Magnetic phase boundary of simple superconductive micronetworks

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The theory of superconductive networks put forward by de Gennes and Alexander can be formulated in terms of two generalized Kirchhoff laws. The current law is generalized to the complex parts of quantum-mechanical current and the voltage law to the flux linkage. These laws are applied to a superconducting quantum interference device (SQUID), to two superconducting loops connected by a superconducting branch (bola), and a balanced Wheatstone bridge. The normal-superconducting phase boundaries are obtained in the limit that the thickness of the wires is neglected.

In the context of granular superconductors, de Gennes[1] discusses the onset of superconductivity in a ring connected to an arm ("lasso") when all lengths are comparable to the temperature-dependent coherence length $\xi(t=T/T_c)$ and penetration depth $\lambda(t)$. The thickness of the wires is neglected so that volume contributions to the free energy are not taken into account. The essential effect the arm has on the second-order phase-transition boundary is to push it to higher temperatures when compared to that of a ring without an arm. Because the arm does not carry a current it aids superconductivity in the ring. Where ring and arm join, the order parameters and their derivatives[1] satisfy boundary conditions which lead to an equation which corresponds, in a broad sense, to Kirchhoff's current law (KCL). Alexander[2] writes de Gennes's KCL in explicit form and applies it to the "lasso," to a rectangular loop divided symmetrically by a connecting branch and to a square lattice.

It is our purpose to extend the above principle to the case where the micronetwork is connected to an external current source, to state the equivalent of Kirchhoff's voltage law, and to obtain the detailed phase boundary for some simple two-node circuits which include the superconducting quantum interference device (SQUID), the bola,[3] and the balanced Wheatstone bridge. We show the common features shared by all these systems. The latter are the important elements in a flux-locking network.

Consider the Ginzburg-Landau (GL) equations

$$(i\vec{\nabla} + 2\pi\vec{A}/\phi_0)^2\psi - \xi^{-2}\psi + (2mB/\hbar^2)|\psi|^2\psi = 0 \ , \quad (1)$$

$$\vec{J} = -(e\hbar/2m)[\psi^*(i\vec{\nabla} + 2\pi\vec{A}/\phi_0)\psi + \text{c.c.}] \ . \quad (2)$$

We assume that at least one dimension of the network is small enough such that a second-order phase transition occurs in a magnetic field and that all branches in Fig. 1(a) are of the same material. The vectors $\vec{A}$ and $\vec{\nabla} = \hat{l}\partial/\partial l$ in (1) are along the path l.

For a branch ab of length L, (1) without the cubic term has the solution

$$\psi(l) = e^{i\gamma_{al}}[\psi_a \sin(L-l) + \psi_b e^{-i\gamma_{ab}} \sin l]/\sin L \ , \quad (3)$$

where l (normalized by ξ) is the curvilinear coordinate along the branch measured from a and

$$\gamma_{al} = \frac{2\pi}{\phi_0}\int_a^l \vec{A}(l')\cdot d\vec{l}' \ . \quad (4)$$

If the branch ab of length L is made into a closed

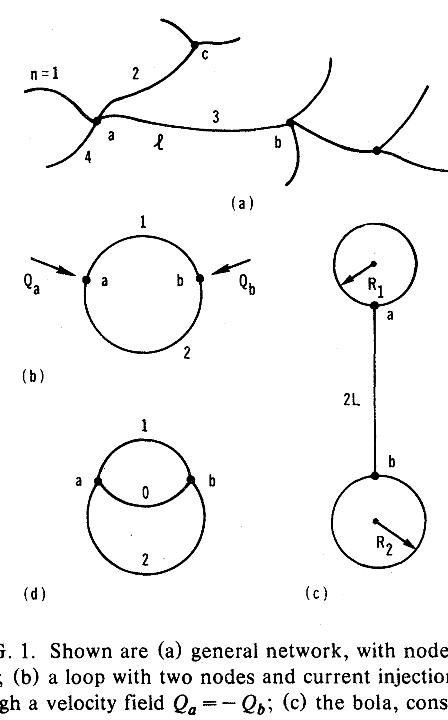

FIG. 1. Shown are (a) general network, with nodes a, b, and c; (b) a loop with two nodes and current injection through a velocity field $Q_a = -Q_b$; (c) the bola, consisting of two "lassos"; (d) the Wheatstone bridge.

loop, then for single valuedness $\psi_b \to \psi_a$ and

$$\gamma_{ab} \to \frac{2\pi}{\phi_0}\oint \vec{A}\cdot d\vec{l}' = \frac{2\pi\phi}{\phi_0} = \gamma \ ,$$

where ϕ is the magnetic flux enclosed by the loop. By use of (3) one can show that the current in a branch element between nodes a and b is

$$J_{ab} = \frac{e\hbar}{m\xi}|\psi_a||\psi_b|\sin(\beta-\alpha-\gamma_{ab})/\sin L \ , \tag{5}$$

where $\psi_a = |\psi_a|e^{i\alpha}$, $\psi_b = |\psi_b|e^{i\beta}$. Equation (5) is similar to the Josephson current through a weak link, where now $1/\sin L$ plays the role of the tunneling probability. If node b is made to approach node a, (5) can be written as

$$\vec{J} = \frac{e\hbar}{m\xi}|\psi_a|^2\vec{Q} \ ,$$

where

$$\vec{Q} = \lim_{L\to 0}(\beta-\alpha-\gamma_{ab})/L = (\vec{\nabla}\alpha - 2\pi\vec{A}/\phi_0)$$

is the gauge-invariant velocity field of the superelectrons. Our KCL requires that at each node the sum of the quantities $\psi^*(i\vec{\nabla}+2\pi\vec{A}/\phi_0)\psi$ belonging to all branches must be zero. Since at each node the ψ's of all connected branches are the same, it follows that the generalized KCL is a condition on the derivatives

$$\sum_n \left(i\frac{\partial}{\partial l} + \frac{2\pi A(l)}{\phi_0}\right)\psi_n(l)\Bigg|_{l=0} = 0 \ , \tag{6}$$

where the sum is over all branches connected to a given node. Using (3) and (6) one obtains[2] at each node

$$\sum_n (-\psi_a \cot L_{an} + \psi_n e^{-i\gamma_{an}}/\sin L_{an}) = 0 \ , \tag{7}$$

where the sum is over all nearest nodes linked to a, L_{an} is the distance along the circuit from node a to next nearest node n, and γ_{an} is (4). Since (3) applies near the second-order phase boundary, the set of linear equations of the form (7), applied to all nodes in a given network, leads to a characteristic determinant which must be zero for a nontrivial solution. This secular equation gives the normal-superconductive phase boundary.

If the network is connected to an external current source at nodes a and b, such that a supercurrent enters the network at a and exits at b, (7) holds for all nodes except a and b. Our generalized KCL gives for node a

$$\psi_a Q_a + i\sum_n (\psi_a \cot L_{an} - \psi_n e^{-i\gamma_{an}}/\sin L_{an}) = 0 \ . \tag{8}$$

$Q_a = -Q_b$ is the velocity field associated with the external current. Overall charge conservation requires that $|\psi_a| = |\psi_b|$.

The number of independent nodal equations is equal to the number of nodes in the network. Applying (7) to a two-nodal network one gets equations of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix}\begin{pmatrix}\psi_a \\ \psi_b\end{pmatrix} = 0 \ . \tag{9}$$

The equivalent to Kirchhoff's voltage law (KVL) in our case states that the sum of the γ_{ab}'s along all branches closing into a loop equals $2\pi\phi/\phi_0$, where ϕ is the magnetic flux threading the loop.

If a supercurrent characterized by a velocity field Q enters a circular loop at node a and exits at b diametrically opposite, the nodal equation [Fig. 1(b) and (8)] for node a is

$$(Q_a + i2\cot\pi R)\psi_a - \frac{i}{\sin\pi R}(e^{-i\gamma_{ab}^{(1)}} + e^{-i\gamma_{ab}^{(2)}})\psi_b = 0 \ , \tag{10}$$

where $\gamma_{ba}^{(1)} = -\gamma_{ab}^{(1)}$, $\gamma_{ba}^{(2)} = -\gamma_{ab}^{(2)}$, and similarly for node b. From the secular equation the phase boundary is then

$$\sin(\gamma/2) = \pm(1-Q^2/4)^{1/2}\sin\pi R \ , \tag{11}$$

with $\gamma = \gamma_{ab}^{(2)} + \gamma_{ba}^{(1)}$. From Eq. (10) we obtain also

$$Q = Q_c \sin\delta \ , \tag{12}$$

where the maximum critical value of Q is

$$Q_c = 2\cos(\gamma/2)/\sin\pi R \tag{13}$$

and $\delta = \alpha - \beta + (\gamma_{ab}^{(1)} + \gamma_{ab}^{(2)})/2$ is the phase difference imposed by the external current source. These relations are those of a SQUID.[4] The critical current is defined in terms of a critical superfluid velocity. A detailed investigation of (11) for $R < \xi(0)/2$ shows that the SQUID alternates between the superconducting and normal states for $T \geq 0$ when H is swept even for $Q = 0$. When $Q \neq 0$ these alternating regions may exist also for $R > \xi(0)/2$.

A plane two-nodal bola[3] is shown in Fig. 1(c) with nodes a and b. The coefficients of (9) are

$$a_{11} = -2\cot 2\pi R_1 - \cot 2L + 2\cos\gamma_1/\sin 2\pi R_1 \ ,$$

$$a_{22} = -2\cot 2\pi R_2 - \cot 2L + 2\cos\gamma_2/\sin 2\pi R_2 \ ,$$

$$a_{12} = a_{21}^* = e^{-i\gamma_{ab}}/\sin 2L \ ,$$

where $\gamma_i = 2\pi\phi_i/\phi_0 = 2\pi(\pi R_i^2 H)/\phi_0$.

For a symmetric bola ($R_1 = R_2 = R$) one obtains

$$\cos\gamma = \cos(2\pi R) \mp \tfrac{1}{2}\sin(2\pi R)\times\begin{cases}\tan(2m\pi R) & \text{(14a)}\\ \cot(2m\pi R) & \text{(14b)}\end{cases}$$

with $m = L/2\pi R$ and L and R normalized by ξ. Solutions for the phase boundary are shown in Fig. 2. For $L = 0$ the solution (14a) is that of an isolated ring. When $m > 0$ the second solution is of higher energy than the first and its phase boundary is probably of no physical significance. One finds that a branch of length $2L$ has the effect of lowering the energy when compared to that of a single loop. The order parameter along the connecting branch is

$$\psi(l) = \psi_a e^{i\gamma_a l} \cos(L-l)/\cos L$$

for the symmetric solution (14a) and

$$\psi(l) = e^{i\gamma_a l} \psi_a \sin(L-l)/\sin L$$

for (14b) (antisymmetric). The solution for the "lasso"[1] is (14a).

The balanced Wheatstone bridge is shown in Fig. 1(d). We impose the condition of zero current in the central branch which leads to

$$\sin(2\pi\phi_1/\phi_0)/\sin(2\pi\phi_2/\phi_0) = \sin L_1/\sin L_2 \ .$$

A circular loop with $L_0 = 2R$ has solutions ($\gamma = 2\pi\phi_1/\phi_0$),

$$\cos\gamma = \pm\cos\pi R - \tfrac{1}{2}\sin\pi R \times \begin{cases} \tan R & (15a) \\ \cot R \ . & (15b) \end{cases}$$

Equation (15a) is similar to (14a), the symmetric

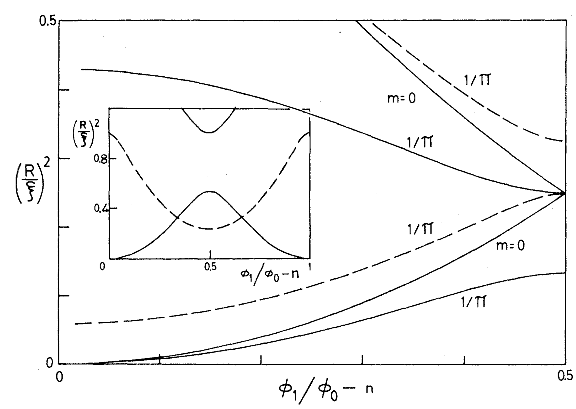

FIG. 2. Shown are the phase boundaries for a bola with $L = 0$ ($m = 0$), and $L = 2R$ ($m = 1/\pi$). The spatially symmetric solution corresponds to the solid line, and the antisymmetric to the broken line. The smallest values of R/ξ correspond to the lowest-energy state; ϕ_1 is the flux through one loop. The solution is periodic and only half of the period is shown. If $R/\xi(0)$ is smaller than the value of the phase boundary at $\phi_1 = 0.5\phi_0$, normal regions appear at absolute zero when the magnetic field is increased. The insert shows the solution for the balanced symmetric Wheatstone bridge over a whole period. The antisymmetric solution has lower energy near $\phi_1 = 0.5\phi_0$, where $2\phi_1$ is the flux linked by paths (1) and (2) shown in Fig. 1(d). The fluxoid quantum number is n.

solution of the bola, but with R replaced by $R/2$ and $m = 1/\pi$. $\psi(l)$ along the branch L_0 resembles that of a bola along the branch $2L$ except that L is replaced by R. Along the circumference

$$\psi(l) = e^{i\gamma_a l}\psi_a[\sin(\pi R - l) \pm e^{-i\gamma}\sin l]/\sin\pi R \ , \quad (16)$$

where γ from (15) is to be substituted. The insert in Fig. 2 shows solutions (15a) and (15b). Near $(\phi_1/\phi_0 - n) \approx 0$ the spatially even solutions has a lower energy, while near $\frac{1}{2}$ the odd solution has a lower energy. For the odd solution $|\psi|^2$ is zero in the center of L_0 and has a maximum on the circumference at $l = \pm\pi R/2$, while for the even solution $|\psi|^2$ has a maximum in the center of L_0 and a minimum at $l = \pm\pi R/2$.

The finite volume of the superconductor will shift the *NS* phase boundary to lower temperatures in an applied magnetic field.[5,6] A long circular cylinder of outside radius R and wall thickness $d = R - R_0$ has a second-order phase transition boundary[5]

$$(1+s)\pi R^2 H/2\phi_0 = n \pm [(1+s)R^2/2\xi^2 - n^2u^2(s)]^{1/2} \ , \quad (17)$$

where $s = (R_0/R)^2$ and the function $u^2(s)$ approaches in the limit that $\epsilon = d/R \ll 1$ the value $\epsilon^2/3$. Assuming that all phase transition boundaries in the H-T plane are of second order (no supercooling), the maximum fluxoid quantum number is[5] $n_{max} < \sqrt{3}R^2/[\xi(0)d]$. For example, when $R/\xi(0) = \frac{1}{2}$ and $d/R = \frac{3}{10}$ then $n_{max} = 2$. Thus solutions exist only up to $n = 2$. It can be seen from (17) for $d/R \to 0$ that there are no solutions for values of the flux in the vicinity of $\phi = (\frac{1}{2})\phi_0$, $(\frac{3}{2})\phi_0$, etc., even at $T = 0$ K if $R/\xi(0) < \frac{1}{2}$. Thus a small, very thin-walled cylinder will alternate between the superconducting and normal states as the magnetic field is increased. These considerations apply to wires and more complicated networks at the phase boundary.

We have shown how the generalized Kirchhoff laws can be applied to simple networks and have calculated the phase boundaries for some two-nodal systems. When the solutions can be classifed as symmetric or antisymmetric, the former has usually the lowest energy. The balanced Wheatstone bridge is an exception.

Our analysis shows that the one-dimensional branches introduced by de Gennes[1] and Alexander[2] can be considered to be weak superconducting links between nodes in the network. The generalization of the nodal equations to cases where external currents are fed into the network allowed us to formulate relations relevant to quantum interference devices in a very simple way. We feel that the present approach to superconductive networks can be fruitfully applied to more complex geometries.

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[1]P. G. de Gennes, C. R. Ser. II 292, 279 (1981).

[2]S. Alexander (unpublished).

[3]The bola is a weapon consisting of two (or more) spherical stones, attached to the end of leather cords. The above circuit is a two-dimensional version of it.

[4]For example, A. C. Rose-Innes and E. H. Rhoderick, *Introduction to Superconductivity* (Pergamon, Oxford, England, 1978), 2nd. ed.

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