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Asymptotic symmetry: Enhancement and stability

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The scaling-limit symmetry of Z_q-invariant spin systems is studied by renormalization-group methods based on qualitative bifurcation theory rather than a series expansion in ϵ. Under the plausible assumption of the absence of secondary bifurcations, it is shown that an ordinary critical phase with $q \neq 2$ or 4 must have full SO_2 invariance in the scaling limit; moreover, any such asymptotically isotropic critical phase is stable under Z_q-invariant perturbations for $q > 4$. Previously obtained results about two-dimensional Kosterlitz-Thouless phases for clock models have a natural interpretation within this group-theoretic bifurcation analysis.

The appearance in two dimensions of a Kosterlitz-Thouless phase[1] in an intermediate-temperature range has been well established both for the XY model perturbed by a Z_q-invariant symmetry-breaking field[2] and for q-state clock models[3] providing q is sufficiently large. The critical q has been estimated by various detailed calculations[2–4] to be between 4 and 5. It has seemed remarkable that these results are so insensitive to the lattice structure or to the source of the anisotropy. In this article, we use symmetry methods to explain this situation by arguing that in the entire dimension range of interest, $2 \leq d \leq 4$, the following two phenomena occur (for q an integer). First, if an ordinary critical phase has Z_q invariance and $q \neq 2,4$, then it is in the same universality class (i.e., has the same block spin scaling limit) as the XY model; that is, local Z_q invariance is enhanced to full SO_2 invariance on a macroscopic scale. Second, any such asymptotically isotropic critical phase is stable under arbitrary Z_q-invariant perturbations for $q > 4$; i.e., for any Z_q-invariant symmetry-breaking field ψ with $q > 4$, there exists an isotropic relevant field ϕ such that $\psi - \phi$ is irrelevant. Thus, for example, the Z_q-perturbed XY model of José *et al.*,[2] at a given temperature, is in the standard XY-model universality class at a perturbed temperature, providing $q > 4$.

The above results may be generalized in a number of ways. For planar-spin models in a critical phase of type[5] k ($k=2$ is ordinary, $k=3$ is tricritical, etc.), Z_q invariance implies asymptotic isotropy for $q \neq 2,4, \ldots, 2k$, and the isotropic universality class is Z_q stable for $q > 2k$. For three-component spin models in an ordinary critical phase, invariance under a sufficiently large subgroup of SO_3 (e.g., the symmetry group of the icosahedron or dodecahedron) implies asymptotic SO_3 invariance.

Our analysis is based on qualitative bifurcation theory and has two main advantages over conventional perturbation expansions in the bifurcation parameter ϵ. First, a determination of the symmetry for small ϵ reduces to an elementary analysis at $\epsilon=0$ rather than to a term-by-term consideration of a series expansion. Second, the symmetry determined for small ϵ remains valid for large ϵ in the absence of secondary bifurcations, which we argue is the case in the models of interest. The technical assumptions implicit in the bifurcation theoretic approach (such as differentiability in ϵ) are quite distinct from the strong analyticity and convergence properties which would be needed to obtain similar conclusions from a term-by-term analysis. On the other hand, the suggestion, implicit in an ϵ-expansion approach, that there is a continuum of actual (scaling invariant) models as ϵ varies, becomes an explicit assumption of bifurcation theory. This assumption, while valid for the hierarchical model (see below), is not well understood for short-range models (where $\epsilon = 4-d$).

Although we will concentrate on planar-spin models, the group-theoretic bifurcation analysis of asymptotic symmetry is applicable to many other systems. Such a system will typically have a Hamiltonian invariant under some full symmetry group G (such as SO_2), but will be in a state invariant only under a (usually discrete) subgroup K (such as Z_q). Symmetry enhancement occurs when the state ρ, which arises as the scaling limit of the original K-invariant state, has complete G invariance. K stability refers to the phenomenon that any K-invariant perturbation of ρ may be perturbed back to a state with scaling limit ρ by a G-invariant relevant (or marginal) field. We next describe the general mechanisms which lead to these phenomena in systems where ρ depends on some bifurcation parameter (such as the dimension d in short-range interaction spin systems) so that group-theoretic bifurcation analysis[6] can be used. The generality of our exposition is called for by the variety of models to which these techniques can be applied.

General mechanism for symmetry enhancement

We divide the mechanism into several parts. (i) ρ is a fixed point of some G-invariant nonlinear scaling transformation, $\rho = N_\nu(\rho)$, with parameter ν. ρ lies on a branch of (nontrivial) K-invariant fixed points which bifurcated from a (trivial) G-invariant fixed point ρ^* at $\nu = \nu^*$.

(ii) Let L^* denote the linearization of N_{ν^*} at ρ^* and let V^* be its eigenspace of eigenvalue 1. If every K-invariant eigenfunction in V^* is also G invariant, then in some neighborhood of (ν^*, ρ^*), every K-invariant fixed point is also G invariant; this is so because the Liapunov-Schmidt procedure[6] applied either to the space of K-invariant ρ's or to the space of G invariant ρ's yields the same fixed points. To determine whether K invariance implies G invariance in V^*, one may decompose the representations of K and of G on V^* into irreducible representations and see whether the trivial representations have the same multiplicity.

(iii) G invariance will persist along any K-invariant branch in the absence of secondary bifurcation; it will persist in the presence of secondary bifurcation if, in the corresponding eigenspace, K invariance again implies G invariance.

General mechanism for stability

(iv) Let L denote the linearization of N_ν at ρ. Let W be the space spanned by its (relevant and marginal) eigenfunctions whose eigenvalues lie outside or on the unit circle in the complex plane; let X be the space spanned by the remaining (irrelevant) eigenfunctions and let W^*, X^* be the analogous objects at ν^*, ρ^*. If K invariance implies G invariance in W^*, then for some neighborhood of (ν^*, ρ^*), the same will be true in W. It follows that for any K-invariant ψ, there exists a G-invariant ϕ in W so that $\psi - \phi$ is irrelevant; thus ρ is (linearly) stable under K-invariant perturbations in the sense described previously.

(v) K stability will persist along any G-invariant branch in the absence of eigenvalues crossing the unit circle; it will persist in the presence of crossing, if in the corresponding eigenspace, K invariance again implies G invariance.

The planar-spin models to which we apply the preceding general mechanisms consist of spins

$$\vec{s}_i = (x_i, y_i) = (s_i \cos\theta_i, s_i \sin\theta_i)$$

and partition function

$$Z = \int \cdots \int e^{-H} \prod_i f^0(\vec{s}_i) d\vec{s}_i \ , \tag{1}$$

with

$$f^0(\vec{s}) = \sum_{l=1}^{q} \delta(\vec{s} - \sqrt{\beta}(\cos 2\pi l/q, \sin 2\pi l/q)) \tag{2}$$

in clock models and

$$f^0(\vec{s}) = e^{h\cos(q\theta)} \delta(s - \sqrt{\beta}) \ , \tag{3}$$

for example, in Z_q-perturbed XY models; the inverse temperature β has been absorbed into f^0 for convenience. H will always be of the form $-\sum J_{ij} \vec{s}_i \cdot \vec{s}_j$, but the J_{ij}'s (and the nature of the i's) will depend on the particular model. For each model, $G = SO_2$ and $K = Z_q =$ (rotations by $2\pi k/q$: $k = 1, \ldots, q$). We will give a detailed analysis for hierarchical models, where the fixed-point equation has an exact simple form, and then treat short-range interaction models from a more field-theoretic point of view. Before that, we briefly discuss Curie-Weiss models, where enhanced symmetry occurs due to a degenerate version of our general mechanism.

In the Curie-Weiss model, $i = 1, \ldots, n$,

$$H = -\frac{1}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} \vec{s}_i \cdot \vec{s}_j$$

and ρ is the limiting probability density of $(\vec{s}_1 + \cdots + \vec{s}_n)/n^\gamma$ as $n \to \infty$. Known results for single-component spins[7] can easily be extended to show that at an ordinary critical point (which occurs in the clock model for $q \geq 4$), $\gamma = \frac{3}{4}$, and ρ is proportional to $\exp[-G_4(\vec{s})]$, where G_4 is a polynomial which is homogeneous of degree 4. The space of such polynomials plays the role of the eigenspace V^* discussed above. It is easily seen that for $q \neq 2,4$, any Z_q-invariant G_4 must be $\text{const}\times(x^2+y^2)^2$ and thus SO_2 invariant; on the other hand the ρ for $q=4$ will have $G_4=\text{const}\times[(x+y)^4+(x-y)^4]$ which is not SO_2 invariant.

The hierarchical model was invented by Dyson[8] as a nontranslation invariant approximation to a one-dimensional model with $J_{ij}=|i-j|^{-\alpha}$. It was later recognized that this model yields a simple, exact fixed-point equation.[9] The sites are indexed by $i=1,2,3,\ldots$ and H is best defined in terms of blocks containing 2^k sites:

$$\vec{S}_i^k=\sum_{j=1}^{2^k}\vec{s}_{(i-1)2^k+j}\,,$$

$$H=-\sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty}\tfrac{1}{2}\lambda^{-k}\vec{S}_i^k\cdot\vec{S}_i^k\right].$$

H may be rewritten as $-\sum J_{ij}\vec{s}_i\cdot\vec{s}_j$ with, for example,

$$J_{1,2^l}=\sum_{k=l}^{\infty}\lambda^{-k}=\text{const}\times\lambda^{-l}=\text{const}\times r^{-\log_2\lambda}$$

so that the related $|i-j|^{-\alpha}$ model has $\alpha=\log_2\lambda=2+2\log_2\nu$, where for future convenience we have defined $\nu=\sqrt{\lambda}/2$. This model exists (in the thermodynamic limit) for $\nu>1/\sqrt{2}$ $(\alpha>1)$ and exhibits a phase transition for $\nu<1$ $(\alpha<2)$. The "effective dimension" is

$$d_{\text{eff}}=2/(\alpha-1)=(\log\nu+\tfrac{1}{2})^{-1}$$

so that $\nu=2^{-1/2},2^{-1/4},1$ correspond, respectively, to $d_{\text{eff}}=\infty,4,2$.

The natural scaling transformation T is defined by $\{\vec{s}_i^{\,1}\}=\{\vec{S}_i^1/\sqrt{\lambda}\}=T(\{\vec{s}_i\})$ and the scaling limit is obtained by letting $k\to\infty$ in $\{\vec{s}_i^{\,k}\}=T^k(\{\vec{s}_i\})=\{\vec{S}_i^k/\lambda^{k/2}\}$. It is easily seen that $\{\vec{s}_i^{\,1}\}$ forms another hierarchical model with the same λ but with f^0 in (1) replaced by f^1 defined by

$$f^1(\vec{s})=\text{const}\times\int\int e^{(\vec{s}\cdot\vec{s})/2}f^0(\vec{t})f^0(\vec{t}\,')\delta\left[\vec{s}-\frac{\vec{t}+\vec{t}\,'}{\sqrt{\lambda}}\right]d\vec{t}\,d\vec{t}\,'\equiv[\mathcal{T}_\lambda(f^0)](\vec{s})\,.$$

If the scaling limit exists (as it should at a critical point), it must be a hierarchical model with f^0 replaced by f^∞ which satisfies $f^\infty=\mathcal{T}_\lambda(f^\infty)$. The (trivial) Gaussian fixed point is $f_G^\infty=\text{const}\times\exp[-\vec{s}\cdot\vec{s}/(\lambda-2)]$ and it is convenient to transform a general fixed point f^∞ into ρ defined by

$$\rho(\vec{s})=\text{const}\times\int\left\{\exp\left[-\left(\left|\frac{\nu^2-\frac{1}{2}}{1-\nu^2}\vec{s}-\vec{t}\right|^2\Big/2\right)\right]\right\}h[(2\nu^2-1)^{1/2}\vec{t}\,]d\vec{t}\,,$$

$$h(\vec{s})=f^\infty(\vec{s})e^{+(\vec{s}\cdot\vec{s})/(\lambda-2)}\,.$$

The fixed point equation for ρ is

$$\rho(\vec{s})=[2\pi(1-\nu^2)]^{1/2}\int(\exp\{-[\,|\nu\vec{s}-\vec{t}\,|^2/2(1-\nu^2)]\})[\rho(\vec{t}\,)]^2d\vec{t}$$

or

$$\rho=M_\nu(\rho^2)\,,$$

and the trivial fixed point is $\rho=1$. The integral operator M_ν is closely related to the two-dimensional harmonic oscillator and can be diagonalized as $M_\nu=e^{-tA}$ with $\nu=e^{-t}$ and

$$A[H_{n_1}(x)H_{n_2}(y)]=(n_1+n_2)[H_{n_1}(x)H_{n_2}(y)]\,,$$

where H_n is the nth Hermite polynomial.

The linearization L^* at the trivial fixed point $\rho^*=1$, is just $2M_\nu$ and thus bifurcations occur when $2e^{-t(n_1+n_2)}=2\nu^{(n_1+n_2)}=1$. The bifurcation appropriate to ordinary critical points corresponds to $n_1+n_2=4$ and occurs at $\nu=2^{-1/4}$ $(d_{\text{eff}}=4)$. The eigenspace V^* is five dimensional, spanned by $\{H_{n_1}(x)H_{n_2}(y):\ n_1+n_2=4\}$, and corresponds to the four-particle states of the harmonic oscillator. By analyzing which angular momenta occur in this eigenspace it is again (as in the Curie-Weiss case) easy to see that Z_q invariance for $q\neq 2,4$ implies full SO_2 invariance. According to part (iii) of our general mechanism, the SO_2 invariance persists to $\nu>2^{-1/4}$ $(d_{\text{eff}}<4)$ if no secondary bifurcation occurs; this is presumably the case all the way to $\nu=1$ $(d_{\text{eff}}=2)$. The space W^* is spanned by eigenfunctions of A with eigenvalue less than or equal to 4 and thus consists of all polynomials of degree four or less; here, Z_q invariance for $q\geq 5$ implies

SO_2 invariance.

By allowing more general f^0's than those of (2) and (3), higher-order critical points with Z_q invariance may be constructed. The fixed point corresponding to a critical point of type k bifurcates at $\nu=2^{-1/(2k)}$ and Z_q invariance implies SO_2 invariance of f^∞ for $q\neq 2,4,\ldots,2k$.

The application of our general mechanism to short-range-interaction models is analogous to that for the hierarchical model. We treat the scaling limit as a continuum (Euclidean) field $\vec{\phi}=(\phi_x,\phi_y)$ with $\rho=\exp[-H(\vec{\phi})]$, bifurcation parameter d, and $d^*=4$. The Gaussian fixed point has

$$H_G(\vec{\phi})=\frac{1}{2}\int[\nabla\phi_x(r)\cdot\nabla\phi_x(r)+\nabla\phi_y(r)\cdot\nabla\phi_y(r)]d^dr\ ,$$

and the scaling transformation at $d=4$ corresponds to replacing $\vec{\phi}(r)$ in H by $\eta\vec{\phi}(\eta r)$, where $n>1$ is fixed. The eigenspace V^* of translation-invariant Hamiltonians is of the form $\{\int G_4(\vec{\phi}(r))d^dr\}$ with $G_4(\vec{s})$ a polynomial homogeneous of degree 4. As in the previous situations we find that Z_q invariance for $q\neq 2,4$ implies SO_2 invariance and this enhanced symmetry persists to $d<4$ as long as no secondary bifurcations occur. This is presumed to be the case at least until $d=2$, since the occurrence of such a bifurcation would be associated with some qualitative change, a phenomenon for which no evidence exists. W^* is similar with G_4 replaced by an arbitrary polynomial of degree 4 or less; once again Z_q invariance for $q\geq 5$ implies SO_2 invariance.

At $d=2$, the Kosterlitz-Thouless (KT) phase of the XY model produces a line of fixed points parametrized by η ($\langle\vec{s}_r\cdot\vec{s}_u\rangle\sim|r-u|^{-\eta}$) or by temperature. We conjecture that this line is a secondary bifurcation (at $\eta=\frac{1}{4}$, corresponding to the transition temperature to the KT phase) from the ordinary XY model fixed-point branch which bifurcated from the Gaussian at $d=4$. If this is so and if the nature of the bifurcating eigenspace at this secondary bifurcation point is as in our general mechanism, we would again expect that the q-state clock models for $q\geq 5$ would exhibit Kosterlitz-Thouless phases in the same universality class as the XY model with corresponding η's. It should also be expected that if a $(2\pi/3)$-invariant system [with f^0 more complicated than (2) or (3)] had a critical point, it would also be in the universality class of the XY model at its transition temperature.

Our view of the Kosterlitz-Thouless phases for $d=2$ is consistent with the usual picture in which the critical points are controlled by a Gaussian-model line of fixed points. However, it suggests in addition that that line is connected to the non-Gaussian XY model fixed points for $2<d<4$ which in turn meets the Gaussian fixed point at $d=4$. This view provides a simple interpretation of the results of those papers[2–4] in which $q>4$ is singled out at $d=2$ with regard to control by the usual (rotationally invariant) KT line of fixed points. Specifically, those results are due to the absence of breaking of the full SO_2 symmetry at the various special points ($d=2$ and $d=4$) through which the KT line of fixed points at $d=2$ is connected to the Gaussian for $d\geq 4$. Thus, although our picture does not *prove* that $d=2$ clock-model critical phases must have full asymptotic symmetry for $q>4$, it does show that this feature and the corresponding feature for $d=4-\epsilon$ are most economically understood by there being a single branch of fixed points for which nothing special (i.e., breaking of the full symmetry for $q\neq 2,4$) occurs on the way from $d=4$ to $d=2$. This picture also implies that the critical phase for $d=3$ is also controlled by the XY fixed point for $q>4$.

Although the group-theoretic approach to critical behavior, described in this article for planar-spin models, is often not capable of predicting *whether* a certain model has critical phases but only of analyzing the symmetry properties of such phases if they occur, it nevertheless may be a useful framework in studying phenomena, such as screening in Coulomb systems and confinement in gauge field theories, which involve enhanced symmetry at long distance. It also suggests that in a lattice gauge theory, replacement of the gauge group by a sufficiently large (not necessarily discrete) subgroup would leave the universality class unchanged. This approach, in the context of short-distance scaling limits in field theory, may also be useful in analyzing enhanced symmetry at high energy, such as in grand unification models.

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