Motion of a magnetic soliton about a lattice soliton in a Heisenberg chain

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As an example of interaction between two solitons belonging to different species, a semiclassical study of the nonlinear dynamics of a coupled magnon-phonon system in a one-dimensional Heisenberg ferromagnet is made, where both the lattice and the spin systems are taken with their respective nonlinear interactions. The lattice (Korteweg–de Vries) soliton is shown to introduce spacial inhomogeneities into the propagation of the magnetic (nonlinear Schrödinger) soliton resulting in (a) a possible trapping of the magnetic soliton in the harmonic field of the lattice soliton and (b) the amplitude and the width of the magnetic soliton becoming time dependent.

I. INTRODUCTION

Solitons as the nonlinear excitations of a magnetic system have been studied extensively during the past few years.[1–7] After it was shown that, in the continuum limit, a Heisenberg chain was an exactly solvable dynamical system,[1] both classical and semiclassical calculations were made to show the existence of low-amplitude solitons in a Heisenberg chain in the presence of the various kind of nonlinear interactions leading to evolution equations with nonlinearities of different orders and thus to different soliton solutions.[8,9]

The spin precessions in a magnetic chain are generally coupled to the oscillations of the spin-carrying atoms or ions about their mean positions. This fact has been used by some authors[10] in extracting a higher-order interaction between the precessing spins through the indirect spin-lattice-spin interaction and thus generating a nonlinear evolution equation with a low-amplitude soliton solution.

The lattice vibrations can, however, be anharmonic, and solitonlike excitations may also exist in the excitation spectrum of lattice vibrations in a chain, as shown by Ichikawa *et al.*[11] This then leads to a strong possibility of interaction between the solitonlike magnetic excitations and the solitonlike lattice excitations in a chain with magnetostrictive interaction. An approximate study to probe into this possibility has been undertaken in this paper, with a view to highlight the perturbative effects of a phonon soliton on the motion of a magnon soliton in a Heisenberg chain.

II. MODEL

We consider a chain of N atoms, each of mass m, oscillating about their mean positions due to a harmonic as well as an anharmonic interaction with their nearest neighbors. Each atom possesses a net spin of magnitude S and the nearest-neighbor spins interact with each other through an anisotropic Heisenberg exchange interaction. We also include the possibility of anisotropy together with an externally applied magnetic field of strength H. For the Hamiltonian describing this system, therefore, we write

$$\mathscr{H}=\frac{1}{2m}\sum_{i=1}^{N}[P_{0_i}^2+(mv_0)^2(u_i-u_{i-1})^2+\tfrac{1}{3}m\kappa\alpha(u_i-u_{i-1})^3]-\tfrac{1}{4}\sum_{i,\delta}J(u_i-u_{i+\delta})(S_i^+S_{i+\delta}^-+S_i^-S_{i+\delta}^+)$$
$$-\tfrac{1}{2}\sum_{i,\delta}\tilde{J}(u_i-u_{i+\delta})S_i^zS_{i+\delta}^z-A\sum_i(S_i^z)^2-\bar{\mu}H\sum_i S_i^z\ . \qquad (2.1)$$

Here P_{0_i} is the linear momentum of the ith atom, $\kappa=mv_0^2$ is the spring constant, α is the strength of nonlinear coupling between atomic displacements u_i, J and $\tilde{J}$ are the exchange coupling coefficients in the xy plane and in the z direction, respectively, A is the single ion anisotropy parameter, and $\bar{\mu}=g\mu_B$ is the magnetic constant. For small atomic displacements, we can expand $J(u_i-u_{i+\delta})$ in a

Taylor series about its equilibrium value J_0 and approximate it by the first two terms only:

$$J(u_i-u_{i+\delta})\approx J_0-(u_{i+\delta}-u_i)J_1 \,,$$
$$\tilde{J}(u_i-u_{i+\delta})\approx \tilde{J}_0-(u_{i+\delta}-u_i)\tilde{J}_1 \,, \tag{2.2}$$

where

$$J_1=\frac{\partial J}{\partial(u_i-u_{i+\delta})} \,. \tag{2.3}$$

We can thus write

$$\mathcal{H}=\mathcal{H}_p+\mathcal{H}_m+\mathcal{H}_{mp} \,, \tag{2.4}$$

where the lattice Hamiltonian $\mathcal{H}_p$ is

$$\mathcal{H}_p=\frac{1}{2m}\sum_i [P_{0_i}^2+(mv_0)^2(u_i-u_{i-1})^2 + \tfrac{1}{3}m\kappa\alpha(u_i-u_{i-1})^3] \,, \tag{2.5}$$

the spin Hamiltonian $\mathcal{H}_m$ is

$$\mathcal{H}_m=-\tfrac{1}{4}J_0\sum_{i,\delta}(S_i^+S_{i+\delta}^-+S_i^-S_{i+\delta}^+)-\tfrac{1}{2}\tilde{J}_0\sum_{i,\delta}S_i^zS_{i+\delta}^z - A\sum_i (S_i^z)^2-\bar{\mu}H\sum_i S_i^z \,, \tag{2.6}$$

and the spin-lattice interaction $\mathcal{H}_{mp}$ is

$$\mathcal{H}_{mp}=\tfrac{1}{4}J_1\sum_{i,\delta}(u_{i+\delta}-u_i)(S_i^+S_{i+\delta}^-+S_i^-S_{i+\delta}^+) + \tfrac{1}{2}\tilde{J}_1\sum_{i,\delta}(u_{i+\delta}-u_i)S_i^zS_{i+\delta}^z \,. \tag{2.7}$$

For the convenience of calculations, we confine our study to a system at sufficiently low temperatures and with spins of sufficiently large magnitude so that the spin operators may be expressed in terms of boson creation and annihilation operators through

$$S_i^+\approx(2S)^{1/2}\left[a_i-\frac{1}{4S}a_i^\dagger a_i a_i\right] \,,$$
$$S_i^-\approx(2S)^{1/2}\left[a_i^\dagger-\frac{1}{4S}a_i^\dagger a_i^\dagger a_i\right] \,, \tag{2.8}$$
$$S_i^z=S-a_i^\dagger a_i \,.$$

We also express the atomic displacements in terms of the phonon normal mode operators b and $b^\dagger$:

$$u_i=\frac{1}{N^{1/2}}\sum_k\left[\frac{\hbar}{2m\omega(k)}\right]^{1/2}(b_k+b_{-k}^\dagger)e^{ikx_i} \,, \tag{2.9}$$

where $\omega(k)$ are the kth phonon normal-mode frequencies

$$\omega^2(k)=4\frac{\kappa}{m}\sin^2\left[\frac{a_0k}{2}\right] .$$

We thus obtain

$$\mathcal{H}_p=\mathcal{H}_p^{(0)}+\mathcal{H}_p^{(1)},\quad \mathcal{H}_m=\mathcal{H}_m^{(0)}+\mathcal{H}_m^{(1)},\quad \mathcal{H}_{mp}=\mathcal{H}_{mp}^{(0)}+\mathcal{H}_{mp}^{(1)} \,, \tag{2.10}$$

$$\mathcal{H}_p^{(0)}=\sum_k \hbar\omega(k)(b_k^\dagger b_k+\tfrac{1}{2}) \,, \tag{2.11}$$

$$\mathcal{H}_p^{(1)}=\sum_{k_1k_2k_3}\Phi(k_1,k_2,k_3)\delta(k_1+k_2+k_3)(b_{k_1}+b_{-k_1}^\dagger)(b_{k_2}+b_{-k_2}^\dagger)(b_{k_3}+b_{-k_3}^\dagger) \,, \tag{2.12}$$

with

$$\Phi(k_1,k_2,k_3)=\frac{\kappa\alpha}{6N^{1/2}}\left[\frac{\hbar}{2m}\right]^{3/2}(2i)^3e^{-i(k_1+k_2+k_3)a_0}[\omega(k_1)\omega(k_2)\omega(k_3)]^{-1/2} \times\sin\left[\frac{k_1a_0}{2}\right]\sin\left[\frac{k_2a_0}{2}\right]\sin\left[\frac{k_3a_0}{2}\right] ,$$

$$\mathcal{H}_m^{(0)}=-\tilde{J}_0S^2N-AS^2N-\bar{\mu}HSN+(\bar{\mu}H+AS+2\tilde{J}_0S)\sum_i a_i^\dagger a_i-J_0S\sum_i(a_{i+1}^\dagger a_i+a_i^\dagger a_{i+1}) \,, \tag{2.13}$$

$$\mathcal{H}_m^{(1)}=\tfrac{1}{4}J_0\sum_i(a_i^\dagger a_{i+1}^\dagger a_{i+1}a_{i+1}+a_i^\dagger a_i^\dagger a_i a_{i+1}+a_i^\dagger a_i^\dagger a_i a_i+a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1}a_i)-\tilde{J}_0\sum_i a_i^\dagger a_i a_{i+1}^\dagger a_{i+1} - A\sum_i a_i^\dagger a_i^\dagger a_i a_i \,, \tag{2.14}$$

$$\mathcal{H}_{mp}^{(1)}=\frac{S}{N^{1/2}}\sum_i\sum_k\left[\frac{\hbar}{2m\omega(k)}\right]^{1/2}(b_k+b_{-k}^\dagger)e^{ikx_i}[J_1(e^{ika_0}-1)(a_{i+1}^\dagger a_i+a_i^\dagger a_{i+1})-2i\tilde{J}_1\sin(ka_0)a_i^\dagger a_i] \,, \tag{2.15}$$

$$\mathscr{H}_{mp}^{(2)}=\frac{1}{N^{1/2}}\sum_i\sum_k\left[\frac{\hbar}{2m\omega(k)}\right]^{1/2}(b_k+b_{-k}^\dagger)e^{ikx_i}\left[\frac{S}{4}J_1(e^{ika_0}-1)(a_ia_{i+1}^\dagger a_{i+1}^\dagger a_{i+1}+a_i^\dagger a_{i+1}^\dagger a_{i+1}a_{i+1}+a_i^\dagger a_ia_ia_{i+1}^\dagger+a_i^\dagger a_i^\dagger a_ia_{i+1})+\tilde{J}_1(e^{ika_0}-1)a_i^\dagger a_ia_{i+1}^\dagger a_{i+1}\right]. \quad (2.16)$$

III. EQUATIONS OF MOTION

Using the Heisenberg equation of motion for an operator $\hat{O}$

$$i\hbar\frac{\partial}{\partial t}\hat{O}=[\hat{O},\mathscr{H}], \quad (3.1)$$

we write down the equation of motion for the phonon field $B_k=b_k+b_{-k}^\dagger$ and the magnon field a_i. For the phonon field B_k we obtain

$$i\hbar\frac{\partial}{\partial t}B_k(t)=[B_k(t),\mathscr{H}_p]+[B_k(t),\mathscr{H}_{mp}]. \quad (3.2)$$

Similarly,

$$i\hbar\frac{\partial}{\partial t}a_l(t)=[a_l(t),(\mathscr{H}_m+\mathscr{H}_{mp})]. \quad (3.3)$$

Equations (3.2) and (3.3) come from a set of coupled equations in the phonon and magnon fields. We make a decoupling approximation here. We assume the exchange coupling coefficient to be a slowly varying function of space both in the xy plane and in the z direction, so that J_1 and $\tilde{J}_1$ are very small compared to J_0 and hence to κ. The second commutator on the right-hand side of Eq. (3.2) is, therefore, of much smaller magnitude than the first and hence we neglect it in this equation. This approximation implies physically that, as far as the nonlinear dynamics of the phonon field is concerned, the effect of its coupling to the magnon field is weak enough to cause any appreciable influence. Under this approximation Eq. (3.2) reduces to a pure phonon equation of motion,

$$i\hbar\frac{\partial}{\partial t}B_k(t)=[B_k(t),\mathscr{H}_p^{(0)}]+[B_k(t),\mathscr{H}_p^{(1)}]. \quad (3.4)$$

Defining a variable $y_k(t)$ as

$$y_k(t)=\left[\frac{\hbar}{2m\omega(k)}\right]^{1/2}B_k(t),$$

and considering now $y_k(t)$ to be a classical variable describing the dynamics of a macroscopic state, one obtains, in the long-wavelength limit[11] ($ka_0\ll 1$) by neglecting the terms of order k^6,

$$\frac{\partial^2}{\partial t^2}y_k(t)+v_0^2\left[k^2-\frac{a_0^2}{12}k^4\right]y_k(t)+\frac{i}{N^{1/2}}\frac{\alpha a_0}{2}v_0^2k\sum_{k'}k'(k-k')y_{k'}(t)y_{k-k'}(t)=0, \quad (3.5)$$

where v_0 is the sound velocity along the chain. Further defining

$$u(k,t)=iky_k(t),$$

and

$$u(x,t)=\frac{1}{N^{1/2}}\sum_k u(k,t)e^{ikx}, \quad (3.6)$$

one obtains the Boussinesq equation

$$\frac{\partial^2}{\partial t^2}u(x,t)-v_0^2\frac{\partial^2}{\partial x^2}u(x,t)-\frac{v_0^2a_0^2}{12}\frac{\partial^4}{\partial x^4}u(x,t)-\frac{\alpha a_0}{2}v_0^2\frac{\partial^2}{\partial x^2}[u(x,t)]^2=0, \quad (3.7)$$

which admits a soliton solution[11]

$$u(x,t)=A_0\operatorname{sech}^2\left\{\left[\frac{\alpha A_0}{a_0}\right]^2\left[x-\left[1+\frac{\alpha a_0}{6}A_0\right]v_0t\right]\right\}. \quad (3.8)$$

Thus, in the above approximation, despite coupling to magnons, the phonons among themselves form a soliton.

Turning now to Eq. (3.3), we get

$$i\hbar\frac{\partial}{\partial t}a_l(t)=[\bar{\mu}H+(2S-1)A+2\tilde{J}_0S]a_l-J_0S(a_{l-1}+a_{l+1})-2A\,|a_l|^2a_l-\tilde{J}_0[a_{l+1}^{\dagger}a_{l+1}+a_{l-1}^{\dagger}a_{l-1}]a_l$$
$$+\tfrac{1}{4}J_0[(a_{l+1}^{\dagger}a_{l+1}+2a_l^{\dagger}a_l)a_{l+1}+(a_{l-1}^{\dagger}a_{l-1}+2a_l^{\dagger}a_l)a_{l-1}+(a_{l+1}^{\dagger}+a_{l-1}^{\dagger})a_la_l]$$
$$+Sa_0u(x_l,t)[J_1(a_{l+1}+a_{l-1})-2\tilde{J}_1a_l]\ . \tag{3.9}$$

In Eq. (3.9) we have ignored the commutator of the field $a_l(t)$ with the Hamiltonian $\mathscr{H}_{mp}^{(2)}$ which contributes such terms to the equation of motion which are negligible in the low-amplitude approximation. We now regard $a_l(t)$ to be a classical variable too, describing the dynamics of a macroscopic magnetic state. Then in the continuum limit ($a_0\to 0$),

$$a_l(t)\to a(x_l,t)\ ,$$

and we obtain the following nonlinear evolution equation for the magnon field

$$i\hbar\frac{\partial}{\partial t}a(x,t)=[\bar{\mu}H-(2S-1)A+2S(\tilde{J}_0-J_0)]a(x,t)+2Sa_0(J_1-\tilde{J}_1)u(x,t)a(x,t)$$
$$-a_0^2S[J_0-a_0J_1u(x,t)]\frac{\partial^2}{\partial x^2}a(x,t)-2(A+\tilde{J}_0-J_0)\,|a(x,t)|^2a(x,t)\ . \tag{3.10}$$

For a pure magnetic problem ($J_1=0=\tilde{J}_1$), Eq. (3.10) can also be obtained from the Landau equation by taking the classical analog of the spin Hamiltonian (2.6). In fact, the equation of motion obtained this way does not contain a term $Aa(x,t)$ present in Eq. (3.10) where it enters as a result of quantum correction brought in by introducing normal ordering of the bose spin-deviation operators a and $a^{\dagger}$. It reflects the necessary fact that for a spin-$\frac{1}{2}$ system, the easy axis anisotropy A should have no effect on the dynamics of the system.

IV. TRAPPING CONDITIONS FOR THE MAGNETIC SOLITONS

For a weak coupling between the lattice and the magnetic excitations, the effects of the former on the latter are expected to be appreciable only in the vicinity of the maximum of the lattice soliton pulse. The function $u(x,t)$ in Eq. (3.10) may, therefore, be expanded in powers of its argument, retaining only the first two terms.

In terms of dimensionless variable x' and t' defined through

$$t'=\frac{2}{\hbar}(A+\tilde{J}_0-J_0)t\ , \tag{4.1}$$

$$x'=\left[\frac{A+\tilde{J}_0-J_0}{Sa_0^2(J_0-a_0A_0J_1)}\right]^{1/2}x\ , \tag{4.2}$$

and defining a field $a'(x',t')$ by

$$a'(x',t')=a(x',t')\exp\left[\frac{i}{2}\frac{\bar{\mu}H+(2S-1)A+2S[\tilde{J}_0+a_0(J_1-\tilde{J}_1)]}{A+\tilde{J}_0-J_0}\right]\ , \tag{4.3}$$

Eq. (3.10) reduces to

$$i\frac{\partial}{\partial t'}a'+\frac{1}{2}\frac{\partial^2}{\partial x'}a'+|a'|^2a'=P(x'-\psi t')^2a'+Q(x'-\psi t')^2\frac{\partial^2}{\partial x'^2}a' , \tag{4.4}$$

where

$$P\approx\frac{-\alpha^2 S a_0 A_0^3 J_0(J_1-\tilde{J}_1)}{(A+\tilde{J}_0-J_0)^2} , \tag{4.5}$$

$$Q=-\frac{1}{2}\frac{\alpha^2 S a_0 A_0^3 J_1}{A+\tilde{J}_0-J_0} , \tag{4.6}$$

and

$$\psi=\frac{(\hbar/2)(1+\frac{1}{6}\alpha a_0 A_0)v_0}{[S a_0^2(A+\tilde{J}_0-J_0)(J_0-a_0A_0J_1)]^{1/2}} . \tag{4.7}$$

In the absence of the spin-lattice coupling, $J_1=0=\tilde{J}_1$, i.e., $P=0=Q$, Eq. (4.4) reduces to the standard nonlinear Schrödinger (NLS) equation with the solution

$$a'(x',t')=2\nu_0 \operatorname{sech} z\, e^{i[(\mu_0/\nu_0)z+\delta_0]} ,$$
$$z=2\nu_0(x'-\xi_0) , \tag{4.8}$$

with the usual time evolution of the parameters μ_0, ν_0, ξ_0, and δ_0:

$$\frac{\partial\mu_0}{\partial t'}=0 , \quad \frac{\partial\nu_0}{\partial t'}=0 ,$$
$$\frac{\partial\xi_0}{\partial t'}=2\mu_0, \quad \frac{\partial\delta_0}{\partial t'}=2(\mu_0^2+\nu_0^2) , \tag{4.9}$$

For a slowly varying exchange function $J(u_i-u_{i+1})$, the coefficients J_1 and $\tilde{J}_1$ are small and may be regarded as the parameters of a perturbation caused by the lattice soliton on the evolution of the magnetic soliton. The right-hand side of Eq. (4.4) can then be taken as a perturbation to the above-mentioned unperturbed NLS equation. It is notable that both the perturbation terms introduce spatial inhomogeneities into the evolution of the magnetic soliton. The P term provides a harmonic potential from the lattice soliton in which the magnetic soliton is expected to get trapped or repelled, depending on the sign of coefficient P. On the other hand, the Q perturbation affects the dispersion term of the evolution equation and is, therefore, expected to make the amplitude, and hence the width, of the magnetic soliton time dependent. An application of the soliton perturbation theory[12] bears out the above expectations as shown in the following.

For Eq. (4.4) we take the solution

$$a'(x',t')=2\nu \operatorname{sech} z\, e^{[i(\mu/\nu)z+\delta]},$$
$$z=2\nu(x'-\xi) \tag{4.10}$$

The perturbation theory then yields the following time evolution of the parameters μ, ν, ξ, and δ:

$$\frac{\partial\mu}{\partial t'}=(-P-\tfrac{8}{3}Q\nu^2+4Q\mu^2)(\xi-\psi t') , \tag{4.11a}$$

$$\frac{\partial\nu}{\partial t'}=-8Q\mu\nu(\xi-\psi t') , \tag{4.11b}$$

$$\frac{\partial\xi}{\partial t'}=2\mu-\pi^2 Q\frac{\mu}{(2\nu)^2}-4Q\mu(\xi-\psi t') , \tag{4.11c}$$

$$\frac{\partial\delta}{\partial t'}=2(\mu^2+\nu^2)-[P-4Q(\mu^2+\nu^2)](\xi-\psi t')^2 + \frac{\pi^2}{48\nu^2}[P-4Q(\nu^2-\mu^2)]+2QI$$

$$I=\int_{-\infty}^{\infty}\frac{z^2(1-z\tanh z)}{\cosh^4 z}dz . \tag{4.11d}$$

Equations (4.11) form a complicated set of coupled differential equations. Rather than solve them explicitly at this stage, we look at the distinct effects of the P and Q perturbations separately. We first consider P perturbation only; i.e., $Q=0$. Equations (4.11) reduce to

$$\frac{\partial\mu}{\partial t'}=-P(\xi-\psi t') , \tag{4.12a}$$

$$\frac{\partial\nu}{\partial t'}=0 , \tag{4.12b}$$

$$\frac{\partial\xi}{\partial t'}=2\mu , \tag{4.12c}$$

$$\frac{\partial\delta}{\partial t'}=2(\mu^2+\nu^2)-P(\xi-\psi t')^2+\frac{\pi^2}{48\nu^2}P . \tag{4.12d}$$

In a frame where the lattice soliton velocity $v_p=(1+\frac{1}{6}\alpha a_0 A_0)v_0$ is zero, ψ vanishes and Equations (4.12) yield the time evolution of magnetic soliton parameters in agreement with the results of Kaup and Newell.[13] Equations (4.12a) and (4.12b) specifically show that the magnetic soliton is trapped (or repelled) in the harmonically perturbing potential generated by the lattice soliton. The magnetic soliton is trapped if $\tilde{J}_1>J_1$. On the other hand, the marked effect of the Q perturbation is that it introduces time dependence in the otherwise constant amplitude of the magnetic soliton.

V. SUMMARY AND CONCLUSIONS

We therefore conclude that, in the event $\tilde{J}_1 > J_1$, the magnetic soliton may get trapped into the harmonic field of the lattice soliton and oscillate about it with a time-dependent amplitude and width.

The physical realization of the situation discussed in this paper is rather remote. Near-Heisenberg chains can be realized in the laboratory, but a chain where the oscillations of the spin-carrying atoms is also restricted along the chain direction is not known to the authors. The asymptotic deformation of the magnetic solitons as a result of the perturbation is under study at present and is planned to be reported elsewhere.

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