PHYSICAL REVIEW B VOLUME 26, NUMBER 9 1 NOVEMBER 1982

Effect of branched loops on the diamagnetism of disordered superconductors

Joseph P. Straley and P. B. Visscher
Department of Physics and Astronomy, University of Alabama, University, Alabama 35486
(Received 14 June 1982)

We discuss a model of a disordered superconducting film consisting of small loops with dead-end arms. de Gennes showed that the arms markedly increase the field at which the loop is driven normal. Our numerical solution of the Ginzburg-Landau equation shows the enhancement of the magnetic moment of the loop to be less remarkable.

I. INTRODUCTION

Disordered superconducting materials have a number of interesting and potentially useful features.[1] One idealization that has been studied is the network of thin superconducting wires. de Gennes[2] has noted the importance of the topology of the network to its physical properties: The diamagnetism is primarily due to currents flowing in closed loops. In a second publication[3] he has noted that the dead-end parts of the network do have a role even though they cannot carry current, because they will tend to prevent the closed loops from being driven normal. Specifically, he treats the case of the circular loop of radius R with a dead-end arm of length L (Fig. 1, inset), and shows that the critical flux at which the diamagnetism of the loop vanishes is given by

$$\cos(2\pi\Phi_c/\Phi_0)=\cos(2\pi R/\xi)-\tfrac{1}{2}\sin(2\pi R/\xi)\tan(L/\xi)\,, \quad (1.1)$$

where $\Phi_0=hc/2e$ is the flux quantum and ξ is the coherence length. This relationship is represented by Fig. 1. The response is periodic in Φ with period Φ_0. Specific conclusions that might be drawn from Eq. (1.1) are that diamagnetism never vanishes for loops of radius greater than $\xi/2$, or for loops with pendant arms longer than $(\pi/2)\xi$.

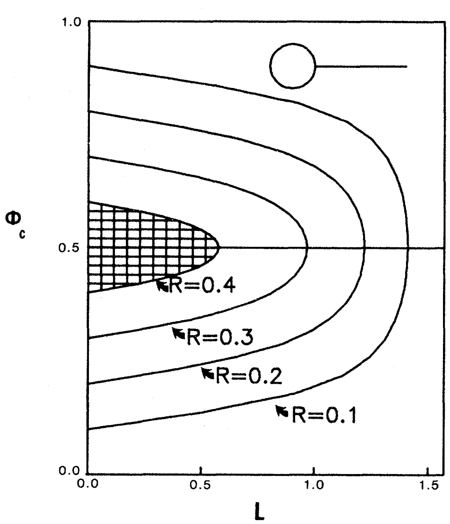

FIG. 1. Critical flux for a loop with a dead-end arm. The lines give the critical flux (measured in units of Φ_0) as a function of the length of the arm (measured in units of ξ) for various values of R (also measured in units of ξ). A loop with radius 0.4ξ is driven normal by combinations of Φ and L that put it in the shaded region. Inset: a loop with an arm.

de Genne's analysis is based on a linearization of the Landau-Ginzburg equations and thus is silent on the question, what happens below the critical flux? We have answered this question by solving the Landau-Ginzburg equations numerically. Our results, as shown in Fig. 2, are the following:

(1) The diamagnetism is zero at $\Phi=\Phi_0/2$ for $R<\xi/2$. For small L it also vanishes in an interval about $\Phi_0/2$.

(2) Although an arm can raise Φ_c above the value $R\Phi_0/\xi$ that is obtained for a simple loop, the diamagnetic moment is quite small when the enhancement of Φ_c is large.

(3) In the film geometry, where the loops and arms are of random sizes, the pendant-arm effect will be difficult to observe.

II. THE GINZBURG-LANDAU EQUATION

Inhomogeneous materials can only be handled practically by the phenomenological Ginzburg-Landau equation, obtained by choosing the order parameter ψ to minimize the free-energy density[4] (Gaussian units)

$$F=-a\,|\psi|^2+\tfrac{1}{2}b\,|\psi|^4+K\left|\left(\frac{1}{i}\vec{\nabla}-\frac{2\pi\vec{A}}{\Phi_0}\right)\psi\right|^2 . \quad (2.1)$$

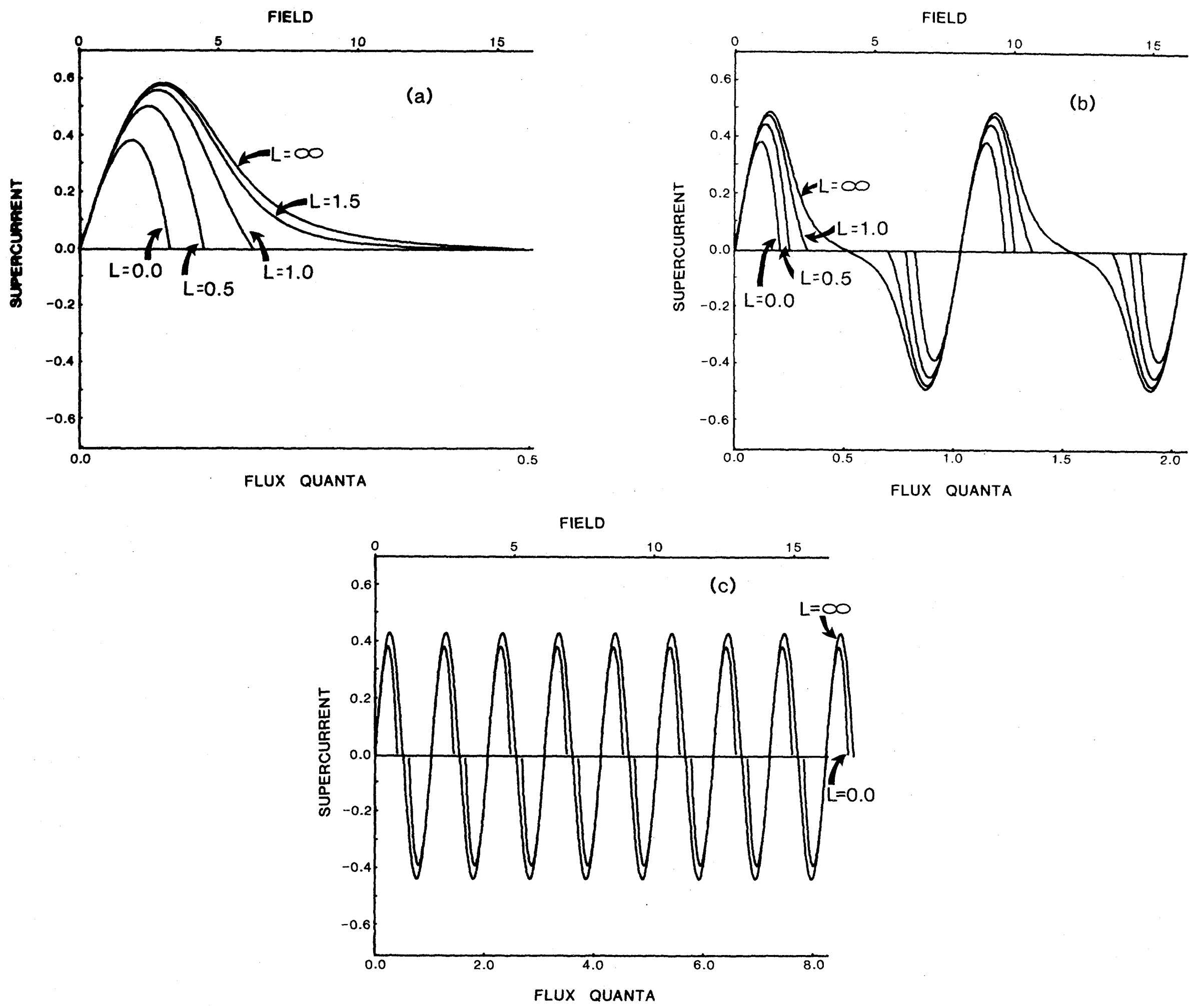

FIG. 2. Induced current J in superconducting loops with dead-end arms. J is the dimensionless current defined by Eq. (3.5), and lengths are measured in units of ξ. (a) Loop radius $R=0.1$, various L, (b) $R=0.2$, (c) $R=0.4$ (upper curve $L=\infty$; lower curve $L=0$). If extended to higher fields, (a) would have the same symmetry as (b) and (c).

Here A is the vector potential and $\Phi_0 = hc/2e$ is the flux quantum. The parameters a, b, and K are determined by the bulk properties of the superconductor, and $a \propto T_c - T$. This approach is intrinsically unable to answer some very interesting questions, such as why the critical temperature of granular aluminum films is higher than that of bulk aluminum[5]; the fact that this effect actually exists serves as a warning against taking the theory too literally. Nonetheless it does predict a variety of size- and field-dependent effects.

The diamagnetism of the superconductor arises from the induced supercurrent density, which is determined by ψ and $\vec{A}$. Thus

$$\vec{j} = \frac{4\pi c K}{\Phi_0} \mathrm{Re}\left[\psi^* \left[\frac{1}{i}\vec{\nabla} - \frac{2\pi\vec{A}}{\Phi_0}\right]\psi\right] . \qquad (2.2)$$

The vector potential has two effects: It gives rise to the supercurrent, but it also increases the third term in the free-energy density, which may drive the material normal.

For present discussion we will consider the thin-wire limit in which the magnetic field of the induced current is negligible. Thus A is due only to the external field, $A = (1/2)rH$. We will return to this point in Sec. V.

The special case of a simple circular loop of radius R is instructive. The vector potential may be taken tangent to the loop with constant magnitude $A=\Phi/2\pi R$, where Φ is the flux through the loop. Equation (2.1) is minimized by

$$\psi=\psi_0 e^{iny/R} , \tag{2.3}$$

where y is the coordinate along the loop, and periodicity in y requires that n be an integer. The magnitude of the order parameter is

$$\psi_0^2=\frac{a}{b}\left[1-\frac{\xi^2}{R^2}\left[n-\frac{\Phi}{\Phi_0}\right]^2\right] , \tag{2.4}$$

where $\xi^2=K/a$ is the coherence length and n is the integer nearest to Φ/Φ_0. This shows that the order parameter, free energy, and supercurrent are periodic in the flux with period Φ_0—the Little-Parks effect.[6]

III. SOLUTION FOR THE LOOP WITH DEAD-END ARM

It is convenient to write the order parameter on the loop in the form

$$\psi=\left[\frac{a}{b}\right]^{1/2} X(y)\exp[i\Theta(y)-i\pi HRy/\Phi_0] , \tag{3.1}$$

where X and Θ are real-valued functions. Then the free-energy density simplifies to

$$bF/a^2=-X^2+X^4/2+\xi^2\,|\,i\Theta' X+X'\,|^2 . \tag{3.2}$$

This is minimized by X and Θ such that

$$X\Theta''+2X'\Theta'=0 , \tag{3.3}$$

$$-\xi^2 X''+\xi^2(\Theta')^2 X-X+X^3=0 . \tag{3.4}$$

The former equation implies that the dimensionless quantity

$$J=\xi\Theta' X^2 \tag{3.5}$$

is independent of y. Its physical significance is that the current circulating in the loop is $4\pi a^2 c\xi SJ/b\Phi_0$, where S is the cross-section area of the wire. We may use this result to eliminate Θ' from Eq. (3.4), giving

$$\xi^2 X''=J^2X^{-3}-X+X^3 . \tag{3.6}$$

The first integral of this equation is readily found

$$\xi^2(X')^2+J^2X^{-2}+X^2-X^4/2=E , \tag{3.7}$$

where E is a constant of integration, and the equation can be solved in terms of Jacobian elliptic functions

$$X^2=X_0^2+\alpha^2\mathrm{sn}^2(\beta y/\xi) \tag{3.8}$$

where X_0 is the minimum value of X,

$$4\beta^2=2-3X_0^2+[(2-X_0^2)^2-8J^2X_0^{-2}]^{1/2} , \tag{3.9}$$

$$2\alpha^2=2-3X_0^2-[(2-X_0^2)^2-8J^2X_0^{-2}]^{1/2} , \tag{3.10}$$

and the modulus of the elliptic function is

$$m=k^2=\alpha^2/2\beta^2 . \tag{3.11}$$

The coordinate y is measured from the point where the minimum occurs.

The order parameter on the arm may be taken to be real

$$\psi=\left[\frac{a}{b}\right]^{1/2} X(y) , \tag{3.12}$$

and is also described by Eqs. (3.6)–(3.11), except that $J=0$. One boundary condition at the point where the arm joins the loop is that the functional forms agree on the value of ψ at that point.

There is also a derivative condition, obtained by requiring the free energy to be extremal with respect to variation of the value of ψ at the junction; it requires that the three outward-directed components of $\psi^*[(\vec\nabla/i)-(2\pi\vec{A}/\Phi_0)]\psi$ sum to zero (this condition includes current conservation but is stronger, because the imaginary part is also considered). These conditions reduced to

$$X_{\text{loop}}=X_{\text{arm}}=X , \tag{3.13}$$

$$2X'_{\text{loop}}=X'_{\text{arm}} . \tag{3.14}$$

There is also a periodicity requirement that ψ on the loop take on the same value at the junction when this is approached from either side.

To understand the behavior of the solutions, it is helpful to think of an analogy, in which Eqs. (3.6) and (3.7) describe the position X at time y of a particle, subject to a potential $J^2X^{-2}+X^2-X^4/2$. The period of small oscillations in this well is $\geq\pi\xi$; oscillatory behavior need not be considered for loops with a circumference less than this, and even for larger loops the solution with the lowest free energy for a given vector potential is monotonic between the inner turning point and the junction with the arm. If we take the origin of y where X is a minimum, y-reversal symmetry implies that $X(y)$ is an even function. Thus the distance $|y|$ to the junction is the same in each direction: The minimum is opposite to the junction that is at

$y=\pm\pi R$. The periodicity condition becomes

$$\Theta(\pi R)=-\Theta(-\pi R)=\pi(\Phi/\Phi_0\pm n)\ ,$$

where n is an integer.

Returning to the mechanical analogy, the particle starts at X_0 at "time" 0 and moves in its potential for a time πR. Its position and velocity X' at that point determine the initial position and velocity of a second particle (which describes ψ on the arm) through the conditions (3.13) and (3.14); it moves in the potential $X^2-X^4/2$ (with $J=0$) and comes to rest at the end of the arm.

In our numerical solution to these equations we chose X_0, R, and J as independent variables. We found it simpler to integrate (3.6) numerically than to code the analytic solution (3.8), though the latter was useful in developing analytical checks and determining the behavior when J and X_0 are both small. The first integration, from $y=0$ to the point where the arm joins, determines X, X', and Θ at that point. The value of Θ determines the magnetic flux through (3.15), and X and X' give the initial conditions for a second integration along the arm. The second integration proceeds until $X'=0$, and the length required for this to occur determines L. If the initial E [as given by (3.7)] is too large there is no turning point: Not all combinations of J and X_0 are physically meaningful. An iteration program adjusted X_0 to get any desired L.

Figure 3 shows the variation of the order parameter X with y on loop and arm for several values of J. In Fig. 3(a) we have chosen $R=0.2\xi$ and $L=\xi$, for which there is a critical flux $\Phi_c=0.322\Phi_0$ at which loop and arm are simultaneously driven normal. In Fig. 3(b) $R=0.2\xi$ and $L=1.5\xi$, which is an example of the case where the loop cannot be driven normal except at the special value $\Phi=\Phi_0/2$.

We can summarize this figure as follows: Loops with short arms can be driven normal by an applied field, just as can loops without pendant arms. Loops with long arms are not driven normal but do not have a large magnetic moment either.

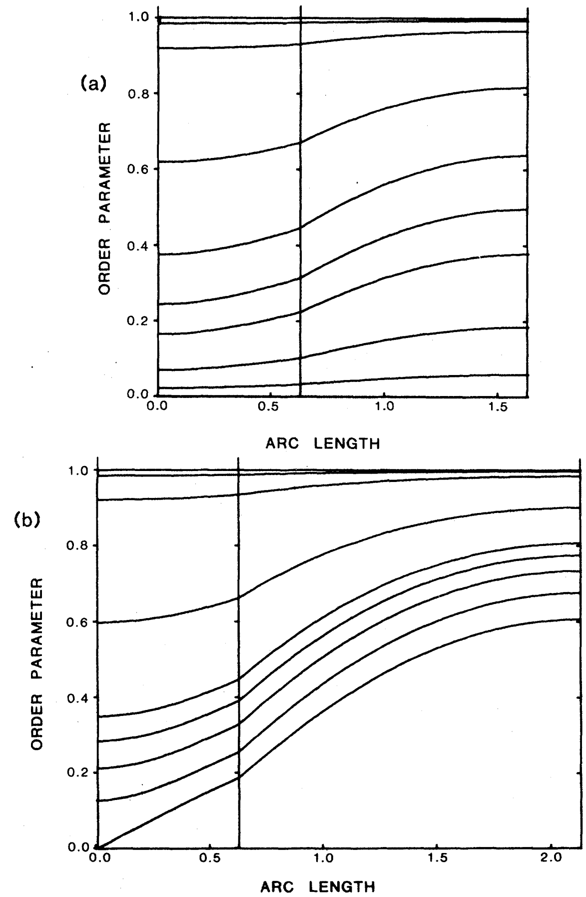

FIG. 3. Order parameter as a function of distance along loop and arm, for loop radius $R=0.2$ (in units of ξ). The central vertical axis is the junction point; to the left we move around the loop (only half the loop is shown), and to the right we move out the arm. (a) $L=1$. From the top, the curves are for $(J,\Phi)=(0,0)$, (0.2,0.041), (0.4,0.094), (0.4,0.198), (0.2,0.253), (0.1,0.280), (0.05,9.297), (0.01,0.316), (0.001,0.321). The curve for ($J=0$, $\Phi=\Phi_c=0.322$) coincides with the horizontal axis. (b) $L=1.5$. Curves from top are $(J,\Phi)=(0,0)$, (0.2,0.041), (0.4,0.093), (0.4,0.209), (0.2,0.276), (0.15,0.296), (0.10,0.323), (0.05,0.369), (0,0.5). (Φ is in units of Φ_0.)

IV. AN APPROXIMATE TREATMENT

The solutions to the Ginzburg-Landau equation found in the previous section can be categorized as follows: (1) ψ nearly constant on loop and arm (for loops with short arms); (2) ψ small on loop, variable over a length $\sim\xi$ of the arm and constant on the rest of the arm; (3) ψ zero everywhere. The free energy is measured relative to case (3).

The first case is a good description over a wide range of current and flux (note the parameter values for the top curves in Fig. 3) and so it is instructive to estimate the free energy corresponding to it. The condensation energy, as given by Eq. (2.1) is

$$F=(-a\psi^2+\tfrac{1}{2}b\psi^4)S(L+2\pi R)+K\frac{\pi^2H^2R^2}{\Phi_0^2}\psi^2S(2\pi R)\ , \qquad (4.1)$$

where S is the cross-section area of the wires. In general the factor $S(L+2\pi R)$ should be the total volume of the superconductor. The last term can be interpreted as one-half the product of the magnetic field and the magnetic-moment $I\pi R^2/c$ of the loop, where the circulating current can be calculated from Eq. (2.2),

$$I = jS = \frac{4\pi c K}{\Phi_0}\frac{\pi H R}{\Phi_0}\psi^2 S \ . \tag{4.2}$$

Minimizing (4.1) with respect to ψ gives

$$\psi^2 = \frac{a}{b}\left[1 - \frac{\xi^2\Phi^2}{R^2\Phi_0^2}\frac{2\pi R}{L+2\pi R}\right] , \tag{4.3}$$

which is similar to (2.4), except that the effect of the field is decreased by a geometric factor, which raises the critical flux to

$$\Phi_c = \Phi_0\frac{R}{\xi}\left[1+\frac{L}{2\pi R}\right]^{1/2} . \tag{4.4}$$

This describes the rise in Φ_c with small L which may be seen in Fig. 1. The result can also be applied to the case of long arms [case (2)], by replacing L by ξ. Equation (4.4) can then be regarded as an effective critical field above which the magnetic moment is small. Although this approach is less accurate than the solutions presented above, it does have the advantage that is readily generalized to the cases of nonconstant cross section and multiple arms.

V. THE INDUCED FIELD

The induced current gives rise to a magnetic field that has been neglected in the foregoing, appropriate to the case where the cross-section S of the wire is small [see Eq. (4.2)]. If S is not negligible, the first effect is that the flux through the loop is less than that due to the external field alone:

$$\Phi = \pi R^2 H_{\text{ext}} - \frac{1}{c} jSL \ ,$$

where in this equation

$$L \approx 2\pi R\left[\frac{\ln(64\pi R^2)}{S} - \frac{7}{2}\right]$$

is the self-inductance of a circular loop.[7] If S is small the dependence of the induced magnetic moment on an external field is similar to Fig. 2 but with a nonuniform shift to the right. The critical field is unaffected since $\psi = 0$ there. However, for larger S the transition becomes discontinuous. This effect is properly treated by including the magnetic field energy density in the Ginzburg-Landau equation and simultaneously minimizing with respect to $\vec{A}$ and ψ.

VI. RELEVANCE TO EXPERIMENT

Looking at Fig. 1 one might be led to expect that dangling arms would lead to dramatic effects in the magnetic properties of superconducting networks. However, Fig. 2 casts some doubt on this, especially in the context of the disordered superconductor.

The field axes in Figs. 2(a)−2(c) have the same scale, showing that the universal periodicity in Φ/Φ_0 leads to rather different field dependences for loops that are not greatly different in size. A disordered film would undoubtedly contain a wide distribution of loop sizes; the superposition of their magnetic responses shows an initial rise for small fields until $H\langle r^2\rangle \approx \Phi_0$; it then decreases to zero since the signs of the diamagnetic moments are random for large fields. The curve is otherwise featureless.

The presence of dead-end arms has two effects. The first is to decrease the effect of the magnetic field. The argument of Sec. IV indicates that the magnetic field is scaled by a geometrical factor:

$$H_{\text{eff}} = H\left[1+\frac{L}{2\pi R}\right]^{-1/2} .$$

The second effect occurs when $L > (\pi/2)\xi$; then the loop is not driven normal. However, as Fig. 2 shows, the magnetic moment is quite small in this region. It is difficult to imagine circumstances in which this effect is measurable.

Experiments on the effects of applied field on disordered superconductors more commonly study the onset of resistance rather than the diamagnetism. Here the only relevant loops are those connected to the external circuit. This is necessarily the "long-arm" case, for which de Gennes's result is that the loops cannot be driven normal: The magnetic field has no effect. Our results suggest that under certain conditions, some of the loops will have very small critical currents, complicating the interpretation of resistive transitions.

ACKNOWLEDGMENTS

This work was supported by NSF through Grant No. DMR 81-14767 (J.P.S.) and Grant No. CHE 81-06122 (P.B.V.).

[1]M. Tinkham, *Inhomogeneous Superconductors—1979 (Berkeley Springs, W.V.),* Proceedings of the Conference on Inhomogeneous Superconductors, edited by D. U. Gubser, T. L. Francavilla, S. A. Wolf, and J. R. Leibowitz (AIP, New York, 1980), pp. 1–12.

[2]P. G. de Gennes, C. R. Acad. Sci. Ser. II 292, 9 (1981).

[3]P. G. de Gennes, C. R. Acad. Sci. Ser. II 292, 279 (1981).

[4]P. G. de Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966), Chap. 6.

[5]R. W. Cohen and B. Abeles, Phys. Rev. 168, 444 (1968).

[6]W. A. Little and R. D. Parks, Phys. Rev. 133, A97 (1964).

[7]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Addison-Wesley, Reading, 1960), pp. 138–139.