. By

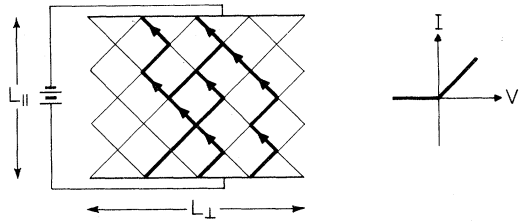

FIG. 1. The directed diode network with the field applied parallel to the anisotropy axis. Periodic boundary conditions in the transverse direction are employed. Diodes are indicated by heavy arrows, and the I-V response of one such element is shown to the right.

iterating this procedure, the potential at the ith node at the nth step is

$$V_i^{(n)} = \sum_j g_{ij} V_j \Big/ \sum_j g_{ij} \ , \tag{1}$$

where g_{ij} is the conductivity between i and j, and V_j is the voltage at j when the nth iteration at i is taking place. Equation (1) is iterated until the voltages reach equilibrium, from which the conductivity can be obtained directly.

For directed percolation, relaxation is not adequate unless the correct state is known. As a first approximation, the correct state may be searched for in the relaxation by turning off any diodes which become back biased, and turning on any previously back-biased diodes if they become forward biased at a later stage. Thus the state of the network "floats" during such a calculation. If such abrupt changes in the conductivity of individual diodes are made, however, long-lived oscillations in the iterative evolution of the state of the network can occur. The reason for this oscillatory behavior stems from the potential for negative feedback between the states of certain "critical" diodes in nested Wheatstone bridge configurations (Fig. 2). A change in the state of one of these diodes may lead to a change in the state of the other diode which can then influence the original diode again. These oscillations are an artifact of the calculational procedure, however, as a unique state of the network exists.[8]

To reduce the oscillations, we relax the network more gradually by effectively smoothing out the break in the I-V response curve at $V=0$. This is accomplished by decreasing the conductivity of each negatively biased diode by a certain factor at each iteration. After many iterations, the conductivity of such a bond is negligible; it has been removed from the network gradually. On the other hand, if one of these apparently back-biased diodes becomes forward biased again, its conductivity is gradually increased at each iteration. As long as such a bond remains positively biased, the conductivity continues to be increased until it is restored to its initial value. This procedure is continued until convergence to a final state has occurred. At this point, overrelaxation[9] is used to speed convergence in the final stages of iteration.

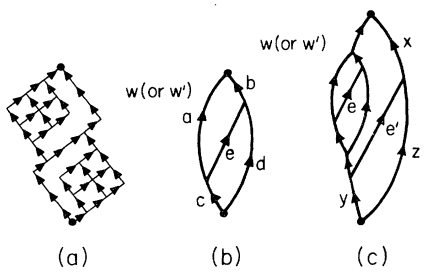

FIG. 2. (a) An example of a Wheatstone bridge on the square lattice in which critical bridge diodes (in series) are back biased, and (b) a schematic of this configuration. (c) Two nested Wheatstone bridges with critical diodes e and e'. The conductivities of the legs of the bridge may be assigned so that if e is forward biased in the inner bridge, the conductivity of the upper-left leg of the outer bridge is w', with $w'z > xy$, so that e' is back biased. On the other hand, if e is back biased, the conductivity of the upper-left leg is w, which may be chosen to satisfy $wz < xy$, so that e' is forward biased.

We have tested this method on judiciously constructed Wheatstone bridges with three levels of nesting. These are convenient test systems because of the feedback between the states of three critical diodes, and because the exact conductivities can be easily found in order to check the relaxation results. Without the smoothing of the diode I-V response, relaxation gives very-long-lived oscillations,[10] while with the smoothing, many fewer oscillations occur before the correct state and conductivity are attained. In actual simulations, only a very small fraction of realizations failed to converge to a final state even after 500 iterations, even though the conductivity had already converged to the correct value.

For obtaining the directed conductivity exponent, t_+, we use finite-size scaling,[11] generalized to account for the anisotropy of the network near the percolation threshold. This is accomplished by writing the following generalized homogeneous functional form for the conductivity, G, in terms of the scaling fields $\Delta p \equiv p_+ - p_{+_c}$, $L_\parallel^{-1}$ and $L_\perp^{-1}$, the inverse of the length scales parallel and perpendicular to the anisotropy axis, respectively:

$$G(\Delta p, L_\parallel^{-1}, L_\perp^{-1}) = \lambda^{a_G} G(\lambda^{a_{\Delta p}} \Delta p, \lambda^{a_\parallel} L_\parallel^{-1}, \lambda^{a_\perp} L_\perp^{-1}) \ . \tag{2}$$

For finite-size scaling, we set $\Delta p = 0$, and $\lambda = L_\parallel^{1/a_\parallel}$. This yields

$$G(\Delta p = 0, L_\parallel^{-1}, L_\perp^{-1}) = L_\parallel^{-t_+/\nu_\parallel} G(0, 1, L_\parallel^{\nu_\perp/\nu_\parallel}/L_\perp) \ , \tag{3}$$

where $t_+ = -a_G/a_{\Delta p}$, $\nu_\parallel = a_\parallel/a_{\Delta p}$, and $\nu_\perp = a_\perp/a_{\Delta p}$. Power-law behavior results only if the argument of the scaling function is constant, i.e., $L_\parallel$ is proportional to $L_\perp^{\nu_\parallel/\nu_\perp}$.

We have obtained data for the conductance $\mathcal{G} = GL_\perp/L_\parallel$ at $p_+ = p_{+_c} = 0.6447$ for three sequences of lattice sizes (Fig. 3), scaled up according to $L_\parallel \sim L_\perp^{\nu_\parallel/\nu_\perp}$, with $\nu_\parallel/\nu_\perp = 1.58$.[5–9] On a double logarithmic scale, we estimate the slope of the conductance data versus $L_\parallel$ to be approximately -0.71 ± 0.05, from which we deduce a directed conductivity exponent of $t_+ = 0.60 \pm 0.10$. This value is quite different from the current estimates of $t \cong 1.30$

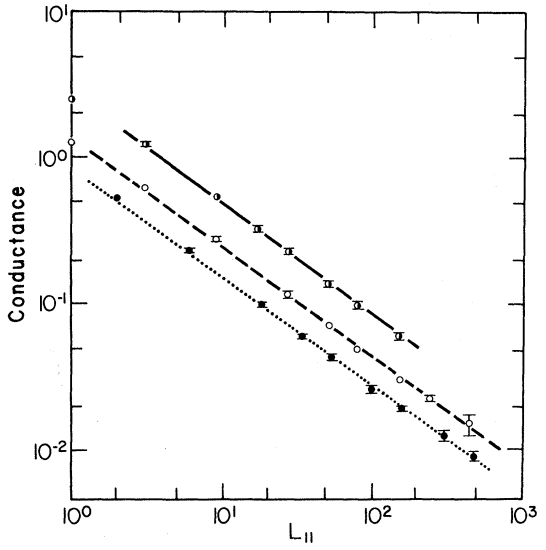

FIG. 3. Double logarithmic plot of the conductance vs $L_{\parallel}$ for three sequences of lattice sizes: (◑), $L_{\perp} \times L_{\parallel} = 2 \times 1 - 48 \times 152$; (○), $1 \times 1 - 48 \times 453$; and (●), $1 \times 2 - 32 \times 478$. Successive points in each sequence are obtained by increasing $L_{\perp}$ by a factor of either 1.5 or 2, and $L_{\parallel}$ by a corresponding factor of $1.5^{1.58} \cong 1.90$ or $2^{1.58} \cong 3.0$.

for the conductivity exponent of the random resistor network,[12] and our result agrees approximately with independent estimates of t_+ by renormalization-group calculations[13] and by analog experiments.[14] Furthermore, the value of t_+ is quite close to the estimated value of the directed percolation backbone exponent,[15] and it would be worthwhile to test whether a general relation holds between these two exponents.

In summary, we have studied the conductivity in directed percolation where the presence of back-biased diodes influences the structure of current-carrying paths. This aspects of the problem has been treated by a numerical relaxation method that simultaneously finds the correct state and voltages in the network. Anisotropic finite-size scaling has been used to extrapolate Monte Carlo data to yield the estimate $t_+ = 0.60 \pm 0.10$ for the directed conductivity exponent on the square lattice.

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