Dynamics of spin vortices in two-dimensional planar magnets

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We investigate the dynamics of a dilute gas of free spin vortices in a two-dimensional planar magnet. An equation of motion for the spin vortex is presented and compared with the corresponding equation for a vortex in a superfluid film. Exploiting a similar analogy with the dynamics of a two-dimensional plasma in a perpendicular magnetic field we calculate the mean-square vortex velocity and the vortex self-diffusion constant in the critical region above the phase transition. When combined with equations obtained previously for the spin-autocorrelation functions, our results provide an approximate description of the vortex contribution to the critical dynamics outside of the hydrodynamic regime.

I. INTRODUCTION

In a classic paper of fundment importance[1] Kosterlitz and Thouless (KT) showed that a variety of two-dimensional systems underwent unusual phase transitions in which there was a singularity in the susceptibility associated with the order parameter. However, below $T_{\rm KT}$, the temperature marking the appearance of the singularity, there was no long-range order; the fluctuations in the order parameter fell off as an inverse power of the distance as opposed to the exponential dependence characteristic of the high-temperature phase. In all of these systems the phase transition was driven by the unbinding of topological defects.

In the case of magnetic systems the KT picture is applicable to easy-plane magnets with rotational symmetry about the hard axis, and the relevant topological excitations are spin vortices. Although there are no ideal two-dimensional easy-plane magnets, recent measurements on the planar compounds K_2CuF_4 (Refs. 2 and 3) and $NiCl_2$ intercalated in graphite[4,5] indicate behavior characteristic of a KT transition modified by weak in-plane anisotropy and interplane interactions.

The dynamic response of magnetic systems undergoing KT transitions reflects the existence of two classes of excitions: spin waves and the aforementioned vortices. The spin-wave contribution to the dynamic correlation functions below $T_{\rm KT}$ has been analyzed by Villain[6] and by Nelson and Fisher.[7] In a series of recent papers[8–10] we have shown how the motion of the vortices affects the response in the critical region above $T_{\rm KT}$ where there is a dilute gas of free vortices. As a first approximation we assumed that the spin-wave and vortex dynamics could be decoupled. (We use "spin wave" to label excitations which evolve into propagating modes below $T_{\rm KT}$.) This being the case the transverse (in-plane) spin-autocorrelation function could be written as

$$\langle S_x^n(t)S_x^n(0)\rangle \equiv \langle S_y^n(t)S_y^n(0)\rangle = C_S(t)C_V(t) \; . \tag{1.1}$$

Here S_x^n and S_y^n are the x and y components of the nth spin $\vec{S}^n$, which we take to be a classical vector of length S, $C_S(t)$ is the spin-wave part, and $C_V(t)$ denotes the vortex contribution. The latter can be written as

$$C_V(t)=\mathrm{Re}\langle \exp[i(\phi_n^L(t)-\phi_n^L(0)]\rangle \; , \tag{1.2}$$

where the brackets denote a thermal average, Re denotes real part, and ϕ_n^L is the polar angle associated with the projection of $\vec{S}^n$ onto the XY plane in the local minimum configuration.[1] The quantity ϕ_n^L is given by

$$\phi_n^L=\sum_j q_j \arctan\,[(y_n-Y_j)/(x_n-X_j')] \; , \tag{1.3}$$

where (x_n,y_n) are the coordinates of the nth spin, and (X_j,Y_j) are the coordinates of the jth vortex with strength (circulation) equal to q_j. For simplicity we will assume $q_j=\pm 1$.

We approximate $C_V(t)$ according to

$$\begin{aligned} C_V(t) &\approx \exp\{-\tfrac{1}{2}\langle[\phi_n^L(t)-\phi_n^L(0)]^2\rangle\} \; , \\ &= \exp\left[-\int_0^t d\tau(t-\tau)\langle\dot\phi_n^L(\tau)\dot\phi_n^L(0)\rangle\right] \; , \end{aligned} \tag{1.4}$$

where $\dot\phi=d\phi/dt$. Neglecting cross correlations between different vortices and replacing the summation by an integration we obtain the result[8]

$$C_V(t)=\exp\left[-2\pi\mathcal{N}_f\ln(L/a)\times\int_0^t d\tau\,(t-\tau)F(\tau)\right], \quad (1.5)$$

where $\mathcal{N}_f$ is the density of free vortices, a is the lattice constant (we assume a square lattice), and L is a macroscopic distance on the order of the dimension of the system. The quantity $F(\tau)$ denotes the vortex velocity autocorrelation function:

$$F(\tau)=\tfrac{1}{2}\langle\vec{V}(\tau)\cdot\vec{V}(0)\rangle . \quad (1.6)$$

For times which are long in comparison with the decay of $F(\tau)$ we can extend the limit on the integral in (1.5) to infinity thus obtaining

$$G_V(t)=\exp[-2\pi\mathcal{N}_f\ln(L/a)D_V\,|\,t\,|\,] , \quad (1.7)$$

where D_V is the vortex self-diffusion constant which is given by

$$D_V=\int_0^\infty F(t)dt . \quad (1.8)$$

In Ref. 10 it was pointed out that Eqs. (1.3)–(1.10) have a simple physical interpretation: The motion of the vortices produces changes in the phase with a consequent loss of correlation.[11]

As noted in Ref. 8 the spin-wave factor in (1.1) is expected to decay much more rapidly than $C_V(t)$ so that the vortex motion probably has no qualitative effect on the transverse spin-autocorrelation function. This situation contrasts with the behavior predicted for the longitudinal (out-of-plane) autocorrelation function.[9] When the longitudinal fluctuations are slowly varying and small in comparison with the in-plane fluctuations the appropriate canonical variables are the polar angles ϕ_n and the conjugate momenta S_z^n.[6,12] From Hamilton's equations one obtains the relation

$$\dot{\phi}=z(J_\perp-J_{||})S_z , \quad (1.9)$$

where z is the number of nearest neighbors and $J_{||}$ and $J_\perp$ ($J_\perp>J_{||}$) are exchange parameters in the microscopic spin Hamiltonian

$$\mathcal{H}=-\sum_{(i,j)}{}'[J_{||}S_z^iS_z^j+J_\perp(S_x^iS_x^j+S_y^iS_y^i)] , \quad (1.10)$$

in which the prime indicates that the sum is restricted to nearest-neighbor pairs.

Assuming that the spin-wave and vortex dynamics are decoupled one obtains an approximate expression for the longitudinal autocorrelation function which is of the form

$$\langle S_z^n(t)S_z^n(0)\rangle=[z(J_\perp-J_{||})]^{-2}\times[f_S(t)+\langle\dot{\phi}_n^L(t)\dot{\phi}_n^L(0)\rangle] . \quad (1.11)$$

Here $f_S(t)$ denotes the spin-wave contribution and $\langle\dot{\phi}_n^L(t)\dot{\phi}_n^L(0)\rangle$ is the vortex part. Note that the spin-wave and vortex excitations contribute *additively* to the longitudinal autocorrelation function in contrast to the *multiplicative* relation in the transverse function (1.1). With the same approximations employed in going from (1.4) to (1.5) one obtains

$$\langle S_z^n(t)S_z^n(0)\rangle=[z(J_\perp-J_{||})]^{-2}\times[f_S(t)+2\pi\mathcal{N}_f\ln(L/a)F(t)] . \quad (1.12)$$

As with the transverse function we expect $f_S(t)$ to decay rapidly in comparison with $F(t)$. This being the case the Fourier transform of $\langle S_z^n(t)S_z^n(0)\rangle$ will consist of a broad background with a spectral width $\sim J_\perp S^2\hbar^{-1}$ coming from $f_S(t)$ and a narrow central peak associated with $F(t)$. The relative weight of the central peak is given by

$$\text{rel. wt.}=\frac{2\pi\mathcal{N}_f\ln(L/a)F(0)}{[z(J_\perp-J_{||})]^2\langle(S_z^n)^2\rangle}=\frac{\pi\mathcal{N}_f\ln(L/a)\langle\vec{V}\cdot\vec{V}\rangle}{[z(J_\perp-J_{||})]^2\langle(S_z^n)^2\rangle} . \quad (1.13)$$

Although the results displayed in Eqs. (1.5), (1.7), (1.12), and (1.13) pertain to the spin-autocorrelation functions they are expected to apply to the wave-vector-dependent response for q values outside of the hydrodynamic regime, i.e., $q\gg\mathcal{N}_f^{1/2}$. It is evident that aside from the vortex density, which is an equilibrium parameter, the function of primary importance in determining the vortex contribution to the dynamical spin-correlation functions is the vortex velocity-autocorrelation function. The main purpose of this paper is to consider the behavior of this function. Utilizing an equation of motion given recently[13] we present a self-consistent calculation of the velocity-autocorrelation function of a gas of free vortices which exploits the formal similarity between the spin-vortex problem and the dynamics of a two-dimensional plasma in a perpendicular magnetic field.

The remainder of this paper is divided into three sections. In Sec. II we consider the equation of motion introduced in Ref. 13. The calculation of the autocorrelation function is developed in Sec. III, and our results are discussed in Sec. IV.

II. EQUATION OF MOTION

As was mentioned in Sec. I the vortex velocity-autocorrelation function plays a central role in determining the effect of the vortex motion on the evolution of the spin-correlation functions. In order to calculate this function we need an equation of motion for the vortices. The purpose of this section is to introduce such an equation. Our approach to vortex dynamics is based on a general theory of the dynamics of magnetic structures introduced by Thiele.[14,15] This theory starts with the Landau-Lifshitz equation

$$\frac{d\vec{M}}{dt}=\gamma\vec{M}\times\frac{\delta w}{\delta\vec{M}}+\alpha\left[\frac{d\vec{M}}{dt}\right]\times\frac{\vec{M}}{M}, \qquad (2.1)$$

where $\vec{M}$ is the local magnetization (magnetic-moment density), γ is the gyromagnetic ratio, w is the volume energy density, $\delta w/\delta\vec{M}$ denotes a variational derivative, and α is the Gilbert damping parameter. Assuming (2.1) is applicable, which is tantamount to the condition that the magnitude of $\vec{M}$ does not change, and making the further assumption that the structure moves without distortion, Thiele obtains the equation

$$(\gamma/m_0)\vec{F}+\vec{G}\times\vec{V}+\overleftrightarrow{D}\cdot\vec{V}=0\ . \qquad (2.2)$$

Here $\vec{V}$ is the velocity of the structure and m_0 is the magnitude of the local magnetic moment per unit area (we assume dependence on x and y only and have integrated over the z direction). The vector $\vec{F}$ denotes the static force (to be discussed below), $\vec{G}$ is the gyrovector, and $\overleftrightarrow{D}$ is the dissipation dyadic.

In terms of $\theta(\vec{r})$ and $\phi(\vec{r})$, the polar coordinates which specify the orientation of the moment at the point $\vec{r}$, $\vec{G}$, and $\overleftrightarrow{D}$ are given by

$$\vec{G}=-\int d^2r\,\sin\theta(\vec{r})\vec{\nabla}\theta(\vec{r})\times\vec{\nabla}\phi(\vec{r})\ , \qquad (2.3)$$

and

$$\overleftrightarrow{D}=-\alpha\int d^2r[\,\vec{\nabla}\theta(\vec{r})\vec{\nabla}\theta(\vec{r}) +\sin^2\theta(\vec{r})\vec{\nabla}\phi(\vec{r})\vec{\nabla}\phi(\vec{r})]\ . \qquad (2.4)$$

Since $\vec{\nabla}\theta$ and $\vec{\nabla}\phi$ are in the XY plane the gyrovector is in the perpendicular direction, while $\overleftrightarrow{D}$ has only xx, xy, and yy components.

In evaluating $\vec{G}$ and $\overleftrightarrow{D}$ we adopt the continuum description of a spin vortex in a two-dimensional, three-component spin system which was given by Hikami and Tsuneto.[16] In their model one has

$$\vec{\nabla}\phi=q\hat{\phi}/r\ , \qquad (2.5)$$

where q ($=\pm1$) is the strength of the vortex and $\hat{\phi}$ is a unit vector in the ϕ direction in a polar coordinate system whose origin coincides with the vortex core. Unlike the two-component planar rotator, where the spin is confined to the XY plane, the vortex in a three-component model has a spin projection perpendicular to the plane. Although the variation of $\theta(\vec{r})$ with r is discussed in Ref. 16 we need only the limiting values

$$\theta=\pi/2,\quad r=\infty \qquad (2.6a)$$

$$\theta=0 \text{ or } \pi,\quad r=0 \qquad (2.6b)$$

to evaluate $\vec{G}$. Using (2.6a) and (2.6b), we obtain

$$\vec{G}=-2\pi qp\hat{z}\ , \qquad (2.7)$$

where $\hat{z}$ is a unit vector perpendicular to the plane and $p=+1$ for vortices with $\theta(0)=0$ and $p=-1$ for those with $\theta(0)=\pi$.

In the case of $\overleftrightarrow{D}$ the dominant contribution to the integral in (2.4) comes from large r where $\nabla\theta$ is small and $\sin^2\theta\approx1$. We find $D_{xy}=D_{yx}=0$ and $D_{xx}=D_{yy}\equiv D_0$ where

$$D_0\approx-\alpha\pi\int\frac{dr}{r} =-\alpha\pi\ln(R/a)\ , \qquad (2.8)$$

in which R is the "outer radius" of the vortex and a is the lattice parameter. Generally, R will be a macroscopic distance on the order of L.

The static force $\vec{F}$ arises from both vortex-vortex interactions and interactions with in-plane applied fields. We will consider the former first. The force on the ith vortex coming from its interaction with the order $N-1$ vortices is given by

$$\vec{F}_i^I=-\frac{dW}{d\vec{R}_i}(\vec{R}_1,\ldots,\vec{R}_N)\ , \qquad (2.9)$$

where $W(\vec{R}_1,\ldots,\vec{R}_N)$ is the interaction energy of an assembly of N vortices situated at $\vec{R}_1,\ldots,\vec{R}_N$. As discussed in Ref. 16 the form of the interaction depends on the anistropy of the exchange. In the isotropic or Heisenberg limit ($J_\perp=J_{\parallel}$) the interaction is independent of the relative separation so that $\vec{F}_i^I=0$. In the XY limit ($J_{\parallel}=0$) we have

$$\vec{F}_i^I=2\pi J_\perp S^2 q_i\sum_{j\neq i}q_j\frac{(\vec{R}_i-\vec{R}_j)}{(\vec{R}_i-\vec{R}_j)^2}\ . \qquad (2.10)$$

The response to in-plane applied fields is less certain. If the vortex moves without distortion (which is probably a reasonable approximation only at very low fields, if at all) then the corresponding static force takes the form

$$\vec{F}^H = \int d^2r \left[\nabla\theta(\vec{r})\frac{d}{d\theta} + \nabla\phi(\vec{r})\frac{d}{d\phi} \right] \times[-m_0 H \sin\theta(\vec{r})\cos\phi(\vec{r})] \quad (2.11)$$

for a field H along the x direction. As in the evaluation of $\overleftrightarrow{D}$ the dominant contribution to the integral comes from large r. Thus we have

$$\vec{F}^H = m_0 H \int d^2r \sin\phi(\vec{r})\vec{\nabla}\phi(\vec{r}) \, . \quad (2.12)$$

In general, $\vec{F}^H$ will have both x and y components whose magnitude depends on the variation of ϕ with $\vec{r}$.

The effect of a static field in the XY plane is to sweep the vortices toward the boundaries of the sample. However, if the field is oscillatory [i.e., $H_0\cos(\omega t)$] the vortices will move back and forth about their equilibrium position. Owing to the viscous drag associated with $\overleftrightarrow{D}$ there will be a corresponding energy loss. When the magnetic forces are dominant the energy loss takes the form

$$\frac{dW}{dt} = -\frac{(\gamma/m_0)D_0}{4\pi^2 + D_0^2} \sum_{i=1}^{N} (\vec{F}_i^H)^2 \, , \quad (2.13)$$

where $\vec{F}_i^H$ is the force on the ith vortex. It is evident that the rate of energy loss is proportional to the square of the field and the density of free vortices. As discussed in Ref. 13 measurements of the energy loss near T_{KT} may provide a means of detecting the presence of vortices.

In the Introduction it was mentioned that there is a formal similarity between the equation of a spin vortex and the dynamics of a two-dimensional plasma in a perpendicular magnetic field.[17] When the Larmor radius is much less than the Debye length the plasma is in the "guiding center" limit. Under these conditions the equation for the velocity of the guiding center, which is the counterpart of the spin vortex, takes the form

$$\vec{V} = \frac{c}{B}\hat{z}\times\vec{\nabla}U \, , \quad (2.14)$$

where B is the magnetic flux density and the electrostatic potential U is a solution to the equation

$$\nabla^2 U(\vec{r}) = -\frac{4\pi|e|}{l} \sum_j q_j \delta(\vec{r} - \vec{R}_j) \, . \quad (2.15)$$

Here $-|e|/l$ is the magnitude of the charge per unit length and $q_j = \pm 1$ depending on whether the center is positively or negatively charged. From (2.14) and (2.15) we obtain the equation of motion

$$\vec{V} = -\frac{2|e|c}{lB}\hat{z}\times\vec{A}(\vec{R}) \, , \quad (2.16)$$

where

$$\vec{A}(\vec{R}) = \sum_j q_j \frac{(\vec{R} - \vec{R}_j)}{(\vec{R} - \vec{R}_j)^2} \, . \quad (2.17)$$

Equation (2.17) is to be compared with the equation of motion of a spin vortex in an XY magnet. From (2.2), (2.7), and (2.10) we obtain

$$\vec{V} = -\frac{D_0\hat{z}\times\vec{V}}{2\pi pq} - \frac{\gamma J_\perp S^2}{m_0 p}\hat{z}\times\vec{A}(\vec{R}) \quad (2.18)$$

for $H=0$. Comparing (2.16) with (2.18) it is evident that the motion of a guiding center resembles that of a dissipation-free spin vortex. However, unlike the guiding center, the spin vortex is characterized by two kinds of charge, p and q, although only q appears in the vortex-vortex interaction [cf. Eq. (2.22)].

There is also a close correspondence between the dynamics of a spin vortex and the equation of motion of a vortex in a superfluid helium film.[18,19] From Ref. 18 we obtain

$$\vec{V} = \frac{-2\pi q D\hbar\rho_s^0}{mk_BT}\hat{z}\times\vec{v}_s - C\vec{v}_s + \vec{v}_s \, , \quad (2.19)$$

assuming the normal fluid is at rest. Here $q(=\pm 1)$ is the sign of the vortex, m is the mass of the helium atom, ρ_s^0 is the "bare" superfluid density, C is a drift constant, and D is a bare diffusion constant associated withe dissipation. The symbol $\vec{v}_s$ denotes the local superfluid velocity which can be written

$$\vec{v}_s = (\hbar/m)\hat{z}\times\vec{A}(\vec{R}) \, , \quad (2.20)$$

where $\vec{A}(\vec{R})$ is given by (2.17).

Equation (2.19) can be expressed in a form resembling (2.18) by taking its cross product with $\hat{z}$ and substituting the result into (2.18). We obtain

$$\vec{V} = \frac{-2\pi q\hbar\rho_s^0 D\hat{z}\times\vec{V}}{mk_BT(1-C)} + \left[1 - C + \frac{[2\pi q h\rho_s^0 D/(mk_BT)]^2}{1-C} \right] \frac{\hbar}{m}\hat{z}\times\vec{A}(\vec{R}) \, , \quad (2.21)$$

which has the same formal structure as (2.18).

The formal correspondence among the three systems also extends to the interactions. In the XY limit the interaction energy of an assembly of spin vortices is given by

$$W = -2\pi J_\perp S^2 \sum_{(i,j)} q_i q_j \ln|\vec{R}_i - \vec{R}_j| \, , \quad (2.22)$$

apart from a constant. The interaction between the guiding centers can be written

$$W=\frac{-2|e|^2}{l}\sum_{(i,j)} q_i q_j \ln|\vec{R}_i-\vec{R}_j| \ , \quad (2.23)$$

whereas the interaction between vortices in a helium film takes the form[20]

$$W=\frac{-2\pi\rho_s^0\hbar^2}{m^2}\sum_{(i,j)} q_i q_j \ln|\vec{R}_i-\vec{R}_j| \ . \quad (2.24)$$

We can exploit this correspondence to obtain an estimate of the bare diffusion constant of the spin-vortex gas. From (2.18), (2.21) (with the drift constant $C=0$), (2.22), and (2.24), we obtain

$$D=\frac{k_B T\gamma|D_0|}{4\pi^2 m_0} \ . \quad (2.25)$$

III. CORRELATION FUNCTIONS

In this section we will outline a calculation of the vortex velocity-autocorrelation function which is based on the results obtained in Sec. II. We will restrict the analysis to the XY limit where the equation of motion has the form displayed in Eq. (2.18). The calculations pertain to the critical region above T_{KT} where there is a dilute gas of free vortices with a density[18]

$$\mathcal{N}_f=a^{-2}\exp[-b(T/T_{\text{KT}}-1)^{-1/2}] \quad (3.1)$$

with $b\approx 1$. In addition to the free vortices there are also bound vortex pairs. Their effect will be introduced later through the mechanism of an effective dielectric constant.

The most important limitation on the calculation pertains to the dissipative term $D_0\vec{V}$. We will assume that the damping parameter is sufficiently small that we can omit this term from the equation of motion. This will be a reasonable approximation whenever the root-mean-square (rms) value of $D_0\vec{V}$ is much less than the rms value of $\vec{F}^I$, i.e.,

$$|D_0|\langle\vec{V}\cdot\vec{V}\rangle^{1/2} \ll (\gamma/m_0)2\pi J_\perp S^2 \times\left[\sum_j \frac{1}{(\vec{R}-\vec{R}_j)^2}\right]^{1/2} , \quad (3.2)$$

or

$$\pi\alpha\langle\vec{V}\cdot\vec{V}\rangle^{1/2} \ll (\gamma/m_0)(2\pi)^{3/2}J_\perp S^2\mathcal{N}_f^{1/2} , \quad (3.3)$$

to logarithmic accuracy. As will be shown below we have

$$\langle\vec{V}\cdot\vec{V}\rangle^{1/2}\approx\pi^{1/2}(\gamma/m_0)J_\perp S^2\mathcal{N}_f^{1/2} , \quad (3.4)$$

neglecting logarithmic terms. Thus the inequality (3.2) is equivalent to

$$\alpha \ll 1 \ . \quad (3.5)$$

Values of α in real magnets vary greatly, being functions of temperature, impurity concentration, etc. However, it is not uncommon to have $\alpha<0.1$ in spin-$\frac{1}{2}$ or S-state systems [e.g., in K_2CuF_4 $\alpha\approx 10^{-2}$ at $0.7T_{\text{KT}}$ (Ref. 21)] so that the inequality (3.5) is not physically unrealistic.

Since the dissipation characterized by α arises in part through interaction with the spin waves, omitting $D_0\vec{V}$ from the equation of motion has the effect of decoupling the spin-wave and vortex degrees of freedom as was done earlier in the calculation of the spin-autocorrelation functions. In the context of superfluid films it is tantamount to assuming that the vortices move at the local superfluid velocity [i.e., $D=C=0$ in (2.19)]. In this limit the vortex motion arises entirely from their mutual interaction. A single vortex remains fixed in position although in reality it will undergo Brownian motion due to its interaction with the spin waves. It is this motion which is being omitted from our calculation.

With D_0 set equal to zero Eq. (2.18) is formally identical to the equation of motion of the guiding center in a two-dimensional plasma. Since the interaction terms (2.22) and (2.23) also have the same structure the connection between the two systems is established through the equations

$$(\gamma/m_0)J_\perp S^2 \rightleftarrows \frac{2|e|c}{lB} , \quad (3.6)$$

$$\frac{|e|^2}{l} \rightleftarrows \pi J_\perp S^2 \ . \quad (3.7)$$

The formal correspondence indicated in (3.6) and (3.7) is very useful for it allows us to apply results obtained in the analysis of the plasma[17] to the problem of spin vortices. To see how this works we outline the calculation of the mean-square vortex velocity. From (2.14) we obtain

$$\langle\vec{V}\cdot\vec{V}\rangle=(c^2/B^2)\langle\vec{\nabla}U\cdot\vec{\nabla}U\rangle \ . \quad (3.8)$$

Following Ref. 17 we expand the electric field in a Fourier series in $\vec{k}$. Neglecting correlations between different Fourier components we find

$$\langle\vec{V}\cdot\vec{V}\rangle=\frac{c^2}{2\pi B^2}\int k\,dk\langle|E(k)|^2\rangle \ . \quad (3.9)$$

The correlation in the electric field amplitudes can be evaluated in the Debye-Hückel approximation[17,18] with the result

$$\langle |E(k)|^2\rangle = \frac{4\pi k_B T}{l(1+k^2\lambda_D^2)} , \tag{3.10}$$

where λ_D is the Debye screening length

$$\lambda_D^2 = \frac{k_B T l}{4\pi \mathcal{N}_f |e|^2} , \tag{3.11}$$

with $\mathcal{N}_f$ the density of guiding centers. Thus we have

$$\langle \vec{V}\cdot\vec{V}\rangle = \frac{4\pi c^2 \mathcal{N}_f |e|^2}{B^2 l^2} \ln(k_{\max}\lambda_D) , \tag{3.12}$$

where $k_{\max}$ is determined by the shortest wavelength in the fluctuating field.

With the use of (3.6) and (3.7) the mean-square spin-vortex velocity is given by

$$\langle \vec{V}\cdot\vec{V}\rangle = \pi \left[\frac{\gamma J_\perp S^2}{m_0} \right]^2 \mathcal{N}_f \ln(2\pi\lambda_D/a) , \tag{3.13}$$

where now $\mathcal{N}_f$ is the density of free vortices and $k_{\max}=2\pi/a$. The Debye screening length for the spin problem is given by

$$\lambda_D^2 = \frac{k_B T_{\rm KT}}{4\pi^2 J_\perp S^2 \mathcal{N}_f} , \tag{3.14}$$

for $T \approx T_{\rm KT}$. Equation (3.14) can be simplified by making use of the approximate equation for $T_{\rm KT}^1$,

$$k_B T_{\rm KT} = (\pi/2) J_\perp S^2 , \tag{3.15}$$

so that

$$\lambda_D^2 = (8\pi \mathcal{N}_f)^{-1} . \tag{3.16}$$

We can obtain an approximate expression for the vortex self-diffusion constant from the theory presented in Ref. 17. The guiding-center diffusion constant is given by

$$D_V^2 = \frac{c^2}{4\pi B^2} \int_{k_{\min}}^{k_{\max}} k^{-1} dk \langle |E(k)|^2\rangle , \tag{3.17}$$

where $k_{\min}(\sim 2\pi/L)$ is determined by the maximum wavelength of the fluctuations in the electric field, which we take to be on the order of the size of the system. Using (3.10) and (3.17) we obtain

$$D_V^2 = \frac{c^2 k_B T}{B^2 l} \ln(L/2\pi\lambda_D) , \tag{3.18}$$

which is equivalent to

$$D_V = \frac{\gamma}{2\pi^{1/2} m_0} (J_\perp S^2 k_B T)^{1/2} \ln^{1/2}(L/2\pi\lambda_D) \tag{3.19}$$

in the spin-wave problem.

We obtained the results for $\langle \vec{V}\cdot\vec{V}\rangle$, D_V, and λ_D neglecting the presence of the bound vortex pairs. Following Refs. 1 and 18 we can account for their influence by using a dielectric constant in the vortex-vortex interaction. This is equivalent to the replacement $J \rightarrow J/\epsilon_c$ where the dielectric constant is approximated by its limiting value at $T_{\rm KT}$, which we denote by ϵ_c. Since Eq. (3.15) is also modified, i.e.,

$$k_B T_{\rm KT} = \frac{\pi J_\perp S^2}{2\epsilon_c} , \tag{3.20}$$

the equation for λ_D, (3.16), remains the same.

In modifying Eq. (3.13) we approximate m_0 by its low-temperature limit $\hbar\gamma a^{-2}$ (we assume a square lattice), and make use of (3.20) to obtain

$$\langle \vec{V}\cdot\vec{V}\rangle = (4/\pi)(a^2 k_B T_{\rm KT}\hbar^{-1})^2 \mathcal{N}_f \ln(2\pi\lambda_D/a) . \tag{3.21}$$

Likewise, in place of (3.19) we find

$$D_V = (2^{1/2}\pi)^{-1}(a^2 k_B T_{\rm KT}\hbar^{-1}) \ln^{1/2}(L/2\pi\lambda_D) , \tag{3.22}$$

which is much greater than the bare diffusion constant (2.25) when $\alpha \ll 1$.

When combined with the equations given in Sec. I, Eqs. (3.21) and (3.22) provide an estimate of the vortex contribution to the spin-autocorrelation functions in the critical region above $T_{\rm KT}$. It is evident that apart from a logarithmic correction[22] D_V is temperature independent whereas $\langle \vec{V}\cdot\vec{V}\rangle$ is proportional to the density of free vortices. Thus the intensity of the vortex central peak, Eq. (1.13), scales as $\mathcal{N}_f^2$.

In Ref. 23 we have applied the Taylor-McNamara formalism to the calculation of the cortex self-diffusion constant in a superfluid film with the result.

$$D_V = 2^{-3/2}(\hbar/m) \ln^{1/2}(L/2\pi\lambda_D) , \tag{3.23}$$

where, as before, m is the mass of the helium atom. Vortex diffusion in helium films has also been stuided by Petschek and Zippelius.[24] In their approach the interactions with the substrate and with thermal excitations such as ripplons (the analog of spin waves) and rotons provide the driving force for the diffusion. They find that the diffusion constant remains finite at $T_{\rm KT}$ with a cusp singularity, i.e.,

$$\frac{D_V(T)}{D_V(T_{\rm KT})} = 1 + d \left| 1 - \frac{T_{\rm KT}}{T} \right|^{1/2} , \tag{3.24}$$

where d is a constant.

IV. DISCUSSION

In the previous sections we have outlined a theory of the vortex dynamics in two-dimensional planar magnets. There are three distinct parts to the theory: the spin-autocorrelation functions, the vortex equation of motion, and the vortex velocity-autocorrelation function. We comment on each of these in turn.

There are two major approximations made in the calculation of the spin-autocorrelation functions. The first and most important is the decoupling of the spin-wave and vortex dynamics. This is achieved by writing the phase angle ϕ as $\phi_S+\phi_V$, where S and V are the spin-wave and vortex contributions, and by treating ϕ_S and ϕ_V as independent dynamical variables. While this is a plausible approximation when the two contributions to the correlation functions have very different decay rates we have no rigorous justification for it nor do we have a systematic way of incorporating higher-order effects. The second approximation involves the neglect of the correlation between different vortices in going from (1.4) to (1.5) and from (1.11) to (1.12). By making this approximation we reduce the problem to the calculation of a single vortex correlation function. We believe this step is reasonable in the low-density limit, $\mathcal{N}_f a^2 \ll 1$, but it too lacks rigorous justification.

However, it is worth noting that closely situated, highly correlated vortex pairs whose motion is determined (largely) by their mutual interaction make a comparatively small contribution to $\dot{\phi}_n^L$. From (1.3) we have

$$\frac{d\phi_n^L}{dt}=\sum_j q_j[\vec{V}_j\times(\vec{r}_n-\vec{R}_j)]_z/(\vec{r}_n-\vec{R}_j)^2 \; . \tag{4.1}$$

If j and k denote a pair of vortices moving under their mutual influence, then $\vec{V}_j=\vec{V}_k$ if the pair has opposite circulation, and $\vec{V}_j=-\vec{V}_k$ if the circulation is the same.[25] In either case they contribute the term

$$-[\vec{V}_j\times(\vec{R}_j-\vec{R}_k)]_z/(\vec{r}_n-\vec{R}_j)^2$$

to $\dot{\phi}_n^L$, which is reduced by a factor $|\vec{R}_j-\vec{R}_k|/|\vec{r}_n-\vec{R}_j|$ relative to the contribution of a single vortex.

The approximations made in obtaining the equation of motion for the spin vortex involve the assumptions of rigid translation and the applicability of the Landau-Lifshitz equation. Judging from the success of the Thiele approach in accounting for the dynamics of magnetic bubbles in thin films[26] we feel that (2.18) is defendable if not rigorous and probably is on a par with the hydrodynamic model of vortices in superfluid helium. The neglect of $D_0\vec{V}$ when $\alpha \ll 1$ seems justified but may leave out some important effects.

The calculations of $\langle \vec{V}\cdot\vec{V}\rangle$ and D_V involve two separate approximations. The equal-time correlation function is obtained after making the physically reasonable Debye-Hückel approximation for $\langle |E(k)|^2\rangle$. As noted in Ref. 23 the equation for D_V given in Ref. 17 can be obtained by postulating that the fluctuations in the Fourier components of the electric field decay as $\exp(-D_V k^2 t)$ and then solving the self-consistent equation

$$\begin{aligned} D_V &= \tfrac{1}{2}\int_0^\infty \langle \vec{V}(t)\cdot\vec{V}(0)\rangle dt \\ &= \frac{c^2}{4\pi B^2}\int_0^\infty dt \int d\,dk \langle |E(k)|^2\rangle e^{-D_V k^2 |t|} \end{aligned} \tag{4.2}$$

for D_V. It is obvious that (4.2) is a crude approximation. Nevertheless, it gives the result expected on dimensional grounds. However, the logarithmic correction in (3.22) and (3.23) appears to be an artifact of the condition $\alpha\equiv 0$ since it originates in the assumption that the fluctuations in the field decay asymptotically as $\exp(-D_V k^2 t)$ for arbitrarily small values of k. If there is a lower cutoff to the decay rate, as well happen when there is dissipation, then D_V is independent of L in the limit $L\to\infty$.

With the assumption of exponential decay for fluctuations in the electric field we can obtain an approximate expression for $\langle \vec{V}(t)\cdot\vec{V}(0)\rangle$ analogous to (4.2). We have

$$\langle \vec{V}(t)\cdot V(0)\rangle = \langle \vec{V}\cdot\vec{V}\rangle (\ln x_{\max})^{-1}\int_{x_{\min}}^{x_{\max}} \frac{dx\, \exp[-(D_V/\lambda_D^2)\,|t|\,x]}{1+x} \; , \tag{4.3}$$

where $x_{\min}=k_{\min}^2\lambda_D^2$ and $x_{\max}=k_{\max}^2\lambda_D^2$. Thus when

$$\lambda_D^2/(D_V x_{\max}) \ll t \ll \lambda_D^2/(D_V x_{\min}) \; , \tag{4.4}$$

$\langle \vec{V}(t)\cdot\vec{V}(0)\rangle$ decays as t^{-1}.[27]

The existence of the various approximations discussed above points to the need for additional theoretical work on all three aspects of the problem. Experimental tests of various features of the theory would also be worthwhile. In particular, the detection of a central peak[28] in the longitudinal spin-correlation function outside of the hydrodynamic regime might provide direct evidence for the existence of a gas of spin vortices.

Note added in proof. An alternative derivation of Eq. (2.18) for a system with $D_0=0$ has been given by Takeno and Homma [S. Takeno and S. Homma, Progr. Theor. Phys. 67, 1633 (1982)].

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[27]Were we to take $k_{\min}=0$, $C_V(t)$ [Eq. (1.5)] would decay as $\exp[-At\ln(D_V t/\lambda_D^2)]$ as $t\rightarrow\infty$ [B. I. Halperin (private communication)].

[28]In the spirit of (4.2) the Fourier transform of $\langle \vec{V}(t)\cdot\vec{V}(0)\rangle$ is proportional to

$$\mathrm{Re}(\tilde{D}-i\omega)^{-1}\ln\left[\frac{(\tilde{D}x_{\max}+i\omega)(x_{\min}+1)}{(x_{\max}+1)(\tilde{D}x_{\min}+i\omega)}\right],$$

so that the central peak has a width on the order of $\tilde{D}\equiv D_V/\lambda_D^2$.