Anderson-localization dimensionality dependence: Further comments

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(Received 25 June 1982)

A conductance in any dimensionality is reduced to a one-dimensional (1D) conductance. In an infinite system a true mobility edge appears at 2D. In a finite system a continuous change (with the Fermi energy) in the conductance dependence on the system length is determined.

According to the scaling theory,[1-3] a one-dimensional (1D) and 2D system resistances R always exponentially increase with the system's length L when L is large enough. Numerical simulations[4] apparently contradicted this theory. Recently a power law $R \propto L^{\gamma}$ was suggested[5,6] for large L and claimed to be experimentally verified.[7]

In this paper I prove a true mobility edge at any dimensionality d higher than one[8] when $L \to \infty$. A finite L provides a continuous change (with the Fermi energy) in R dependence on L, from $\ln R \propto L$ to $\ln R \propto L^{(4-d)/3}$ (when $d \leq 4$) to $\ln R \propto L^{2-d}$. I also demonstrate that localization in any dimensionality can be described by a 1D potential, which is the initial potential averaged over all possible cross-section positions of given numbers of impurities.

It is well known that a readily available quantity is an ensemble average[9] $\langle R \rangle$ of a resistance R, but a physically meaningful quantity is[1-3] $\langle \ln R \rangle$. However, one can relate the localization of a 1D representative R to $\langle R \rangle$. According to Ref. 2, an ultimate localization implies $\langle \ln R \rangle \approx \frac{1}{2} \ln \langle R \rangle$. A straightforward refinement of the corresponding reasoning[2] implies $\langle \ln R \rangle \gtrsim \langle R \rangle$ whenever $R \gg 1$. On the other hand, as it was noticed by Azbel and Soven,[10] $\langle R \rangle = \langle \exp(\ln R) \rangle \geq \exp \langle \ln R \rangle \approx R$, where R is a representative resistance; in the absence of a strong localization $R \approx \langle R \rangle$ as usual. Thus,

$$\ln \langle R \rangle \geq \ln R \gtrsim \tfrac{1}{2} \ln \langle R \rangle \ . \tag{1}$$

By Eq. (1) the localization length $L_0 \equiv \langle \ln R \rangle / 2L$ is determined by $\ln \langle R \rangle$ within the accuracy of a factor between 1 and 2, and is related to the averaged system "band" characteristics. Also, it was suggested[11] and numerically verified[10] that a drastic difference between a representative and ensemble average was related to impurity concentration fluctuations, while a readily available ensemble average with respect only to different impurity positions (whereas the impurity number N is fixed) provides a very accurate value[10] of L_0 even when the averaged quantity is a wave-function transfer matrix θ.

A reduction of a representative R to certain ensemble averages (which are straighforwardly evaluated) can be performed in any dimensionality.

By Ref. 2, when a dimensionless conductance G is small, then in any dimensionality

$$G \approx \mathrm{Tr}(t^{\dagger} t) \ , \tag{2}$$

where t is a transmission matrix. The matrix t can be related to a matrix θ, which transfers incoming and outgoing waves (later on denoted correspondingly by subscripts $\alpha = +1$ and -1) through a given resistance. The matrix elements[11] $\theta_{++} = t^{-1}$, $\theta_{+-} = -t^{-1}\rho$ (ρ is a reflection matrix) provide an Hermitian $(\theta\theta^{\dagger})_{++} = t^{-1}(1 + \rho\rho^{+}) t^{-1\dagger}$. Therefore, when localization implies an almost complete reflection: $\rho\rho^{+} \approx 1$, then $(\theta\theta^{\dagger})_{++} \approx 2t^{-1}t^{-1\dagger}$ and thus

$$G \approx 2\,\mathrm{Tr}[(\theta\theta^{\dagger})_{++}]^{-1} \ . \tag{3}$$

So, whenever $G \ll 1$, it reduces to a multiplicative transfer matrix. When, for instance, a current flows along a lattice axis x, a magnetic field is zero or parallel to x, and n is an ordinary number of a lattice plane ($1 \leq n \leq L$), then

$$\theta = w^{-(L+1)} w\theta_L w\theta_{L-1} \cdots w\theta_1 w^{-1} \ , \tag{4}$$

Here, θ_n characterizes a site n and w describes propagation between adjacent planes

$$w = \exp(iE); \quad E = \begin{pmatrix} \hat{k} & 0 \\ 0 & -\hat{k} \end{pmatrix}; \quad k^{\nu\mu} = \delta_{\mu\nu} k^{(\nu)} \ ; \tag{5}$$

ν, μ denote channels; $k^{(\nu)}$ is a longitudinal wave vector.[12]

Similar to Eq. (1) G from Eq. (3) yields to[13]

$$2\,\mathrm{Tr}\langle Q \rangle^{-1/2} \gtrsim G \gtrsim 2\,\mathrm{Tr}\langle Q \rangle^{-1}; \quad Q = (\theta\theta^{\dagger})_{++} \ . \tag{6}$$

Thus, whenever $G \ll 1$, in a general case of arbitrary length, cross section, magnetic field, randomness, $\ln G^{-1}$ is at most twice less and never larger than $-\ln \mathrm{Tr}\langle Q \rangle^{-1}$. The average $\langle Q \rangle$ implies (see later) the localization length L_0 consistent with

$$L_0^{-1} = L^{-1} \mathrm{Re} \ln \langle \theta_{++} \rangle_p \ , \tag{7}$$

where a subscript "p" denotes the averaging with respect only to impurity positions in cross-section

planes (while their number is kept fixed). According to Eqs. (4) and (5)

$$\langle\theta\theta^{\dagger}\rangle = w^{-(L+1)}\langle w\theta_L\langle w\theta_{L-1}\cdots\langle w\theta_1\theta_1^{\dagger}\bar{w}\rangle\cdots\theta_{L-1}^{\dagger}\bar{w}\rangle\theta_L^{\dagger}\bar{w}\rangle\bar{w}^{-(L+1)} \equiv w^{-(L+1)}P_L w^{(L+1)} \tag{8}$$

(where $w^{+}=\bar{w}=w^{-1}$; a bar denotes a complex conjugation). By Eq. (8),

$$P_n = \langle w\theta_n P_{n-1}\theta_n^{\dagger}\bar{w}\rangle;\quad P_0 \equiv \hat{i}\ , \tag{9}$$

where $\hat{i}$ is a unity matrix.

Since P_{n-1} is independent of the impurity situation at the nth plane, so the averaging in Eq. (9) is performed only with respect to θ_n and is independent of n. Thus,

$$P_n = \langle\tilde{M}\rangle P_{n-1}\ . \tag{10}$$

Matrix indices of P_n are a_1a_2; $a\equiv(\alpha,\nu)$; α is $+1$ or -1, and $\langle\tilde{M}\rangle$, which is independent of n, equals

$$\langle\tilde{M}\rangle_{b_1b_2}^{a_1a_2} = \exp[i(E_{a_1}-E_{a_2})]\langle\theta_{nb_1}^{a_1}\bar{\theta}_{nb_2}^{a_2}\rangle\ . \tag{11}$$

By Eq. (4), $w\theta_n$ is a transfer (by one site) matrix for an amplitude A_α^ν of a wave function $\psi=\sum A_\alpha^\nu \times\exp(i\alpha k^{(\nu)}x)$. So $\tilde{M}$ by Eqs. (11) and (5) is a transfer matrix for $A_\alpha^\nu\bar{A}_\beta^\mu$. Consider, for simplicity, periodic boundary conditions in the cross-section r plane (the period along the axis q being L_q). Then $A_\alpha^\nu\propto\exp(i\mathfrak{K}_\nu r)$, where $\mathfrak{K}_\nu$ is a transverse wave vector (whose projection on the q axis is an integer of $2\pi/L_q$) and thus $A_\alpha^\nu\bar{A}_\beta^\mu\propto\exp[i(\mathfrak{K}_\nu-\mathfrak{K}_\mu)r]$. An averaged system is homogeneous; therefore, $\langle\tilde{M}\rangle_{\mu_1\mu_2}^{\nu_1\nu_2}$ preserves $\mathfrak{K}_{\mu_1}-\mathfrak{K}_{\mu_2}=\mathfrak{K}_{\nu_1}-\mathfrak{K}_{\nu_2}\propto\nu_1-\nu_2$ (later it is proven by a direct calculation). By Eq. (9), $P_0^{\mu_1\mu_2}\propto\delta_{\mu_1\mu_2}$, so the conserved $\mu_1-\mu_2=0$, and thus P_n are diagonal with respect to channel numbers. Therefore, Eq. (10) provides

$$P_n^{\nu\mu}=\delta_{\nu\mu}\langle\tilde{M}\rangle_{\nu\nu}^{\nu\nu}P_{n-1}^{\nu\nu}\equiv\delta_{\nu\mu}M^n\ ; \tag{10a}$$

where $M\equiv M(\nu)\equiv\langle\tilde{M}\rangle_{\nu\nu}^{\nu\nu}$. By Eqs. (6) and (8), $\langle Q\rangle^{-1}=[(M^L)_{++}]^{-1}$. Suppose the eigenvalues of M are $\lambda_{\nu\alpha}$ (where $\alpha=\pm1$) and $L\gg1$. Then, e.g., $\mathrm{Tr}\langle Q\rangle^{-1}\simeq\sum_\nu(\max_\alpha\lambda_{\nu\alpha})^{-1}$, and by Eq. (6),

$$\ln G^{-1}\sim\min_\nu\max_\alpha\ln|\lambda_{\nu\alpha}|\ . \tag{12}$$

Thus, essentially the conductance reduces to the sum of one-channel conductances related to different $k^{(\nu)}$'s, of which the most transparent survives and determines the localization length.

By Eq. (8), P is Hermitian: $P_{\alpha\beta}=\bar{P}_{\beta\alpha}$. So, by Eq. (10), $M_{\beta_1\beta_2}^{\alpha_1\alpha_2}=\bar{M}_{\beta_2\beta_1}^{\alpha_2\alpha_1}$. Since $\beta_1^2=\beta_2^2=1$, this allows for the presentation of the M eigenfunction $p_{\beta_1\beta_2}=\bar{p}_{\beta_2\beta_1}$ as $p_{\beta_1\beta_2}=p_0+p_1\beta_1+\bar{p}_1\beta_2+p_2\beta_1\beta_2$ (with real p_0,p_2). As in a 1D case,[9] the calculation reduces to a cubic equation. To determine M explicitly, follow Ref. 11 and consider a dimensionless Schrödinger equation for a "disk" scatterer:

$$\Delta\psi+k^2\psi=\delta(x)V(r)\psi\ , \tag{13}$$

where r denotes coordinates, orthogonal to x (and V and boundary conditions are periodic with respect to r). At $x=0$, by Eq. (13),

$$\delta\psi|_{x=0}\equiv\psi|_{x=0^+}-\psi|_{x=0^-}=0\ , \tag{13a}$$

$$\delta\left(\frac{\partial\psi}{\partial x}\right)\Bigg|_{x=0}=V(r)\psi(0)\ . \tag{13b}$$

A wave function ψ at $x\neq0$ is

$$\psi=S^{-1/2}\sum a_\alpha^\nu\exp(i\alpha k^{(\nu)}x+i\mathfrak{K}_\nu r)\ , \tag{14}$$

where $k^{(\nu)}=(k^2-\mathfrak{K}_\nu^2)^{1/2}$; $S=\prod L_q$ is a cross-section area. A straightforward matching of Eq. (13a) and (13b) by wave function at $x<0$ and $x>0$ implies

$$\theta_{na'}^{a}=\delta_{aa'}-i\alpha\Gamma_{\nu\nu'}\ , \tag{15a}$$

$$\Gamma_{\nu\nu'}=V_{\nu'-\nu}/2k_\nu\ ,$$

$$V_\nu=S^{-1}\int\exp(i\mathfrak{K}_\nu r)V(r)\,dr\ ; \tag{15b}$$

in Eq. (15b), $0\leq r_q\leq L_q$.

Consider, for simplicity, one kind of impurities

$$V(r)=\sum_{j=1}^{N}V^{(0)}(r-r_j)\ , \tag{16}$$

where r_j is the jth impurity location. Just for simplicity, allow for any number of impurities per site. Then

$$\langle V_\mu\rangle=V_\nu^{(0)}\sum\langle\exp(i\mathfrak{K}_\mu r_j)\rangle=V_\mu^{(0)}S^{-1}\left(\sum_{j=1}^{N}\sum_{r_{jq}=1}^{L_q}\exp(i\mathfrak{K}_\mu r_j)\right)_{\mathrm{av}}=V_0^{(0)}\delta_{\mu0}N_{\mathrm{av}}\ ,$$

where a subscript "av" denotes an average over the number of impurities per site; while

$$\langle V_\mu\bar{V}_\nu\rangle=V_\mu^{(0)}\bar{V}_\nu^{(0)}\left\langle\sum_{j,j'}\exp(i\mathfrak{K}_\mu r_j-i\mathfrak{K}_\nu r_{j'})\right\rangle=V_\mu^{(0)}\bar{V}_\nu^{(0)}[\delta_{\mu0}\delta_{\nu0}(N^2)_{\mathrm{av}}+(\delta_{\mu\nu}-\delta_{\mu0}\delta_{\nu0})N_{\mathrm{av}}]\ .$$

As stated previously, $\nu_1-\nu_2$ is conserved in $M_{\mu_1\mu_2}^{\nu_1\nu_2}$. By Eqs. (11), (15a), (15b), and (16),

$$M^{\alpha_1\alpha_2}_{\beta_1\beta_2}(\nu)=\exp[i(\alpha_1-\alpha_2)k^{(\nu)}]\{(\delta_{\alpha_1\beta_1}-\tfrac{1}{2}i\alpha_1k^{(\nu)-1}V_0^{(0)}N_{\rm av})(\delta_{\alpha_2\beta_2}+\tfrac{1}{2}i\alpha_2k^{(\nu)-1}V_0^{(0)}N_{\rm av})$$
$$+\tfrac{1}{4}\alpha_1\alpha_2k^{(\nu)-2}(V_0^{(0)})^2[(N^2)_{\rm av}-(N_{\rm av})^2]\}\ . \qquad (17)$$

The first term in figure brackets, by Eqs. (11), (15a), and (15b) is related to a periodic set of scattering planes with the averaged potential $V_0^{(0)}$ in the plane. Such a one-channel (for a given ν) periodic system implies allowed and forbidden bands with the mobility edge between them and an (independent of S) "localization length" on the forbidden side.

Now consider the second term in the figure brackets in Eq. (17). When $N_{\rm av}\gg 1$ then $(N^2)_{\rm av}$ differs from $(N_{\rm av})^2$ only in virtue of fluctuations, so $(N^2)_{\rm av}-(N_{\rm av})^2\sim N_{\rm av}$ and provides a relatively ($\propto 1/N_{\rm av}$) small perturbation. When $N_{\rm av}\sim 1$, then $|V_0^{(0)}|^2\times[(N^2)_{\rm av}-(N_{\rm av})^2]$ is relatively small compared to $|V_0^{(0)}|N_{\rm av}$ (their relation is $\propto S^{-1}$). This perturbation does not alter the energy-gap solutions but implies $\ln G^{-1}\propto L/S$ for allowed band solutions. In a multichannel 1D wire with $S\ll L$ this provides the localization L_0 changing from L_0 independent of S to $L_0\propto S$. In any dimensionality above 1, when $S>L$, this provides a mobility edge. A quantative formula for L_0 immediately follows, by Eq. (12), from the eigenvalues of Eq. (17), whose evaluation reduces to a cubic equation.

Equation (17) is related to a one-channel wave vector $k^{(\nu)}$ and a 1D set of potentials with the average $V_0^{(0)}N_{\rm av}\propto N_{\rm av}/S$ and a dispersion $(V_0^{(0)})^2\times[(N^2)_{\rm av}-(N_{\rm av})^2]$, cf. Ref. 9. This agrees with the claim[11] (proven[10] in 1D) that an ensemble average of a transfer matrix with respect to all possible cross-section positions of impurities (with the impurity numbers in cross-section planes being fixed) provides a very accurate value of the localization length. Such an averaging, in virtue of Eqs. (15a) and (15b) reduces θ to that of a 1D potential $V_0^{(0)}N=S^{-1}\times N\int V^{(0)}(r)\,dr$, whose randomness is related to a random number N of impurities in a cross section. I demonstrate the results by an example of a continuous disk impurities distribution with the effective impurity potential $S^{-1}\int V^{(0)}(r)\,dr\equiv S^{-1}v$ and an average x distance between impurities $(nS)^{-1}$, where n is an impurity density in the original ramdon system. (This dD-1D scaling was first presented in Ref. 11.) Suppose $v>0$. Then, according to a 1D formula,[10]

$$L_0(k)=\max_\nu\tilde{L}_0(k^2-\mathfrak{K}_\nu^2)^{1/2}\ , \qquad (18)$$

$$\tilde{L}_0^{-1}(k^{(\nu)})\approx|{\rm Re}(vn-k^{(\nu)2})^{1/2}+v^2n/8S|nv-k^{(\nu)2}|\ , \qquad (18a)$$

where $k^{(\nu)}=(k^2-\mathfrak{K}_\nu^2)^{1/2}$. By Eq. (18a), $\tilde{L}_0(k^{(\nu)})$ has a sharp minimum in the vicinity of $k=(vn)^{1/2}$. In any many-channel case (when $k^{(\nu)}$ almost continuously changes from 0 to k), L_0 first monotonically increases with k:

$$L_0(k)\simeq(vn-k^2)^{-1/2}\ . \qquad (19a)$$

At $k\simeq(nv)^{-1/2}$, $L_0\sim(8S/v^2n)^{1/3}$. Then $L_0(k)$ becomes related to $k^{(\nu)}\simeq(vn)^{1/2}$ and remains constant in the interval $\delta k\sim(v^5n)^{1/6}(8S)^{-2/3}$. Thereafter,

$$L_0(k)\simeq 8S(k^2-nv)/(v^2n)\ . \qquad (19b)$$

When $S\to\infty$ and a dimensionality $d\geq 2$, Eqs. (19a) and (19b) provide a true mobility edge for the exponential localization.

Note that small Fermi energy may belong in a localized state and that the Fermi energy changes with impurity density magnetic field pressure. Equation (12) provides the G^{-1} exponent. When the localization is weak (i.e., $L_0=\infty$), then the right-hand side of Eq. (6) (with the above calculated $\langle Q\rangle$) provides the resistance G^{-1}, which cannot increase with the system length L quicker than linearly. This is consistent with Refs. 5–7, and does not exclude the power-law localization.

To summarize. A multichannel localization length L_0 and its dependence on a cross section S and a random system parameters is mapped onto a one-channel[10] L. When $S\to\infty$, in any dimensionality above one L_0 becomes infinite at a (determined) mobility edge.

*On a sabbatical from Tel-Aviv University, Tel-Aviv, Israel.

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[13]Suppose the eigenvalues of an Hermitian $Q \equiv (\theta\theta^{\dagger})_{++} = t^{-1}(1+\rho\rho^{\dagger})t^{-1\dagger}$ are H_m. Then $\mathrm{Tr}Q^{-1} = \sum H_m^{-1}$. The reasoning which provides Eq. (1) is applicable to any representative $R \gg 1$. Thus, when $H_m \gg 1$, then $\langle H_m \rangle^{-1/2} \gtrsim H_m^{-1} \gtrsim \langle H_m \rangle^{-1}$, and $\mathrm{Tr}Q^{-1} = \sum H_m^{-1} \gtrsim \sum \langle H_m \rangle^{-1} = \mathrm{Tr}\langle Q \rangle^{-1}$; $\mathrm{Tr}Q^{-1/2} \lesssim \sum \langle H_m \rangle^{-1/2} = \mathrm{Tr}\langle Q \rangle^{-1/2}$, in agreement with Eq. (6).