# Comment on "Reconciliation of high-temperature series and renormalization-group results by suppressing confluent singularities"

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We have used a slightly modified version of Roskies's quadratic map method [Phys. Rev. B 24, 5305 (1981)] to estimate the confluent correction amplitudes for the susceptibility $\chi$ and correlation length $\xi$ of a model that interpolates between Ising and Gaussian limits. Our best estimates for the corresponding leading exponents are $\gamma=1.237\pm0.003$, $\nu=0.630\pm0.003$, and $\eta=2-\gamma/\nu=0.036\pm0.002$. The correction to scaling amplitude ratio $B(\xi)/B(\chi)=0.8\pm0.1$ is slightly larger than that predicted from $\phi^4$ continuum model analysis [M.- c. Chang and A. Houghton, Phys. Rev. Lett. 44, 785 (1980) and C. Bagnuls and C. Bervillier, Phys. Rev. B 24, 1226 (1981)].

Roskies has described recently[1] a simple procedure for obtaining critical exponent estimates from high-temperature series on the assumption that the leading correction to scaling exponent $\theta$ is precisely 0.5 and that higher-order corrections, including possible analytic background terms, are negligible. The key idea is to eliminate the confluent corrections by a quadratic mapping; we use

$$1-x=p^2(1-z)^2/(p-z)^2 , \qquad (1)$$

where $x$ is either the normalized inverse-temperature $K/K_c$ or the related $v/v_c$ with $v=\tanh K$. The relation (1) maps the critical point $K=K_c$ onto $z=1$ independent of the parameter $p$ and with $2\sqrt{2}-1\leq p\leq\sqrt{2}+1$ this mapping is in the acceptable region established by Roskies.[1] Furthermore, we follow Roskies's prescription for the loose-packed lattices which is to map the antiferromagnetic singularity at $x=-1$, $z=-(\sqrt{2}-1)p/(p-\sqrt{2})$ as far from the origin as the possible singularity at $x=\infty$, $z=p$. This requires

$$p=2\sqrt{2}-1 . \qquad (2)$$

Our method differs from that described by Roskies only in that we do not impose *a priori* a critical point location but rather adjust $K_c$ or $v_c$ until the $D$log Padé approximants in the $z$ plane are singular at precisely the expected critical point $z=1$.

We have applied the method to what is appropriately described as a double-Gaussian model. The model as a function of inverse temperature $K$ and magnetic field $h$ is defined by the partition function

$$Z=\prod_i \left[\int dS_i f(S_i)\right] \times\exp\left[K\sum_{nn} S_iS_j+h\sum_i S_i\right] , \qquad (3)$$

where the spin distribution on each lattice site $i$ is

$$f(S)=\exp[-(S-\sqrt{y})^2/2w^2] +\exp[-(S+\sqrt{y})^2/2w^2] ,$$
$$y=1-w^2 . \qquad (4)$$

The second moment of this distribution is identically unity and the width $w$ or related parameter $y$ enables one to interpolate between Ising ($y=1$) and Gaussian ($y=0$) limits. High-temperature series for the zero-field susceptibility $\chi=\sum_i\langle S_oS_i\rangle$ and correlation length squared $\xi^2=\sum_i r_{oi}^2\langle S_oS_i\rangle/\chi$ have been derived to order $K^{21}$ on the bcc lattice[2] and will be reported elsewhere.[3] The general coefficient of $K^n$ in these series is a polynomial in $y$ of degree approximately $n$ so that the series can be analyzed by two variable approximant methods.[4] However, here we restrict ourselves to analyzing a single-variable series in $K$ with $y$ fixed at the discrete values $y=1.0$, 0.95, 0.9, ..., 0.6. For $y\lesssim0.6$ the leading correction to scaling amplitude is so large that the neglect of higher-order corrections is probably not justified.

The results described below were obtained by standard *D*log Padé analysis of *z* plane series for $\chi$, $K\chi$, $\xi^2$, and $\xi^2/K$ supplemented by a Newton-Raphson search for $K_c$ to yield a pole at $z=1$. If the *K* plane function that is transformed via (1) and (2) has the expected critical behavior

$$F(K) \approx A(1-K/K_c)^{-\lambda} \times [1+B(1-K/K_c)^{\theta}+\cdots], \quad \theta=0.5, \tag{5}$$

then from the residue at $z=1$ we obtain the exponent $\lambda$ and from the background the amplitude *B*. We find the exponents $\lambda=\gamma$ or $2\nu$ remarkably independent of *y*, that is, universal. We also find the strong correlations between $\lambda$ and *B* shown in Fig. 1 just as expected on the basis of simple ratio analysis.[2] Finally, we note that the $\chi$ and $\xi^2$ exponents must be strongly correlated if we accept the universality hypothesis that the correction to scaling amplitudes vanish at the same *y* value for both functions. With our preferred estimates $\gamma=1.237$ and $\nu=0.630$, this special $y=0.85$. Also, this choice yields $\eta=2-\gamma/\nu=0.0365\pm0.0015$. Our analysis appears to be consistent in that if we impose no conditions on the amplitudes but demand instead that the $\chi$ and $\xi^2$ series diverge at the same $K_c$, we obtain $\eta=0.0357\pm0.0015$. For the spin-$\frac{1}{2}$ model, i.e., $y=1$, our preferred estimates $\gamma=1.237$ and $\nu=0.630$ correspond to a critical $v_c=\tanh K_c \approx 0.156\,086$, whereas the choice $\gamma=1.240$ corresponds to $v_c \approx 0.156\,090$ in agreement with Roskies.[1]

As a further test of universality and/or analysis consistency we show the differential ratio of the correction to scaling amplitudes in Fig. 2. This ratio varies by order 20% which is well outside the apparent uncertainties one would have guessed from the data of Fig. 1. The most likely explanation is that either the bias $\theta=0.5$ or the neglect of higher-order corrections have resulted in an "effective" but otherwise spurious fit for both the amplitudes and, to a lesser extent, the exponents $\gamma$ and $\nu$. The quadratic mapping by itself does not allow one to estimate the magnitude of these effects and thus the present calculation can serve principally as a benchmark for other analyses[3,4] in which more degrees of freedom are allowed in the fits.

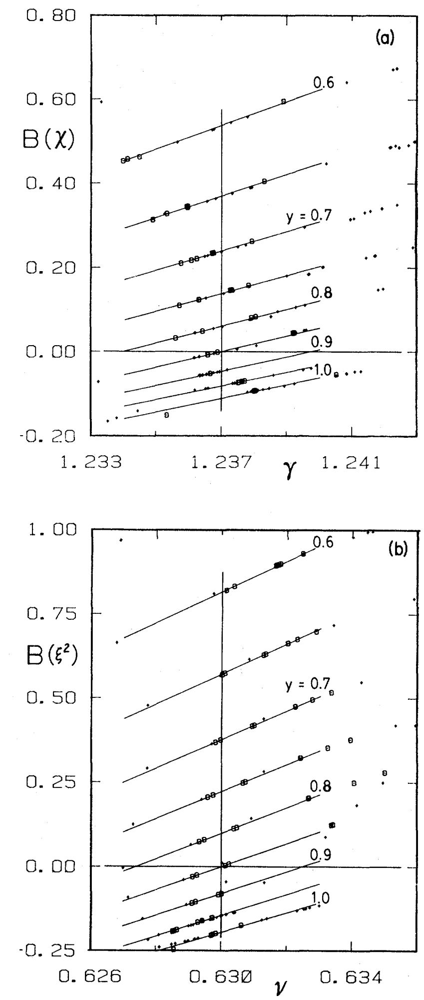


FIG. 1. Correlation plots of correction to scaling amplitude vs leading exponent from near diagonal *D*log Padé estimates based on series of order 18–21. Estimates based on the full 21 term series are circled. Solid lines are drawn as a guide to the eye in the most probable range $\gamma=1.237\pm0.003$ and $\nu=0.630\pm0.003$.

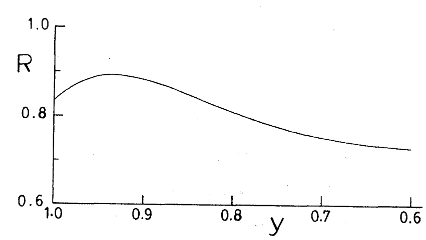


FIG. 2. Differential amplitude ratio $R=(dB(\xi)/dy)/(dB(\chi)/dy)$ vs double-Gaussian model parameter *y*. Amplitude estimates come from the solid-line intersections shown in Fig. 1 at the exponent values $\gamma=1.237$ and $\nu=0.630$.

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[1]R. Z. Roskies, Phys. Rev. B 24, 5305 (1981).

[2]B. G. Nickel, in *Phase Transitions: Cargèse 1980,* edited by M. Levy, J.-C. Le Guillou, and J. Zinn-Justin (Plenum, New York, 1982), pp. 291–324.

[3]B. G. Nickel and J. J. Rehr (unpublished).

[4]M. E. Fisher, in *Statistical Mechanics and Statistical Methods in Theory and Application,* edited by U. Landman (Plenum, New York, 1977), pp. 3–32; J. -H. Chen, M. E. Fisher, and B. G. Nickel, Phys. Rev. Lett. 48, 630 (1982).