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Theory of crystallization waves in ^{4}He at finite frequency

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We determine the spectrum of crystallization waves and Rayleigh waves at the interface between superfluid and solid ^{4}He, taking into account the effect of elasticity and the finite-growth coefficient of the crystal. For $k \gtrsim 10^6$ cm^{-1} the coupling between these two modes significantly changes the entire spectrum.

I. INTRODUCTION

Andreev and Parshin,[1] in investigating theoretical properties of the interface between superfluid and solid ^{4}He, predicted periodic crystallization-melting waves, in which the flow of the superfluid supplies the inertia and the surface tension provides the restoring force. They conjectured that the surface of the crystal is atomically rough because of the large zero-point motion of the kinks and steps and therefore crystallization or melting occurs without dissipation, at very low temperatures. These crystallization waves were observed experimentally by Keshishev, Parshin, and Babkin[2] at wavelengths comparable to or one order of magnitude smaller than the capillary length ($10 \lesssim k \lesssim 200$ cm^{-1}) on the rounded parts of the crystals.[3] In this wavelength region the solid and liquid can be considered incompressible, the case considered by Andreev and Parshin, as the frequency of the phonons is much greater than that of the crystallization waves.

A second type of wave that can propagate along a solid-liquid interface is an elastic Rayleigh wave, a sound wave localized in the surface, with no mass transfer between the solid and liquid, in which the elasticity of the solid provides the restoring force. In the case of helium the fast melting and crystallization causes the Rayleigh waves to behave as if the solid were in contact with vacuum.[4] Our purpose in this paper is to study the effect of elasticity on shorter-wavelength crystallization waves, the coupling of the crystallization and Rayleigh modes at finite frequency, the damping of the modes, and their relation to ordinary Rayleigh waves when the rate of crystal growth becomes small. As we shall show the coupling between the two boundary modes becomes predominant at wavelength $k \gtrsim 10^6$ cm^{-1} (which is longer than those of the ripplons at a free superfluid ^{4}He surface, because the frequency of a crystallization wave at given k is much greater than that of a ripplon). Owing to mixing of the crystallization and Rayleigh waves, the spectrum of crystallization waves for large k remains at lower frequency than predicted by the simple Andreev-Parshin calculation.

II. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

To discover the effect of elasticity on the crystallization waves and melting on the elastic Rayleigh waves we make the following simplifying assumptions: We generally treat the ^{4}He crystal as an isotropic continuous elastic body, and take the surface tension of the interface as a constant independent of curvature of the boundary and strain of the solid. We also neglect the normal component of liquid ^{4}He, and thus limit our discussion to low temperatures.

If we consider only linearized equations, the velocity and the pressure of the liquid can be described in terms of a velocity potential ψ by

$$\vec{v}_2 = \vec{\nabla}\psi, \quad \delta P_2 = -\rho_2 \dot{\psi} \tag{1}$$

(we denote quantities for the solid and the liquid by subscripts 1 and 2, respectively). The potential ψ obeys the wave equation

$$\nabla^2 \psi - \frac{1}{c_2^2}\ddot{\psi} = 0 \; . \tag{2}$$

The displacement $\vec{u}(\vec{r})$ of the solid from equilibrium can be decomposed into longitudinal and transverse components ($\vec{\nabla}\times\vec{u}_l = 0$ and $\vec{\nabla}\cdot\vec{u}_t = 0$) that satisfy the following wave equations:

$$\nabla^2 \vec{u}_l - \frac{1}{c_l^2}\ddot{\vec{u}}_l = 0, \quad \nabla^2 \vec{u}_t - \frac{1}{c_t^2}\ddot{\vec{u}}_t = 0 \; . \tag{3}$$

At the interface we have three boundary condi-

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tions. The first is particle conservation across the boundary

$$\rho_1(\dot{u}_\perp - \dot{\zeta}) = \rho_2(v_{2\perp} - \dot{\zeta}) \equiv j \ , \tag{4}$$

where $\dot{\zeta}$ is the velocity of the surface and j is the mass current across the interface, normal to the surface. The second is the usual condition of mechanical equilibrium across the interface

$$\sigma_{ik} n_k + \left[P_2 + \alpha \left[\frac{1}{R'} + \frac{1}{R''} \right] \right] n_i = 0 \ , \tag{5}$$

where σ_{ik} is the stress tensor of the solid at the boundary, n_i the unit vector normal to the surface, α the surface tension, and R' and R'' are the principal radii of curvature of the boundary. The condition for chemical equilibrium at $T=0$ is[5]

$$\frac{\epsilon_1 + P_2 + \alpha \left[\frac{1}{R'} + \frac{1}{R''} \right]}{\rho_1} = \mu_2 \ , \tag{6}$$

where ϵ_1 is the energy density of the stressed solid and μ_2 is the chemical potential per unit mass. Here we have assumed that the newly formed solid has the same strain as the old one (going beyond the isotropic continuum approximation). If the solid is unstressed the quantity on the left-hand side of (6) reduces, from the mechanical equilibrium condition, to $(\epsilon_1 + P_1)/\rho_1$ which is the usual chemical potential μ_1. The derivation of (6) is given, for those not familiar with the details, in the Appendix. If we neglect dissipation in crystal growth at the interface, then the chemical equilibrium condition (6) is the third boundary condition. If the solid is not stressed and there is dissipation in the growth at the interface, as in Ref. 1 (see also Ref. 6), we can assume that the rate of crystal growth is proportional to the difference of the chemical potentials,

$$j = K(\mu_1 - \mu_2) \ , \tag{7}$$

where K is the crystal-growth coefficient. We may generalize this assumption to the case of a stressed solid, by replacing (6) by

$$\frac{\epsilon_1 + P_2 + \alpha \left[\frac{1}{R'} + \frac{1}{R''} \right]}{\rho_1} - \mu_2 = \frac{j}{K} \ . \tag{8}$$

In the absence of dissipation $K \to \infty$ and one recovers (6).

In equilibrium with a flat interface, (5) and (6) reduce to

$$\sigma_{ik}^0 + P_2^0 \delta_{ik} = 0 \ , \tag{9}$$

$$\epsilon_1^0 + P_2^0 - \mu_2^0 \rho_1^0 = 0 \ . \tag{10}$$

Where there are small changes, from (5) and (9) we have

$$\delta\sigma_{ik} n_k + \left[\delta P_2 + \alpha \left[\frac{1}{R'} + \frac{1}{R''} \right] \right] n_i = 0 \ . \tag{11}$$

In lowest order the change of energy density is

$$\delta\epsilon_1 = -\epsilon_1^0 u_{ii} + \sigma_{ik}^0 u_{ik} \ , \tag{12}$$

where

$$u_{ik} = \frac{1}{2} \left[\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right]$$

is the strain tensor. From (8) and (10), with the help of (9) and (12), we obtain

$$\left[1 - \frac{\rho_1^0}{\rho_2^0} \right] \delta P_2 + \alpha \left[\frac{1}{R'} + \frac{1}{R''} \right] = \frac{j}{K} \ , \tag{13}$$

which gives the pressure change in the liquid in terms of the curvature of the surface, and mass flow across the interface.

Let us assume the interface to be located in the x-y plane at $z=0$, and that the solid is in $z<0$, and consider boundary waves propagating at the interface in the x direction. Then one of the radii of curvature (which we take to be R'') is infinite and the other is related to ζ by

$$\frac{1}{R'} = -\frac{\partial^2 \zeta}{\partial x^2} \ . \tag{14}$$

The conditions (11) and (13) become

$$\delta\sigma_{zz} + \delta P_2 - \alpha \frac{\partial^2 \zeta}{\partial x^2} = 0 \ , \tag{15}$$

$$\delta\sigma_{zx} = \delta\sigma_{zy} = 0 \ , \tag{16}$$

and

$$\left[1 - \frac{\rho_1}{\rho_2} \right] \delta P_2 - \alpha \frac{\partial^2 \zeta}{\partial x^2} = \frac{\rho_1}{K} j \tag{17}$$

(hereafter we omit the symbols indicating the equilibrium quantities). We look for waves localized at the interface and put

$$\begin{aligned}
&\psi = \psi_0 \exp(-\kappa_2 z + ikx - i\omega t) \quad (z > 0) \ , \\
&\zeta = \zeta_0 \exp(ikx - i\omega t) \ , \\
&\vec{u}_l = \vec{u}_{l0} \exp(\kappa_l z + ikx - i\omega t) \quad (z < 0) \ , \\
&\vec{u}_t = \vec{u}_{t0} \exp(\kappa_t z + ikt - i\omega t) \quad (z < 0) \ ,
\end{aligned} \tag{18}$$

where the κ_i's which are inverses of the penetration depths, are defined by

$$\kappa_i^2 = k^2 - \frac{\omega^2}{c_i^2} \tag{19}$$

to satisfy the wave equations (2) and (3). Combining (4), (15), (16), and (17), and using the familiar stress-strain relation

$$\delta\sigma_{ik} = \rho_1(c_l^2 - \tfrac{4}{3}c_t^2)u_{ll}\delta_{ik} + 2\rho_1 c_t^2(u_{ik} - \tfrac{1}{3}\delta_{ik}u_{ll}) \,, \tag{20}$$

we obtain the dispersion relation of the boundary waves:

$$F(k,\Omega) \equiv F_R(k,\Omega) + iAF_I(k,\Omega) = 0 \,, \tag{21}$$

where

$$F_R(k,\Omega) \equiv \Omega^2(1-R)^2 f(k,\Omega) + k^2\tilde{\alpha}[\kappa_l\Omega^4 - R\kappa_2 f(k,\Omega)] \,, \tag{22}$$

$$F_I(k,\Omega) = \Omega[R\kappa_l\Omega^4 - \kappa_2 f(k,\Omega)] - \Omega^3\kappa_l\kappa_2 k^2\tilde{\alpha} \,, \tag{23}$$

and

$$f(k,\Omega) \equiv 4k^2\kappa_t\kappa_l - (k^2+\kappa_t^2)^2 \,, \tag{24}$$

$$\Omega \equiv \frac{\omega}{c_t},\quad R \equiv \frac{\rho_2}{\rho_1},\quad \tilde{\alpha} \equiv \frac{\alpha}{\rho_1 c_t^2},\quad A \equiv \frac{\rho_2}{Kc_t} . \tag{25}$$

III. CRYSTALLIZATION WAVES AND RAYLEIGH WAVES

In this section we analyze the solutions of (21) neglecting for the moment the dissipation term which varies as K^{-1}. First we look for solutions to

$$F_R(k,\Omega) = 0 \tag{26}$$

for small Ω ($\ll k$). Since $f(k,\Omega) \sim \Omega^2 k^2 \neq 0$ we can neglect the $\kappa_l\Omega^4$ in (22), and let $\kappa_i \to k$. Then we find

$$\Omega_1^2 = \frac{R}{(1-R)^2}\tilde{\alpha}k^3 . \tag{27}$$

This is the crystallization-wave solution for small[1] k [$\ll (1-R)^2/\tilde{\alpha}R$] with gravity neglected. More familiarly (27) is

$$\omega_1^2 = \frac{\rho_2}{(\rho_1-\rho_2)^2}\alpha k^3 . \tag{28}$$

The first correction to (28) is obtained by expanding (22) in powers of Ω^2/k^2, which yields

$$\omega^2 = \frac{\rho_2}{(\rho_1-\rho_2)^2}\alpha k^3 \times \left[1 - \frac{1}{2}\left\{S_2 + \frac{1}{1-S_l}\frac{\rho_1}{\rho_2}\right]\frac{\rho_2}{(\rho_1-\rho_2)^2}\frac{\alpha}{c_t^2}k\right] , \tag{29}$$

where $S_2 \equiv c_t^2/c_2^2$ and $S_l \equiv c_t^2/c_l^2$; in this approximation the effect of elasticity on the crystallization wave is to reduce the frequency by a term of relative order $(\omega/c_t k)^2$.

Equation (26) has a second solution, Ω_2, comparable to k. In the limit of small k the effect of the surface tension, the second term of (22), is negligible and Ω_2 satisfies

$$f(k,\Omega_2) = 0 \,, \tag{30}$$

which is just the equation describing Rayleigh waves at a free surface.[4] This becomes perhaps more evident if we neglect the surface tension, for then (17) (with $K=\infty$) tells us $\delta P_2 = 0$, and therefore $\delta\sigma_{zz} = 0$, which is the boundary condition for the free surface. At finite k the surface tension works as a restoring force and pushes up the frequency. As k increases, the frequency finally exceeds $c_t k$, the penetration depth becomes infinite, as is clear from (19), and the wave is no longer localized at the interface. This happens at the critical wave number $k = k_c$, determined by $F_R(k_c,k_c) = 0$:

$$k_c = \frac{c_t^2(\rho_1-\rho_2)^2}{\alpha(\rho_1\sqrt{1-S_l} + \rho_2\sqrt{1-S_2}} . \tag{31}$$

This is about half of the wave number at which the transverse sound and lowest-order crystallization wave (28) branches cross. In the limit of large k, the surface tension term of (22) dominates and the solution becomes linear in k with a constant velocity, determined by

$$\kappa_l\Omega_3^4 - R\kappa_2 f(k,\Omega_3) = 0 . \tag{32}$$

When the crystal does not melt, that is, $A \to \infty$ ($K=0$), (21) reduces to

$$F_I(k,\Omega) = 0 \,, \tag{33}$$

which is the equation for $j=0$. The solution is a Rayleigh wave without melting, and with the effect of surface tension. At small k the surface tension can be neglected and we have

$$R\kappa_l\Omega_4^4 - \kappa_2 f(k,\Omega_4) = 0 . \tag{34}$$

Observe that $c_t\Omega_4/k$ is simply the velocity of the usual Rayleigh wave of a solid in contact with a liquid. The surface tension also increases Ω, and at

k_c', which is given by $F_I(k_c',k_c')=0$, the frequency exceeds $c_t k$:

$$k_c' = \frac{c_t^2(\rho_2\sqrt{1-S_l}+\rho_1\sqrt{1-S_2})}{\alpha\sqrt{1-S_l}\sqrt{1-S_2}} \ . \tag{35}$$

This wave number, $\sim 10^9$ cm^{-1}, is more than $1/(1-R)^2$ times larger than (31) and is far above our concern.

Numerical results for the entire spectrum are shown in Fig. 1. The parameters we use here are $c_l=4.8\times10^4$ cm sec^{-1}, $c_t=2.3\times10^4$ cm sec^{-1}, $c_2=3.7\times10^4$ cm sec^{-1}, $\alpha=0.16$ erg cm^2 ($\tilde{\alpha}=1.6\times10^{-9}$ cm), and $R=0.91$.

The two branches corresponding to crystallization and Rayleigh waves are shown as solid lines. For comparison we show the crystallization-wave spectrum using (27) as a thin solid line, and the Rayleigh-wave spectrum with no melting [Eq. (33)] as the dashed line. At higher k these two modes become mixed, with the lower (crystallization branch) being significantly reduced in frequency for $k \gtrsim 10^6$ cm^{-1}, and the Rayleigh-wave branch enhanced in frequency for all k.

IV. FINITE-GROWTH COEFFICIENT AND DAMPING OF THE WAVES

So far we have dealt with the case of nondissipative crystal growth and melting. In general, at least at finite temperatures, this is not the case.[7] If $K^{-1}>0$ the dissipative term produces a damping of the modes which we now examine. When K is large enough, and hence A small, the imaginary part of Ω is given by

$$\mathrm{Im}\Omega = -\frac{F_I(k,\Omega_0)}{F_R'(k,\Omega_0)} \ , \tag{36}$$

where Ω_0 is the solution of $F_R(k,\Omega_0)=0$. For the crystallization wave at small k we find the same result as Andreev and Parshin,[1]

$$\mathrm{Im}\Omega = -A\frac{1}{2(1-R)^2}k \ . \tag{37}$$

For the Rayleigh-wave branch at small k,

$$\mathrm{Im}\Omega = -A\frac{1}{4}\frac{R}{(1-R)^2}\frac{\sqrt{1-S_l\lambda^2}}{1-\lambda^2+\left[1-\dfrac{\sqrt{1-S_l\lambda^2}}{\sqrt{1-\lambda^2}}-S_l\dfrac{\sqrt{1-\lambda^2}}{\sqrt{1-S_l\lambda^2}}\right]}\lambda^2 k \ , \tag{38}$$

where $\lambda\equiv\Omega_2/k$.

The dependence of the real and imaginary parts on A is shown in Fig. 2 for several k. As A becomes larger the real part of the solution moves toward the solution for a Rayleigh wave without melting. This happens when the imaginary part (38) is comparable to the difference of the solutions of (30) and (34). If we take the values (see Fig. 1) $\Omega_2/k=0.94$ and $\Omega_4/k=0.79$ we expect the crossover to occur at

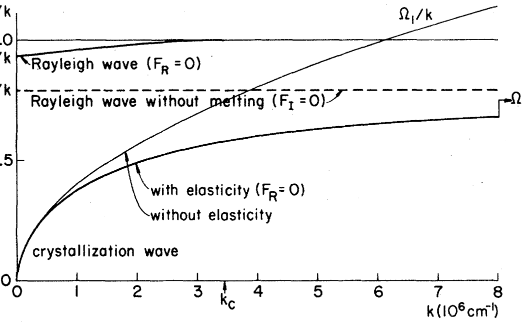

FIG. 1. Spectrum of crystallization and Rayleigh waves of ^{4}He, with elasticity and melting, neglecting dissipation. The value $\Omega/k=1.0$ corresponds to transverse sound. The solid lines show crystallization and the Rayleigh waves with perfect melting, the solutions of (26), and the dashed-line Rayleigh waves without melting, (33). The thin solid line (Ω_1/k) is Andreev-Parshin's spectrum given by (27). The limiting velocities divided by c_t [see (30), (32), and (34)] are also indicated.

$$A \sim 0.01 \ . \tag{39}$$

At long wavelength ($k \lesssim 2\times10^6$ cm^{-1}) on the crystallization branch

$$\mathrm{Im}\Omega/\mathrm{Re}\Omega \propto k^{-1/2} \ , \tag{40}$$

so that as k decreases the mode eventually becomes critically damped. This occurs when

$$A \gtrsim 2(1-R)^2\frac{\Omega_1}{k} = 2R^{1/2}(1-R)\tilde{\alpha}^{1/2}k^{1/2} \ . \tag{41}$$

The shorter the wavelength, the more well defined the mode is.[8] Nevertheless we note from (40) that the mean free path of a crystallization wave is $\propto \mathrm{Re}\Omega/k\mathrm{Im}\Omega \propto k^{-1/2}$, so that the longer-wavelength modes propagate further. At large k the damping falls off, and this branch moves upward, becoming the Rayleigh wave without melting. As we can easily see from (41) or Fig. 2, the crystallization wave is very sensitive to the crystal-growth coefficient especially in the small-k region.

V. DISCUSSION

As seen from Fig. 1 the coupling between crystallization waves and Rayleigh waves has a significant effect on the spectrum at $k \gtrsim 10^6$ cm^1. Surface tension raises the Rayleigh-wave frequency; from (31) and (35) we see that this effect is much more significant for the fast melting case ($K=\infty$). We may understand this by noting that since melting produces large amplitude displacement of the boundary, the resulting effect of surface tension is much greater in this case. To calculate the relative amplitudes of solid $\delta\zeta_1$, liquid $\delta\zeta_2$, and the boundary displacement $\delta\zeta$ perpendicular to the surface, we use (4) and (17) to obtain for $K=\infty$,

$$\frac{\delta\zeta_1}{\delta\zeta}=\frac{\rho_1-\rho_2}{\rho_1}-\frac{k^2\kappa_2\alpha\rho_2}{\omega^2\rho_1(\rho_1-\rho_2)}\ , \tag{42}$$

$$\frac{\delta\zeta_2}{\delta\zeta}=-\frac{k^2\kappa_2\alpha}{\omega^2(\rho_1-\rho_2)}\ . \tag{43}$$

Comparing with the result $\delta\zeta_1=\delta\zeta_2=\delta\zeta$ for $K=0$, we see that the boundary displacement for a given displacement in the solid $\delta\zeta/\delta\zeta_1$ is much larger for $K=\infty$, and hence so is the ratio of the surface tension to elastic forces.

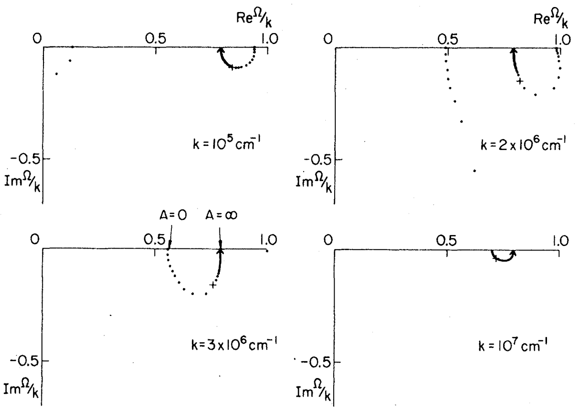

FIG. 2. Reduced frequencies in complex plane, solutions of (21), for several wave numbers, as a function of the inverse of dimensionless crystal-growth coefficient A. From point to point along the curves, A is changed by 0.001, and the points for $A=0.01$ are indicated by crosses. The ends of the arrows are for $A\to\infty$, that is, the Rayleigh waves without melting. The sets of points at lower ReΩ/k for $k=10^5$ and 2×10^6 cm^{-1} show the complex frequencies of the crystallization waves at these wave numbers.

For shorter wavelengths, comparable to the lattice constant, the effect of the lattice becomes important[9] and our continuum model breaks down. We also have to take into account several effects such as[10] the curvature dependence of the surface tension and the actual phonon spectrum at very short wavelengths.

According to the experiment of Keshishev *et al.*[2] the crystal-growth coefficient at relatively high temperatures, which is attributed to rotons, is $1/mK_{\mathrm{AP}}=3.4\times10^5e^{-7.8/T}$ cm sec^{-1}, which corresponds to

$$A\approx13\times e^{-7.8/T}\ , \tag{44}$$

where m is mass of a He atom and K_{AP} is the crystal-growth coefficient as used in Refs. 1 and 2, related to our current definition of K (and that of Ref. 6) by $K=m\rho_1K_{\mathrm{AP}}$. The value given by (44) is about 0.005 at $T=1$ K. As we recall from (39) the Rayleigh wave, for $A\lesssim0.01$, propagates with melting; we expect this to continue up to rather high temperatures. However, with A given by (44) the long-wavelength crystallization modes will be highly damped at higher temperature.

Andreev and Parshin[1] pointed out that crystallization waves (CW) would produce a temperature dependence of the surface tension proportional to $T^{7/3}$. Quantitatively this effect is, from (28),

$$\alpha_{\mathrm{CW}}=-\frac{1}{4\pi}\Gamma\left[\frac{7}{3}\right]\zeta\left[\frac{7}{3}\right]\left[\frac{(\rho_1-\rho_2)^2}{\rho_2\alpha_{T=0}}\right]^{3/2}\frac{k_B^{7/3}}{\hbar^{4/3}}T^{7/3}$$
$$\approx-6\times10^{-4}T^{7/3}\ , \tag{45}$$

in units of erg cm^{-2}, with T in kelvin. The Rayleigh waves (RW) also contribute to the surface tension an amount

$$\alpha_{\mathrm{RW}}=-\frac{\zeta(3)}{2\pi}\frac{1}{C_R^2}\frac{k_B^2}{\hbar^2}T^3\approx-9\times10^{-4}\ T^3\ , \tag{46}$$

in units of erg cm^{-2} (using $\lambda=0.94$), and the reflection of sound at the interface has a similar effect.[10] However, as seen from Fig. 1, the spectrum of the crystallization wave deviates from $\omega\propto T^{3/2}$ considerably for $k\gtrsim10^6$ cm^{-1}. On the one hand the decrease in the spectrum will lead to a more rapid increase in the magnitude of α_{CW} with T than given by (45) ($\sim50\%$ at $T\gtrsim0.1$ K). Also this contribution to the surface tension will be modified by damping of the crystallization mode. However the net effect is probably too small to be observed practically for $T \lesssim 1$ K.

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APPENDIX

We give here a brief derivation of the chemical equilibrium condition of a stressed solid in contact with a liquid [Eq. (6)], first given by Gibbs.[5] Let us consider a stressed solid in a liquid reservoir whose pressure is P_2 and chemical potential μ_2. Suppose that the solid grows or melts by an infinitesimal amount, and that the newly formed solid has the same strain as the adjacent old one when the solid grows. The change of energy of the solid ΔE_1 is related to the energy density of the distorted solid at the interface by

$$\Delta E_1 = \int d^2f\, \delta\zeta(\vec{r})\epsilon_1(\vec{r}) \ , \qquad \text{(A1)}$$

where the integration is over the interface and $\delta\zeta$ is the displacement of the surface due to growth or melting. Considering the liquid reservoir to be very large and P_2 and μ_2 constant we can express the energy change of the liquid as

$$\Delta E_2 = -P_2\Delta V_2 + \mu_2\Delta M_2 \ , \qquad \text{(A2)}$$

where ΔV_2 and ΔM_2 are the change of volume and mass, respectively. In addition we must consider the energy of the interface. We assume that the interfacial energy E_s is proportional to the area of the surface and independent of stress or curvature of the surface. Its change is given by

$$\Delta E_s = \int d^2f\, \delta\zeta(\vec{r}) \left[\frac{1}{R'(\vec{r})} + \frac{1}{R''(\vec{r})} \right] \alpha \ , \qquad \text{(A3)}$$

where α is the surface tension. The total volume and the total mass of the system are conserved in growth or melting:

$$\int d^2f\, \delta\zeta(\vec{r}) + \Delta V_2 = 0 \ , \qquad \text{(A4)}$$

$$\int d^2f\, \delta\zeta(\vec{r})\rho_1(\vec{r}) + \Delta M_2 = 0 \ . \qquad \text{(A5)}$$

With the help of (A4) and (A5) we obtain the change of total energy,

$$\begin{aligned} \Delta E &= \Delta E_1 + \Delta E_2 + \Delta E_s \\ &= \int d^2f\, \delta\zeta(\vec{r}) \left[\epsilon_1(\vec{r}) + P_2 + \alpha \left[\frac{1}{R'(\vec{r})} + \frac{1}{R''(\vec{r})} \right] - \mu_2\rho_1(\vec{r}) \right] \ . \end{aligned} \qquad \text{(A6)}$$

When the solid and the liquid are in equilibrium, ΔE is zero for arbitrary infinitesimal change of $\delta\zeta(\vec{r})$. Therefore the quantity in brackets is identically zero at any point of the interface,

$$\epsilon_1 + P_2 + \alpha \left[\frac{1}{R} + \frac{1}{R''} \right] - \mu_2\rho_1 = 0 \ . \qquad \text{(A7)}$$

Note that when $\sigma_{ik} \propto \delta_{ik}$, we can define P_1 as

$$P_1 = 1/3\sigma_{ii} \qquad \text{(A8)}$$

and

$$\mu_1 = \frac{\epsilon_1 + P_1}{\rho_1} \ . \qquad \text{(A9)}$$

Then, with the help of (5), (A7) reduces to

$$\mu_1 = \mu_2 \ . \qquad \text{(A10)}$$

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to flat faces because, in general, Eq. (7) is not valid for those faces.

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[7] The value of the observed crystal-growth coefficient extrapolated to $T=0$ does not diverge (Ref. 2) and the crystallization is still dissipative.

[8] However, this may not be valid if K varies with frequency. Recent measurements [T. E. Huber and H. J. Maris, Phys. Rev. Lett. 47, 1907 (1981)] of Kapitza resistance between solid and liquid support the assumption that the crystallization-melting process is rapid enough to maintain phase equilibrium up to rather high frequency, and phonons are transmitted by the surface tension (Ref. 4).

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