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Renormalization effects near the vortex-unbinding transition of two-dimensional superconductors

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The theory of topological phase transitions predicts a universal relation between the vortex-unbinding temperature T_c and the areal superelectron density $n_s(T_c)$ in two-dimensional superconductors. Evaluating $n_s(T_c)$ with the theory of dirty superconductors leads to a relation between T_c and the normal-state sheet resistance $R_N^{\square}$. This latter relation, however, must be modified to account for the distinction between the unrenormalized n_s^0 and the renormalized value n_s^R, in the critical region near T_c. This can be expressed via a nonuniversal parameter $\epsilon_c = n_s^0(T_c)/n_s^R(T_c)$, which depends in turn for homogeneous films on N_0, the density of statistically independent vortices within an area ξ^2. For the Hg-Xe thin films studied earlier, we find that $\epsilon_c = 1.2 \pm 0.1$ and $N_0 = 0.05 \pm 0.03$.

There has been much attention paid in recent years to phase transitions in systems with two spatial dimensions, especially in those systems where the ordered phase can be described by a two-component order parameter. Thin-film superconductors are examples of such systems and should exhibit a phase transition from a vortex plasma state just below the BCS critical temperature T_{c0}, to a state consisting of bound vortex-antivortex pairs below a second lower critical temperature T_c.[1–3] Our purpose here is to emphasize the need to use the properly renormalized physical quantities in achieving an accurate description of the critical regime near T_c, something that has often been overlooked in previous work (including that by the present authors).[4] In particular, we highlight the fact that the well-known dependence of T_c/T_{c0} on the normal-state sheet resistance $R_N^{\square}$, first derived by Beasley, Mooij, and Orlando, is only approximate.[5] The more exact result includes a nonuniversal renormalization factor (which we determine from our earlier experimental results on quench-condensed Hg-Xe films) which depends on the physical character of the vortices in the particular physical system.

The essential feature of the ordered state below T_c is the existence of thermally excited bound vortex-antivortex pairs. Consider one such pair at a low temperature $T << T_c$, when such pairs are rather rare. If the members of the pair are separated by a distance r, then the interaction between them is of the logarithmic form

$$U(r) = 2\pi K_0 \ln(r/\xi) \ , \tag{1}$$

where $K_0 = n_s^0(T)\hbar^2/4m$, the Ginzburg-Landau coherence length $\xi(T)$ is the effective radius of the vortex core, m is the electronic mass, and $n_s^0(T)$ is the unrenormalized superelectron density per unit area.

At temperatures close to T_c, however, a given vortex pair with separation $r >> \xi$ is likely to contain between its members many similar but smaller bound pairs, which act as a polarizable medium which partially screens the primary vortex-antivortex interaction. This can be expressed in terms of renormalized quantities

$$K(r) = n_s(r,T)\hbar^2/4m = K_0/\epsilon(r,T)$$

in the form[6]

$$U(r) = \int_\xi^r 2\pi K(r') d(\ln r') \ . \tag{2}$$

The quantities $K(r)$, $n_s(r,T)$, and the effective dielectric constant $\epsilon(r,T)$ all incorporate the effects of vortex pairs with separation $\leq r$; they have renormalized these vortex pairs out of the problem. If $K^R \equiv K(r \to \infty)$ and $n_s^R \equiv n_s(r \to \infty)$ are the fully renormalized quantities, then the vortex unbinding temperature T_c is determined by the universal relation[7]

$$\pi K^R(T_c) = \pi\hbar^2 n_s^R/4m = 2k_B T_c \ , \tag{3}$$

since unbinding will occur first for vortices with separation $r \to \infty$.

The problem, then, is one of determining $\epsilon(r)$, i.e., of relating the renormalized and unrenormalized quantities. It is useful in this regard to introduce the "vortex fugacity"[8]

$$Y_0(T) = N_0 \exp(-E_c/k_B T) \ , \tag{4}$$

where E_c is the core energy of the vortex and N_0 is the density of statistically independent vortices within an area of size ξ^2. Physically, $Y_0^2(T)$ is proportional to the probability of thermally exciting a vortex pair with separation ξ. One would like to get from this to the renormalized quantity $Y(r,T)$, which likewise gives the probability of thermal excitation of a pair with separation r. Not surprisingly, $Y(r)$ and $K(r)$ form a complex set of tightly coupled integral equations,[9] which will not be reproduced here.

However, if one defines the quantity

$$X(r) = 2k_BT/[\pi K(r)] - 1 \quad (5)$$

[note that $X(r) \to 0$ as $T \to T_c$] then to lowest order in $X(r)$ and $Y(r)$, the integral equations can be reduced to the relatively simple Kosterlitz recursion relations[9] in the form

$$\frac{dY}{dL} = 2XY; \quad \frac{dX}{dL} = 8\pi^2 Y^2 \ , \quad (6)$$

where $L = \ln(r/\xi)$. These can be solved analytically,[10] but for the present it is sufficient to note that

$$X^2(L) - 4\pi^2 Y^2(L) = \text{const} = X_0^2 - 4\pi^2 Y_0^2 \ . \quad (7)$$

This forms a set of hyperboli (see Fig. 1), each of which represents the path of the renormalization transformations for a given set of initial values X_0 and Y_0. Three contours are shown. The straight line $X = -2\pi Y$ corresponds to $T = T_c$. As $L \to \infty$ the system approaches the fixed point $X = 0$ ($\pi K^R/2 = k_B T_c$) and $Y = 0$ (no thermally excited vortices for $r \to \infty$). For $|X_0| < 2\pi Y_0$, corresponding to $T > T_c$, the system starts to approach $(X, Y) = (0, 0)$, but turns away and ends at $X = \infty$ ($K^R = 0$, i.e., the interaction is totally screened) and $Y = \infty$ (lots of excited vortices far apart). For $|X_0| > 2\pi Y_0$, corresponding to $T < T_c$, the system approaches the fixed point $X(\infty) = -(X_0^2 - 4\pi^2 Y_0^2)^{1/2}$, $Y(\infty) = 0$. The existence of a line of fixed points for $T \le T_c$ is a well-known but curious feature of this system.

One can now express the effective dielectric constant $\epsilon(r,T)$ in terms of these quantities:

$$\epsilon(r,T) = K_0(T)/K(r,T) = \{[1 + X(r)]/[1 + X_0(T)]\} \ . \quad (8)$$

A particularly useful parameter is[11]

$$\epsilon_c \equiv \epsilon(r \to \infty, T_c) = [1 + X_0(T_c)]^{-1} = [1 - 2\pi Y_0(T_c)]^{-1} \ . \quad (9)$$

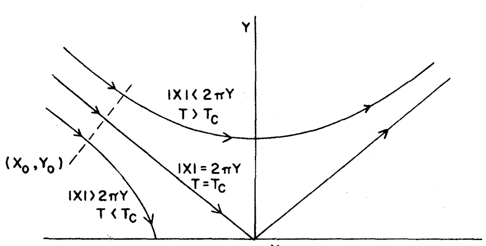

FIG. 1. System of hyperboli that solves Eq. (6). Only the upper half is shown since negative values of Y are nonphysical. Three renormalization contours are shown, corresponding to $T >$, $=$, and $< T_c$, starting from values (X_0, Y_0) indicated by the dashed line.

We can evaluate this further if we treat the vortex core as a normal cylinder of radius ξ and height d (the film thickness), and take the condensation energy to be $H_c^2(T)/8\pi$ per unit volume [$H_c(T)$ is the thermodynamic critical field]. Then[12]

$$E_c(T_c) = \xi^2(T_c) d H_c^2(T_c)/8 = \Phi_0^2 d/64\pi^2\lambda^2 = \pi n_s(T_c)\hbar^2/16m = k_B T_c/2 \quad (10)$$

by application of Eq. (3). Here $\Phi_0 = hc/2e$ is the flux quantum and $\lambda(T)$ is the bulk magnetic penetration depth. It is unclear whether n_s above is renormalized, but it will not make a significant difference here. Thus from Eq. (4),

$$\epsilon_c = (1 - 3.8 N_0)^{-1} \ . \quad (11)$$

We have obtained an experimental value of the same parameter by analyzing a set of nonlinear I-V characteristics below T_c of thin quench-condensed Hg-Xe films which we believe to be homogeneous. The details of fabrication and characterization have been published elsewhere.[4] We find that over a wide range of current, the characteristics fit the form $V \sim I^{a(T)}$, i.e., they fit a straight line on log-log paper (see Fig. 1 of Ref. 4). An analysis of vortex nucleation theory in the presence of a transport current[2,10] had led to the association

$$a(T) = 1 + \pi K^R(T)/k_B T = 1 + \pi n_s^R(T)\hbar^2/4mk_B T \ . \quad (12)$$

This follows from the fact that the electrical resistance is due to flux flow of free vortices liberated by transport current in a process in which vortices unbind by activated escape over the saddle point in the vortex-antivortex potential in the presence of a supercurrent. The algebraic form of the I-V characteristic is thus a direct consequence of the logarithmic form of the vortex-antivortex interaction.

Figure 2 is a plot of the exponent $a(T)$ measured directly from the I-V curves of three samples of Hg-

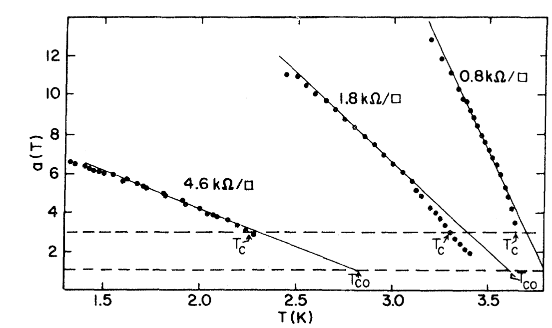

FIG. 2. Plot of the exponent $a(T)$ of the nonlinear I-V characteristic for three samples. The straight lines are fits to Eq. (13) which are then extrapolated to $a(T_{c0}) = 1$. The values $a(T_c) = 3$ and $a(T_{c0}) = 1$ are indicated by dashed lines.

Xe of different sheet resistance $R_N^{\square}$. The vortex unbinding temperature T_c can be determined as the temperature given by the identity $a(T_c)=3$. We can determine T_{c0} by recognizing that the renormalization is significant only in a critical region close to T_c. For lower temperatures, the unrenormalized quantities can be used, and since $K_0(T) \sim n_s^0(T) \sim 1-T/T_{c0}$ for temperatures that are not too low, we have

$$a(T) \approx a_0(T) = 2\epsilon_c(T_{c0}-T)/(T_{c0}-T_c)+1 \ . \qquad (13)$$

Note that for the data in Fig. 2, there is indeed a range of temperatures over which $a(T)$ is rather linear with T. We can then extrapolate from this regime to determine T_{c0} from $a_0(T_{c0})=1$.

Closer to T_c, in the critical regime, $a(T)$ appears to curve somewhat downward, in a way reminiscent of the square-root cusp expected for n_s^R at T_c.[2] We recognize, however, that the association of the measured values of $a(T)$ with the fully renormalized n_s^R is not strictly correct. They correspond more properly to a measuring distance $r_c \sim 1/I$, the distance to the saddle point over which the current-induced vortex nucleation occurs. The distinction is likely to be most significant extremely close to T_c, so that a sharp cusp may be difficult to see using finite measuring currents.

In Fig. 3 we plot the value of $\tau_c \equiv 1-T_c/T_{c0}$ as a function of $R_N^{\square}$ for a number of samples, including those from Fig. 2. Clearly, τ_c increases with increasing $R_N^{\square}$. Beasley, Mooij, and Orlando[5] (BMO) derived an expression which showed this general behavior (the lower line, labeled B, in Fig. 3), but neglected to take into account the distinction between renormalized and unrenormalized quantities. This can be remedied simply by writing the universal relation [Eq. (3)] in the form

$$\pi K^R(T_c) = \pi K_0(T_c)/\epsilon_c$$
$$= \Phi_0^2 d/[16\pi^2\lambda^2(T_c)\epsilon_c] = 2k_B T_c \ . \qquad (14)$$

If we then follow BMO in evaluating $\lambda(T_c)$ according to the theory of dirty superconductors, we find that ϵ_c enters the end result only in the combination $R_N^{\square}\epsilon_c$, so that their expression becomes

$$(T_c/T_{c0})\{\Delta(T_c)/\Delta(0)\tanh[\Delta(T_c)/2k_BT_c]\}^{-1}$$
$$=2.18R_c/R_N^{\square}\epsilon_c \ , \qquad (15)$$

or in the more familiar approximate form,

$$T_c/T_{c0} = (1+0.17R_N^{\square}\epsilon_c/R_c)^{-1} \ , \qquad (16)$$

where $R_c = \hbar/e^2 = 4108\,\Omega$.

The lower line in Fig. 3 is the solution of Eq. (15) with $\epsilon_c=1$, i.e., the original result of BMO. Note that all the points but one fall above this line, suggesting that indeed renormalization is important. For the upper line, we have assumed that the parameter ϵ_c is independent of $R_N^{\square}$ (although it is not universal) and obtained the best single-parameter fit to the points with the value $\epsilon_c = 1.2 \pm 0.1$. Using Eq. (11) for ϵ_c we find $N_0 = 0.05 \pm 0.03$, a result that at first seems rather small. On the other hand, a low density of thermal vortex excitations appears to be essential to the applicability of the theory, in particular the neglect of pair-pair interactions in the derivation of the renormalization equations.[9]

In summary, we have identified the effect of the renormalization of the superelectron density near T_c on the analysis of BMO relating T_c, T_{c0}, and $R_N^{\square}$, and applied this analysis to the interpretation of data on Hg-Xe films. We find that $\epsilon_c = \epsilon(r \to \infty, T_c) = 1.2$, corresponding to $N_0 = 0.05$, a result consistent with the low-density approximations made in the derivation of the theory. Since the value of ϵ_c depends on the character of the vortex core, it is expected to be nonuniversal. Thus, although the universal relationship for $n_s^R(T_c)$ is the same in Josephson coupled arrays,[13] proximity-coupled arrays,[14,15] granular superconductors,[16,17] and homogeneous superconductors, since they belong to the same universality class, great care must be taken in interpreting the dependence of T_c/T_{c0} on $R_N^{\square}$.

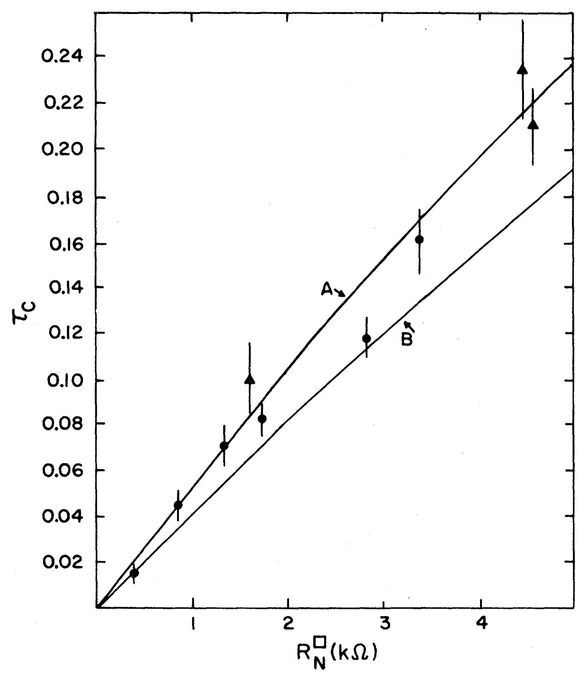

FIG. 3. Plot of $\tau_c = (T_{c0}-T_c)/T_{c0}$ vs the normal-state sheet resistance $R_N^{\square}$ measured at $T=4.2$ K. There is no substantial temperature dependence of $R_N^{\square}$ up to about 10 K. Lines A and B are obtained from Eq. (15) with $\epsilon_c=1.2$ and 1, respectively.

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