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Bridge between the solutions and vacuum states of the Korteweg–de Vries equation and that of the nonlinear equation $y_t + y_{xxx} - 6y^2 y_x + 6\lambda y_x = 0$

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We analyze a Bäcklund transformation that relates the solutions of two different equations, the Korteweg–de Vries (KdV) equation and the nonlinear equation $y_t + y_{xxx} - 6y^2 y_x + 6\lambda y_x = 0$, which includes the modified KdV equation as a special case. It is shown here that the Miura transformation represents a special case of our transformation. Following our previous interpretation of the vacuum states of the KdV solutions, we have found and interpreted the physical meaning of the vacuum parameters of the stated nonlinear equation.

I. INTRODUCTION

As shown previously, using a differential-geometrical approach,[1,2] one may arrive at a system of Bäcklund transformations,[2,3]

$$u^* = b \ , \tag{1a}$$

$$u^* = u(x,t) \ , \tag{1b}$$

$$u^* = -u(x,t) - y^2 + \lambda \ , \tag{1c}$$

of the Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + 12 u u_x = 0 \ , \tag{2}$$

where λ and b are constants and the function y satisfies the conditions

$$y_x = -2u(x,t) - y^2 + \lambda \ , \tag{3a}$$

$$y_t = -4[u(x,t) + \lambda] y_x + 2u_{xx} - 4u_x y \ . \tag{3b}$$

Through our Bäcklund transformation (1), we obtained[1] a set of analytical solutions to (2); in particular, the nontrivial solution corresponding to transformation (1c) is

$$u^* = (\lambda - b) - (\lambda - 2b) \times \frac{(Ce^{\sqrt{\lambda-2b}\,r} - e^{-\sqrt{\lambda-2b}\,r})^2}{(Ce^{\sqrt{\lambda-2b}\,r} + e^{-\sqrt{\lambda-2b}\,r})^2} \ , \tag{4}$$

where

$$r = x - 4(b+\lambda)t \ . \tag{5}$$

As an example, for $\lambda - 2b > 0$ and taking $C = 1$, we arrive at a one-soliton solution:

$$u^* = b + (\lambda - 2b) \times \text{sech}^2\{\sqrt{\lambda - 2b}\,[x - 4(b+\lambda)t]\} \ . \tag{6}$$

From inspection, we know from (6) that the soliton velocity is

$$v = 4(b + \lambda) \tag{7}$$

and the amplitude is

$$A = \lambda - 2b \ . \tag{8}$$

Our analysis[1] of solutions like (6) indicates that the directions and magnitude of the velocity, as well as the amplitude, depend on the value of b. In fact, b represents the value of u^* at $x \to \pm\infty$, and we call b the vacuum parameter. We are led by our previous result to believe that b is a physical observable. It is then natural to ask the following questions: Do "vacuum states" exist in the soliton solutions of other nonlinear equations? If the answer to this question is affirmative, is there any relationship between the vacuum parameter b of the set of KdV solutions and other vacuum parameter(s) of another type of nonlinear equation?

II. BÄCKLUND TRANSFORMATION OF THE KdV EQUATION AND THE EQUATION $y_t + y_{xxx} - 6y^2 y_x + 6\lambda y_x = 0$

In this investigation, we provide answers to the above questions for the nonlinear equation

$$y_t + y_{xxx} - 6y^2 y_x + 6\lambda y_x = 0 \ , \tag{9}$$

where λ is a parameter. Before we present the final result of our analysis, we would like to list the following general properties concerning our Bäcklund transformation [specified by Eq. (3)].

(i) The close-ideal condition in our differential-geometrical approach[1,2] guarantees that the integrability condition

$$y_{xt}=y_{tx}$$

is satisfied for Eq. (3). In other words, if u is a solution to the KdV equation, the integrability condition holds.

(ii) Loh[4] studies the relation between the functions u, u^*, and y of our Bäcklund transformation under the special case $\lambda=0$. He has shown that if y is a solution to the modified KdV equation,

$$y_t+y_{xxx}-6y^2y_x=0 , \tag{10}$$

then

$$u=-\tfrac{1}{2}(y^2+y_x) \tag{11}$$

is a solution to the KdV equation (2). Also conversely, if u is a solution to the KdV equation, then y is a solution to the stated modified KdV equation. If y is a solution to the modified KdV equation, $-y$ is also its solution. Substituting $-y$ into relation (11), the solution to u is shown to be identical to the Bäcklund transformation u^* in (1).

In this study, we generalize the above deduction to the case where $\lambda\neq 0$ in general. In our proof, we first rewrite (3) into the form

$$u(x,t)=-\tfrac{1}{2}(y^2+y_x-\lambda) . \tag{12}$$

Differentiating the above equation, we obtain

$$u_t+u_{xxx}+12uu_x$$
$$=-\frac{1}{2}\left[\frac{\partial}{\partial x}+2y\right](y_t+y_{xxx}-6y^2y_x+6\lambda y_x) . \tag{13}$$

Based on Eq. (13), we arrive at the following theorem

Theorem 1. If y satisfies the equation

$$y_t+y_{xxx}-6y^2y_x+6\lambda y_x=0 , \tag{9}$$

then u of (12) satisfies the KdV equation (2). We observe that the Miura transformation[5] (11) becomes the special case of our relation (12) when $\lambda=0$. According to Theorem 1, if we start with $-y$, then the result for u using transformation (12) is identical to u^* in (1c). The proof is elementary.

Using relations (2) and (3), differentiating, and canceling u, we obtain

$$y_t+y_{xxx}-6y^2y_x+6\lambda y_x=0 .$$

In other words, we establish the following theorem.

Theorem 2. If u is a solution to the KdV equation, then y, satisfying the stated integrability condition and Eq. (3), is a solution to Eq. (9).

According to the above two theorems, we have deduced that Eqs. (3a) and (3b) represent a Bäcklund transformation that relates the KdV equation and Eq. (9). With the knowledge of the solutions to the modified KdV equation, it is well known that the Miura transformation[5] enables us to obtain solutions to the KdV equation. As shown previously, the reverse statement cannot be established because the explicit expression for y_t is not given in Miura's transformation. Our transformations (3a) and (3b) bridge the solutions to the KdV and the nonlinear equation (9) (which includes the modified KdV equation as a special case) in both forward and backward directions.

III. SOLUTIONS AND VACUUM STATES

First, we recall that we have already obtained in our previous investigation the solutions for y to the Bäcklund transformation (3) [see Eq. (14) of Ref. 1]. With the use of results listed in the preceding section, it is rather obvious that the solutions to Eq. (9) are

$$y=\pm\sqrt{\lambda-2b} , \tag{14a}$$

$$y=\pm\frac{1}{r+C},\quad \lambda-2b=0 \tag{14b}$$

$$y=\pm\sqrt{\lambda-2b}\,\frac{Ce^{\sqrt{\lambda-2b}\,r}-e^{-\sqrt{\lambda-2b}\,r}}{Ce^{\sqrt{\lambda-2b}\,r}+e^{-\sqrt{\lambda-2b}\,r}} , \tag{14c}$$

where

$$r=x-4(b+\lambda)t .$$

One can readily show that relations (14) for y are solutions to Eq. (9) by direct substitution.

In order to find out the vacuum states and hence the vacuum parameter, we take the nontrivial solution (15c) for illustration. When

$$\lambda-2b>0$$

and taking

$$C=1 ,$$
$$y=\pm\sqrt{\lambda-2b}$$
$$\times\tanh\{\sqrt{\lambda-2b}\,[x-4(b+\lambda)t]\} . \tag{15}$$

The soliton solution represented by (15) appears in the forms of Fig. 1(a) (taking positive sign) and Fig. 1(b) (taking negative sign).

On the other hand, when

$$\lambda-2b>0$$

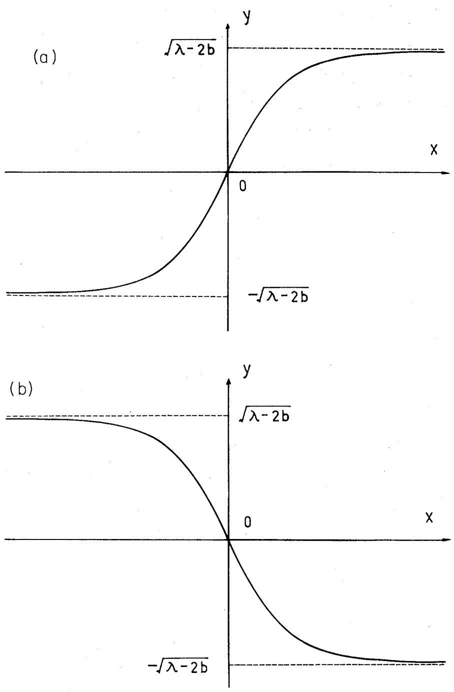

FIG. 1. Soliton solution for Eq. (15): (a) positive sign, (b) negative sign.

but

$$C=-1\ ,$$
$$y=\pm\sqrt{\lambda-2b}\times\coth\{\sqrt{\lambda-2b}\,[x-4(b+\lambda)t]\}\ , \quad (16)$$

whereas for $-k^2=\lambda-2b<0$ and $C=\pm1$, y can be expressed simply as

$$y=\pm k\tan(kr)\ . \quad (17)$$

Solutions represented by (16) and (17) have singular points. It is easy to analyze (16) and (17) in the same way as before, and the analysis is omitted here.

From solution (15), we observe that the velocity of the soliton $v=4(b+\lambda)$ can take on positive, negative, or zero values. Since the boundary conditions at $x\to\pm\infty$ are given by

$$y_{\pm\infty}\to\pm\sqrt{\lambda-2b}\ , \quad (18)$$

we can take

$$d=\sqrt{\lambda-2b} \quad (19)$$

as the vacuum parameter, representing the vacuum state of the soliton. In fact, d is the fluctuation amplitude. Note that the velocity depends on b and λ as in the case of KdV solutions, but the vacuum parameter for the KdV solution(s) is simply b. As $y_{+\infty}\neq y_{-\infty}$, the set of solutions (15) are called topological soliton solutions.[6]

We note also that if $\lambda=0$, Eq. (15) becomes

$$y=\pm\sqrt{-2b}\tanh[\sqrt{-2b}\,(x-4bt)]\ . \quad (20)$$

Clearly, the parameter b cannot take on zero value in this modified KdV solution, supporting our conclusion drawn in our previous paper.

In passing, we note that the KdV solution corresponding to (15) is

$$u^*=b+(\lambda-2b)\times\mathrm{sech}^2\{\sqrt{\lambda-2b}\,[x-4(b+\lambda)t]\}\ . \quad (21)$$

This soliton solution is nontopological.

IV. CONCLUSION

We have shown in this paper that via our Bäcklund transformation (3), a rather powerful relation as specified by Theorems 1 and 2 is established between the solutions to the KdV equation and another nonlinear equation (9), which is transformed to the modified KdV equation if the parameter $\lambda=0$. Following our new interpretation on the vacuum parameter introduced recently, we have found the vacuum parameter ($d=\sqrt{\lambda-2b}$) for Eq. (14), and d has been shown to have a definite physical meaning. This result supports the idea that different vacuum states of nonlinear processes represented by (14) have different effects on the observable physical state.

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